Role of fluctuations in random compressible systems at marginal dimensionality

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In a unified treatment we have studied the role of fluctuations in uniaxial random systems at marginal dimensionality $d^*=4$ with the $n=1$ component order parameter being coupled to elastic degrees of freedom. Depending on the ratio of the nonuniversal parameters of quenched disorder Δ_0 and of elastic fluctuations $\tilde{v}_0$, a first- or second-order phase transition is found to occur, separated by a tricritical point. A complete account of critical properties and of macroscopic as well as of microscopic elastic stability is given for temperatures $T>T_c$. Universal singularities of thermodynamic functions are determined for $t=(T-T_c)/T_c\rightarrow 0$ including the tricritical point: for $\tilde{v}_0/\Delta_0>-2$, they are the same as in a rigid random system; for $\tilde{v}_0/\Delta_0=-2$, they are different due to lattice compressibility being related, however, to the former by Fisher renormalization. Fluctuation corrections in one-loop approximation have been evaluated in a nonuniversal critical temperature range, $t_x \ll t \ll 1$, sufficiently far from the universal critical regime. The latter apparently becomes extremely narrow due to the order of magnitude being obtained for t_x whose origin is in a peculiar degeneracy of the renormalization-group (RG) equations in leading order. In solving these RG recursion relations in the nonuniversal regime exactly, analytic expressions for thermodynamic functions have been obtained improving a recent approach for rigid random systems. The nonuniversal critical behavior can be characterized by effective exponents varying continuously with Δ_0 and $\tilde{v}_0$. We have also estimated numerically limits of the validity of the one-loop approximation. The temperature dependence of the elastic constants has been obtained for the entire region of temperatures. The critical behavior of random compressible systems, unlike that of pure compressible systems, is finally shown to remain stable against weak lattice anisotropy.

I. INTRODUCTION

A sensible assessment of the importance of fluctuations at phase transitions can apparently be obtained by the concept of the upper critical (marginal) dimensionality d^* introduced in the renormalization-group (RG) theory of critical phenomena.[1] For a given system of spatial dimensionality d the marginal dimensionality d^* serves as a borderline between classical mean-field ($d>d^*$) and nonclassical critical ($d<d^*$) behavior. The RG equations at d^* turn out to be solvable exactly in the asymptotic critical limit with the result that mean-field (MF) or Landau-type behavior is only modified by singularities weaker than any power of the reduced temperature $t=(T-T_c)/T_c$, where T_c denotes the true critical temperature. It is extremely important, however, that the marginal critical behavior of pure and rigid systems may be changed significantly by perturbations somehow typical for real materials, e.g., by randomness in the local MF transition temperature or by lattice compressibility, provided the specific heat of the unperturbed system diverges. The marginal dimensionality d^* is assumed to remain unchanged under this type of perturbation.

Particularly interesting in that context are uniaxial dipolar ferromagnets and uniaxial ferroelectrics, where the spontaneous magnetization M and the spontaneous polarization P, respectively, act as $n=1$ component order parameters. The anisotropic interaction of dipoles aligned along a single axis leads to a marginal dimensionality $d^*=3$ which thus coincides with the actual spatial dimensionality, i.e., $d=d^*=3$. The theory of these uniaxial dipolar systems therefore provides for direct experimental verification of rigorous results obtained by RG methods.

It is well established by now that critical fluctuations in pure uniaxial dipolar systems on a rigid lattice give rise to fractional powers of logarithmic correction factors displayed in the asymptotic critical behavior, e.g., of the order-parameter susceptibility $\chi\propto|t|^{-1}|\ln|t||^{1/3}$, of the specific heat $C\propto|\ln|t||^{1/3}$, of the nonlinearity coefficient $f_2^{-1}\propto|\ln|t||$ in the equation of state, etc.[2–6]

Clear evidence for logarithmic corrections in the specific heat apparently came first from experiments on $LiTbF_4$ as a model system for uniaxial dipolar ferromagnets.[7] Logarithmic corrections in the order-parameter susceptibility have recently also been observed unambiguously in trissarcosine calcium chloride (TSCC), a model system for uniaxial ferroelectrics.[8] Since work directed towards detecting logarithmic corrections in the electric susceptibility, in the specific heat, and in the spontaneous polarization of unaxial ferroelectrics remained inconclusive for a long time, it was pointed out[4] that the nonlinearity coefficient $f_2=(\partial^3E/\partial P^3)_{t>0;E\to 0}$ in the equation of state is more favorable for observing such logarithmic corrections in agreement with data on the uniaxial ferroelectric triglycine sulfate (TGS). In replacing the electric field E conjugate to the polarization P by the magnetic field H conjugate to the magnetization M an analogous behavior is to be expected for $f_2=(\partial^3H/\partial M^3)_{t>0;H\to 0}$ in the equation of state of pure uniaxial dipolar ferromagnets, which was indeed found with a remarkable accuracy in recent experimental studies[9] on $LiTbF_4$.

The role of fluctuations in random uniaxial systems on a rigid lattice at $d^*=d=4$ has aroused interest,[10] since a new type of marginal critical behavior could be anticipated from the logarithmically diverging specific heat of the pure rigid system owing to a general argument by Harris[11] as well as subsequent RG calculations.[12] By using such RG methods Aharony[13] has shown that these systems with quenched random disorder indeed exhibit a new asymptotic critical behavior characterized by exponential rather than logarithmic corrections to the MF behavior. Thus, the order-parameter susceptibility for $t>0$, e.g., behaves as $\chi\propto|t|^{-1}\exp[(D|\ln|t||)^{1/2}]$, where the difference of the values of $D=\frac{6}{53}$ for uniaxial short-ranged $d=4$ systems and of $D=9/[81\ln(4/3)+53]$ for uniaxial dipolar $d=3$ systems is small numerically. The specific heat remains finite exhibiting a cusplike singularity only. Theoretical studies in the limit of small impurity concentrations[13,14] apparently suggest the asymptotic critical behavior with exponential singularities to occur in a temperature range, $t\ll t_x$, with some extremely small characteristic temperature t_x, the origin of which is in the peculiar degeneracy of the RG scaling relations to leading (one-loop) order.

Thorough experimental studies[15–17] of the magnetic susceptibility in randomized uniaxial dipolar ferromagnets as, e.g., $LiTb_pY_{1-p}F_4$, indeed failed to detect exponential corrections. In a region $10^{-3}<t<10^{-1}$, a better fit was achieved[15] using $\chi\propto t^{-\gamma_{\rm eff}}$, where at low impurity concentrations the effective nonuniversal exponent $\gamma_{\rm eff}$ exhibits a clear concentration dependence.[16] A fluctuation-dominated temperature range $t_x\ll t\ll 1$, sufficiently far from the asymptotic critical region $t\ll t_x$, explored recently by Vause and Bruno,[18] actually seems to belong to this experimentally accessible regime. Nonasymptotic forms of the equation of state, of the specific heat, etc., parametrized by the impurity concentration are seemingly consistent with the experimental data.[15]

The role of elastic degrees of freedom in the critical behavior of pure systems has long been a subject of debate. At present, however, modifications of the marginal critical behavior of pure rigid systems under elastic fluctuations being coupled electro(magneto)strictively to the order parameter are believed to be understood. It is known, for instance, that an isotropic compressible $n=1$, $d=d^*$ system becomes elastically unstable under enhanced fluctuations due to the logarithmically diverging specific heat, and the phase transition becomes weakly first order quite analogous to pure compressible Ising models of dimensionality $d<d^*$ with an infinite specific heat.[19–22] Depending on boundary conditions and certain physical properties of the systems, either ideal marginal critical behavior or changes in the fractional powers of the logarithmic correction factors are found in a pseudocritical region[23,24] resulting, e.g., in a type of Fisher-renormalized[25] cusplike singularity of the specific heat $C\propto|\ln|t||^{-2/3}$.

In the present paper RG methods are used to study the combined effect of quenched disorder and of elastic fluctuations on phase transitions at marginal dimensionality, being somehow typical for real uniaxial dipolar materials exhibiting elastic anisotropy. We will consider the behavior of thermodynamic functions and of elastic properties both in the true critical region $t\to 0$ and in a fluctuation-dominated nonuniversal regime $t_x\ll t\ll 1$ with t_x being related to a peculiar degeneracy of the RG recursion relations in leading order. RG methods have previously been applied in order to investigate effects of the competition of quenched disorder and of elastic degrees of freedom on the critical behavior of systems with short-range interaction in $d=4-\epsilon$ dimensions.[26] All our calculations in the main part of this paper will be performed for a short-range random compressible Ising model at $d=d^*=4$, and therefore the results do not rely on the ϵ expansion, although certain analogies with the case $d=4-\epsilon$ may be expected to exist. Moreover, many of the present results can be applied or easily extended to uniaxial dipolar systems at $d=d^*=3$, due to a close correspondence existing between the fluctuation corrections to leading order in these two cases.[5] Some of our results being presented here have been contained, either implicitly or explicitly, in earlier work; however, it is certainly useful to give a unified presentation of the role of fluctuations on random compressible systems at marginal dimensionality, also to focus the attention of experimentalists on their importance.

In Sec. II of this paper, the underlying model is introduced and the RG equations are presented. The solutions of these equations for $d=d^*=4$ are analyzed in Sec. III. Depending on the ratio $\tilde{v}_0/\Delta_0$ of the strength of elastic fluctuations to quenched disorder, a first-order or second-order transition is found to take place,[27] being separated by a tricritical point. A complete account of critical properties and of elastic stability is given for $T>T_c$. In the asymptotic vicinity of the phase transition including the tricritical point, e.g., the universal singularities of the order-parameter susceptibility and of the specific heat are determined. In Sec. IV we calculate fluctuation corrections outside the asymptotic critical region in one-loop approximation, thus, finding an improvement upon a recent approach by Vause and Bruno[18] for rigid random systems in solving the RG recursion relations exactly. These results are expected to be of relevance in interpreting experimental data. Elastic properties are discussed in Sec. V in the nonuniversal and the universal critical region. In particular, the first-order transition occurring for $\tilde{v}_0/\Delta_0 < -2$ is shown to be associated with a macroscopic elastic instability. A microscopic instability due to weak lattice anisotropy developing in pure compressible systems, however, is shown not to occur in random compressible systems. We analyze the effect of boundary conditions on critical properties, particularly at the tricritical point using the concept of Fisher renormalization.[25] Section VI finally is devoted to a brief summary of our results and to a short discussion of their relevance for uniaxial dipolar systems at $d=3$, also with respect to the applicability to real uniaxial dipolar materials.

II. MODEL HAMILTONIAN AND RECURSION RELATIONS

We will consider a random compressible system with short-range interactions at the marginal dimensionality $d^*=d=4$. The one-component order parameter is coupled to elastic deformations of the lattice and to quenched disorder. In the long-wavelength limit the Hamiltonian of such a system can be written in terms of continuum variables as

$$H=\int d^dx \Bigg[\tfrac{1}{2}[r+\varphi(x)]\sigma^2+\tfrac{1}{2}(\nabla\sigma)^2+\tfrac{1}{4}\tilde{u}\sigma^4 + \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}\lambda_{\alpha\beta\gamma\delta}e_{\alpha\beta}e_{\gamma\delta}+\sum_{\alpha,\beta}P_{\alpha\beta}(x)e_{\alpha\beta} + \sum_{\alpha,\beta}g_{\alpha\beta}e_{\alpha\beta}\sigma^2\Bigg] . \tag{2.1}$$

Here $\sigma(x)$ denotes the one-component order parameter field (spin density in the case of magnetic phase transitions, polarization density in the case of ferroelectrics, etc.), while $e_{\alpha\beta}(x)$, $\alpha,\beta=1,2,3,4$, are the components of the strain tensor. In (2.1), $\lambda_{\alpha\beta\gamma\delta}$ are bare values of the elastic moduli and the $g_{\alpha\beta}$ denote coupling constants between the order-parameter field and elastic deformations. Quenched disorder is introduced by the local shifts of the mean-field transition temperature and via induced local stresses, represented by the random fields $\varphi(x)$ and $P_{\alpha\beta}(x)$, respectively. The probability distribution of these two quantities will be specified later. The temperature dependence of $r\propto(T-T_0)$ is linear, as usual, where T_0 is the averaged mean-field transition temperature. All other parameters in (2.1) are taken to be temperature independent. Spatial variations are allowed for wave numbers smaller than the cutoff $\Lambda=1$.

Due to long-range spatial correlations of elastic deformations the critical behavior of compressible systems is known to be sensitive to the boundary conditions used. In the present work we choose free boundary conditions and separate the homogeneous deformations from the microscopic modes (acoustic waves) by writing

$$e_{\alpha\beta}(x)=e^0_{\alpha\beta}+\frac{1}{2\sqrt{V}}\sum_{k\,(\neq 0)} i[k_\alpha u_\beta(k)+k_\beta u_\alpha(k)]e^{ikx} , \tag{2.2}$$

where V is the volume of the system. Shape-dependent surface modes are ignored.

An effective Hamiltonian for the order-parameter fluctuations can be obtained by inserting the deformations which minimize (2.1) at a given random-field configuration. Since deformations are included in harmonic approximation, this procedure is equivalent to Fourier-transforming and integrating over deformations in the partition function. As a result we obtain the effective Hamiltonian

$$H_{\rm eff}=\tfrac{1}{2}\sum_k (r+k^2)\,|\sigma_k|^2+\frac{\tilde{u}}{4V}\sum_k(\sigma^2)_k(\sigma^2)_{-k} +\frac{1}{2\sqrt{V}}\sum_k\phi(k)(\sigma^2)_{-k}-\frac{1}{4V}\sum_{k\,(\neq 0)}v(\hat{k})(\sigma^2)_k(\sigma^2)_{-k} -\frac{v_m}{4V}\left[\sum_k|\sigma_k|^2\right]^2 , \tag{2.3}$$

where σ_k and $(\sigma^2)_k$ are the Fourier components of $\sigma(x)$ and $\sigma^2(x)$, respectively, and $\hat{k}=k/|k|$.

Quenched disorder is now represented in $H_{\rm eff}$ by the random field $\phi(k)$ which can be expressed as a linear combination of the random fields in the original Hamiltonian H. Assuming $\varphi(x)$ and $P_{\alpha\beta}(x)$ to be Gaussian random fields with zero means and δ correlations, we find $\phi(k)$ also to be governed by a Gaussian distribution, specified by

$$[\phi(k)]=0 , \qquad [\phi(k)\phi(-k)]=\begin{cases}\Delta(0), & k=0 ,\\ \Delta(\hat{k}), & k\neq 0 .\end{cases} \tag{2.4}$$

Here $[\cdots]$ denotes averaging over random-field configurations.

The last two terms of (2.3) represent effective interactions due to acoustic waves and homogeneous deformations, respectively. The coefficients of these couplings are

$$v(\hat{k})=2\sum_{\alpha,\beta,\gamma,\delta}[D^{-1}(k)]_{\alpha\beta}g_{\alpha\gamma}k_\gamma g_{\beta\delta}k_\delta \tag{2.5}$$

and

$$v_m=2\sum_{\alpha,\beta,\gamma,\delta}g_{\alpha\beta}(\lambda^{-1})_{\alpha\beta\gamma\delta}g_{\gamma\delta} , \tag{2.6}$$

where $[D^{-1}(k)]_{\alpha\beta}$ is the inverse of the dynamical matrix

$$[D(k)]_{\alpha\beta}=\sum_{\gamma,\delta}\lambda_{\alpha\gamma\beta\delta}k_\gamma k_\delta \tag{2.7}$$

and $(\lambda^{-1})_{\alpha\beta\gamma\delta}$ are the elastic compliance constants.

Note, that $\Delta(\hat{k})$ and $v(\hat{k})$ depend only on the direction of the wave number and $\tilde{u}$, $\Delta(\hat{k})$, $v(\hat{k})$, and v_m are all positive. Furthermore, the quartic part of $H_{\rm eff}$ must be positive definite in order of $H_{\rm eff}$ to have a finite absolute minimum. Considering homogeneous variations of the order parameter $[\sigma(x)=\sqrt{V}P]$ we thus find the condition

$$\tilde{u}-v_m>0 . \tag{2.8}$$

Here, instead of a field-theoretical approach applied before,[10] the critical properties of the model will be determined using a Wilson-type renormalization-group approach to the effective Hamiltonian $H_{\rm eff}$. A similar program was also carried out before, where our model was studied under periodic boundary conditions in $d=4-\epsilon$ dimensions.[26] The choice of free boundary conditions causes only minor changes in the derivation of recursion relations. Fluctuating homogeneous deformations merely induce an effective interaction of infinite range as given by the last term in (2.3) which affects properties at $k=0$ in the thermodynamic limit. In the diagrams of the perturbation expansion the vertex v_m appears only as a treelike decoration with zero wave number. Consequently, v_m does not modify the transformation of $\widetilde{u}$, $\Delta(\hat{k})$, and $v(\hat{k})$: their recursion relations are the same as in the case of periodic boundary conditions. It is convenient, however, to eliminate $\widetilde{u}$ in favor of

$$u=\widetilde{u}-\langle v\rangle \; , \qquad (2.9)$$

where $\langle(\cdots)\rangle=\Omega_d^{-1}\int d\Omega_d(\cdots)$ refers to the angular average with the surface area Ω_d of the unit sphere in d dimensions. The recursion relation for v_m can be obtained from that for $v(\hat{k})$ in simply replacing $v(\hat{k})$ by v_m at each place where $\hat{k}$ is the external wave number. In summary, the one-loop recursion relations can finally be written in the form of the following differential equations:

$$\frac{dr}{dl}=2r+(3u-\langle\Delta\rangle+\langle v\rangle-v_m)\frac{K_4}{1+r} \; , \qquad (2.10)$$

$$\frac{du}{dl}=-3u(3u-2\langle\Delta\rangle)K_4+5(\langle v\rangle^2-\langle v^2\rangle)K_4 +4(\langle v\rangle\langle\Delta\rangle-\langle v\Delta\rangle)K_4 \; , \qquad (2.11)$$

$$\frac{d\Delta(\hat{k})}{dl}=-2\Delta(\hat{k})[3u-\langle\Delta\rangle-v(\hat{k})+\langle v\rangle]K_4 +2\langle\Delta^2\rangle K_4 \; , \qquad (2.12)$$

$$\frac{dv(\hat{k})}{dl}=v(\hat{k})[-6u+2\langle\Delta\rangle+v(\hat{k})-2\langle v\rangle]K_4 \; , \qquad (2.13)$$

$$\frac{dv_m}{dl}=v_m[-6u+2\langle\Delta\rangle+v_m-2\langle v\rangle]K_4 \; . \qquad (2.14)$$

Here $l=\ln b$ and $K_4=1/8\pi^2$ arises from angular integration. In one-loop approximation $\eta=0$.

Finally, instead of r we introduce

$$t\equiv r+\frac{K_4}{2}(3u-\langle\Delta\rangle+\langle v\rangle-v_m)[1-r\ln(1+r)] \; , \qquad (2.15)$$

which, in one-loop order, measures the distance from the critical surface. Equation (2.10) may then be replaced by

$$\frac{d\ln t}{dl}=2-(3u-\langle\Delta\rangle+\langle v\rangle-v_m)K_4 \; . \qquad (2.16)$$

The renormalization-group equations will then be used to calculate the contributions of critical fluctuations to physical quantities of interest, like the free energy, $F=-[\ln Z_\phi]$, the inverse of the order-parameter susceptibility $\chi^{-1}=\lim_{k\to 0}(\delta^2F/\delta[\sigma_k]\delta[\sigma_{-k}])$, etc., where $Z_\phi=\int\{\delta\sigma\}\exp(-H_{\text{eff}})$ denotes the partition function for a given random-field configuration. Here, this aim may be achieved by integrating the recursion relations to one-loop order[28] up to $l=l^*$, where l^* is defined by the condition

$$t(l^*)=1 \; . \qquad (2.17)$$

Hence, to one-loop order the following expressions are obtained[28] for the susceptibility

$$\chi=e^{2l*} \; , \qquad (2.18)$$

and for the singular part of the free energy

$$F_{\text{sing}}=-\tfrac{1}{4}K_4\int_0^{l^*}e^{-4l}t^2(l)dl \; . \qquad (2.19)$$

The value of l^* depends on the physical temperature $t_0=t(0)$ via (2.17) serving as a matching condition.[29]

III. UNIVERSAL BEHAVIOR CLOSE TO THE CRITICAL POINT

Weak elastic anisotropy does not alter the critical behavior of random compressible systems, in contrast to pure compressible systems,[22,24] if $\hat{k}$-independent fixed points of Eqs. (2.10)–(2.14) remain stable under $\hat{k}$-dependent perturbations. Therefore, we first focus on the case, when the parameters of H_{eff} do not depend on $\hat{k}$, i.e.,

$$v(\hat{k})\equiv v, \quad \Delta(\hat{k})\equiv\Delta \; . \qquad (3.1)$$

We will return to the discussion of the general case at the end of this section.

Conditions given by (3.1) are realized in models with isotropic elastic properties. In this case $g_{\alpha\beta}=g\delta_{\alpha\beta}$ and there are only two independent elastic constants, namely, $C_{11}=\lambda_{\alpha\alpha\alpha\alpha}$ and $C_{44}=4\lambda_{\alpha\beta\alpha\beta}$ $(\alpha\neq\beta)$. Relations (2.5)–(2.6) then are reduced to

$$v=2g^2/C_{11} \quad \text{and} \quad v_m=2g^2/B \; , \qquad (3.2)$$

where $B=C_{11}-[(d-1)/2d]C_{44}=C_{11}-\frac{3}{8}C_{44}$ is the bulk modulus. The recursion relations can now be written as

$$\frac{d\ln t}{dl}=2-(3u-\Delta+v-v_m)K_4 \; , \qquad (3.3)$$

$$\frac{du}{dl}=-3u(3u-2\Delta)K_4 \; , \qquad (3.4)$$

$$\frac{d\Delta}{dl}=-2\Delta(3u-2\Delta)K_4 \; , \qquad (3.5)$$

$$\frac{dv}{dl}=-v(6u-2\Delta+v)K_4 \; , \qquad (3.6)$$

$$\frac{dv_m}{dl}=-v_m(6u-2\Delta+2v-v_m)K_4 \; . \qquad (3.7)$$

Fixed points of these equations are located along two lines in parameter space:

$$\begin{aligned}&\text{(i)} \;\; 3u^*-2\Delta^*=0, \;\; v^*=v_m^*=0 \; ,\\ &\text{(ii)} \;\; 3u^*-2\Delta^*=0, \;\; v^*=0, \;\; v_m^*=2\Delta^* \; ,\end{aligned} \qquad (3.8)$$

which will be referred to as fixed lines, i.e., rigid fixed line and renormalized fixed line, respectively. At each fixed point critical exponents can be defined in the familiar way. Thus from (3.3) we get, e.g., for the inverse of the exponent of the correlation length

$$\frac{1}{\nu}=2-(\tfrac{3}{2}u^*-v_m^*)K_4 . \tag{3.9}$$

For a given value of u^* the exponents ν of the rigid and renormalized fixed points satisfy the relation

$$\frac{1}{\nu_{\rm rig}}+\frac{1}{\nu_{\rm ren}}=4=d , \tag{3.10}$$

which implies that the exponents of these fixed points are related by Fisher renormalization.[25]

Although the degeneracy, indicated by the existence of fixed lines, is an artifact of the one-loop approximation and will be lifted by higher-order corrections, the study of the one-loop trajectories and fixed points clearly provides useful information about the global flow.

The domains of attraction of the fixed lines can be explored introducing a new combination of the parameters,

$$\tilde{v}=v-v_m , \tag{3.11}$$

which also appears in Eq. (3.3) for t. The transformation of $\tilde{v}$ is described by

$$\frac{d\tilde{v}}{dl}=-\tilde{v}(6u-2\Delta+\tilde{v})K_4 , \tag{3.12}$$

which is the same as Eq. (3.6) for v. Since $C_{11}>B>0$, an important difference follows, however, from Eq. (3.2) which implies that $\tilde{v}<0$, while $v>0$. At the fixed point we have

$$\tilde{v}^*=-v_m^* . \tag{3.13}$$

The fixed lines in the reduced parameter space $(u,\Delta,\tilde{v})$ are shown in Fig. 1. The renormalized fixed line together with the line $\tilde{v}=\Delta=0$ defines a plane the equation of which is given by

$$\tilde{v}+2\Delta=0 . \tag{3.14}$$

Combining (3.5) and (3.12) we obtain

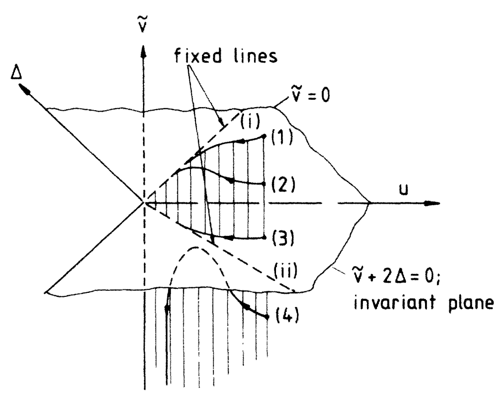

FIG. 1. Schematic view of typical renormalization-group trajectories of an isotropic system for given nonuniversal bare parameters (u_0,Δ_0) and various values $\tilde{v}_0$, i.e., (1) $\tilde{v}_0=0$, (2) $\tilde{v}_0>-2\Delta_0$, (3) $\tilde{v}_0=-2\Delta_0$, (4) $\tilde{v}_0<-2\Delta_0$.

$$\frac{d}{dl}(\tilde{v}+2\Delta)=-(\tilde{v}+2\Delta)(6u-4\Delta+\tilde{v})K_4 . \tag{3.15}$$

This implies the plane defined by (3.14) to be an invariant plane of the renormalization-group transformation. Linear stability analysis shows that rigid fixed points are locally stable. Renormalized fixed points, however, are unstable against variations which make the system leaving the invariant plane.

Therefore, we conclude that the asymptotic character of a renormalization-group trajectory is determined by its position relative to the invariant plane. The following three cases are possible (see also Fig. 1):

(i) In the region *above the invariant plane* $(0>\tilde{v}>-2\Delta)$ trajectories go to the rigid fixed line, where all couplings to elastic degrees of freedom vanish. Systems with bare parameters in this region behave at the transition point like random systems on a rigid lattice.

(ii) *On the invariant plane* $(\tilde{v}=-2\Delta)$ trajectories are attracted by the renormalized fixed line, where the coupling to homogeneous deformations is relevant $(\tilde{v}^*=-v_m^*\neq 0)$ and leads to a new (renormalized) behavior.

(iii) *Below the invariant plane* $(\tilde{v}<-2\Delta)$ large negative values of $\tilde{v}$ and v_m are generated and the trajectories run away to infinity. This is an indication of a first-order transition similar to the case of pure compressible systems. A complete analysis of this region not being our present concern may require more suitable matching techniques.[29] (See also the discussion in Sec. V.)

In the case of rigid random systems at the marginal dimensionality it was pointed out[13] that the one-loop approximation breaks down in the vicinity of the fixed line, where $3u-2\Delta$ becomes comparable with Δ^2. The degeneracy is already lifted on the two-loop level and only the trivial fixed point survives. The same happens in compressible random systems. Fixed lines have their origin in the degenerate structure of the one-loop equations for u and Δ. In isotropic systems the transformation of u and Δ is independent of v and v_m at all orders of the perturbation expansion, and so their recursion relations coincide with those in a rigid random system. Consequently, we may take the asymptotic solutions for u and Δ from the two-loop calculation by Aharony.[13] After appropriate change of the notations $(u=4v_{\rm Aharony},\ \Delta=-8u_{\rm Aharony})$ we obtain

$$\begin{aligned} K_4u&=\frac{2}{3}\left[\frac{3}{53l}\right]^{1/2}+O(1/l) ,\\ K_4\Delta&=\left[\frac{3}{53l}\right]^{1/2}+O(1/l) . \end{aligned} \tag{3.16}$$

After the insertion of these expressions, (3.12) becomes

$$\frac{d\tilde{v}}{dl}=-2\left[\frac{3}{53l}\right]^{1/2}\tilde{v}-K_4\tilde{v}^2 . \tag{3.17}$$

For $l\to\infty$ we find three kinds of asymptotic solutions, which also indicate how trajectories are modified close to the fixed lines of the one-loop equations.

(i) If $|\tilde{v}|\ll\Delta$, (3.17) can be linearized and we get

$$\tilde{v}\propto\exp[-4(3l/53)^{1/2}] . \tag{3.18}$$

This behavior is exhibited by trajectories running above the invariant plane. Two-loop corrections drive them to the trivial fixed point along the rigid fixed line.

(ii) If $\tilde{v}$ is of order $l^{-1/2}$, the right-hand side of (3.17) must vanish, and we have

$$K_4\tilde{v} = -2\left[\frac{3}{53l}\right]^{1/2} + O(1/l) \ . \tag{3.19}$$

This is characteristic of trajectories lying in the invariant plane. They arrive at the trivial fixed point along the renormalized fixed line.

(iii) If $K_4\tilde{v} < -2(3/53l)^{1/2} = -2K_4\Delta$, trajectories run away to minus infinity.

Returning to (3.6) we find that positive solutions decay faster than u and Δ. Linearization yields

$$v \propto \exp[-4(3l/53)^{1/2}] \ \text{ as } l \to \infty \ . \tag{3.20}$$

In order to find the leading singularities in the susceptibility and the specific heat the procedure sketched in the previous section will be used. Since $t(l)$ is needed only for $l \gg 1$, it is permissible to insert the asymptotic form of u, Δ, and $\tilde{v}$ into (3.3). We obtain

$$\ln[t(l)/t] = 2l \mp 2\left[\frac{3l}{53}\right]^{1/2} + O(\ln l) \ , \tag{3.21}$$

where $t \equiv t_0 \propto (T - T_c)/T_c$. The upper (lower) sign refers to rigid (renormalized) behavior. Integration is stopped at $l = l^*$ defined by $t(l^*) = 1$. Under this condition (3.21) can be solved by iteration and we find

$$l^* = \frac{|\ln t|}{2} \pm \left[\frac{3}{106}|\ln t|\right]^{1/2} + O(\ln|\ln t|) \ . \tag{3.22}$$

Now it is straightforward to calculate the susceptibility and the singular part of the specific heat,

$$C_{\text{sing}} = -2F_{\text{sing}}/t^2 \ , \tag{3.23}$$

using (2.17) and (2.18). The results are summarized in Table I. Expressions for rigid behavior were previously known from the studies by Aharony.[13] The present analysis shows in addition, that this behavior is stable against not too large coupling to elastic deformations. Both in the rigid and the renormalized case, mean-field behavior is modified by weak exponential corrections. Singularities are, however, stronger in the latter case. Especially in the specific heat, the finite cusp of rigid behavior is replaced by a weak divergence at the critical point. It is obvious from (3.22), that the results are correct to logarithmic accuracy, i.e., up to a factor of some power of $|\ln t|$ (Ref. 30).

It is clear from (3.8) that condition (2.8) is not satisfied at the renormalized fixed line, i.e., this line is located in the domain where H_{eff} has no finite absolute minimum and it has to be stabilized by higher-order terms. We expect, that these terms will modify the invariant surface separating the regions of first-order and second-order transitions, the asymptotic behavior on this surface will, however, not be changed. Arguments, supporting this expectation will be exemplified later in Sec. V.

Finally, we turn to the discussion of the general case, when $\Delta(\hat{k})$ and $v(\hat{k})$ are anisotropic. Assuming weak anisotropy, we linearize the recursion relations (2.11)–(2.13) with respect to the deviations from angular averages

$$\begin{aligned} \delta v(\hat{k}) &= v(\hat{k}) - \langle v \rangle \ , \\ \delta\Delta(\hat{k}) &= \Delta(\hat{k}) - \langle \Delta \rangle \ . \end{aligned} \tag{3.24}$$

In this approximation u, $\langle \Delta \rangle$, and $\langle v \rangle$ obey the isotropic recursion relations (3.4)–(3.6), whereas

$$\begin{aligned} \frac{d}{dl}\delta v(\hat{k}) &= -\delta v(\hat{k})(6u - 2\langle \Delta \rangle)K_4 \ , \\ \frac{d}{dl}\delta\Delta(\hat{k}) &= -\delta\Delta(\hat{k})(6u - 2\langle \Delta \rangle)K_4 + 2\delta v(\hat{k})\langle \Delta \rangle K_4 \ . \end{aligned} \tag{3.25}$$

Inserting the asymptotic solutions (3.16) for u and $\langle \Delta \rangle$, we find that both $\delta v(\hat{k})$ and $\delta\Delta(\hat{k})$ decay like $\exp[-4(3l/53)^{1/2}]$ when $l \to \infty$. (In order to be consistent, polynomials multiplying the exponential function have been ignored.) In view of (3.16) and (3.20) we conclude, that the assumption of weak anisotropy remains valid in the course of renormalization. Anisotropy decays much faster than u and $\langle \Delta \rangle$, so that it cannot influence critical behavior. We expect this to be also true in systems of finite degree of anisotropy, although the opposite case is not ruled out by our present considerations.

IV. NONUNIVERSAL BEHAVIOR

As pointed out by Aharony[13] and later exploited in detail by Vause and Bruno,[18] in rigid random systems at marginal dimensionality, one-loop equations may be relevant in weakly random systems in spite of the fact that the asymptotic behavior can be determined only in a

TABLE I. Anomalous temperature dependence $t = (T - T_c)/T_c$ of the inverse of the susceptibility χ^{-1} and of the singular part of the specific heat C_{sing} for $T \to T_c$ in the case of rigid and of renormalized behavior with $\tilde{v}_0 < 0$ and $\Delta_0 > 0$.

Nonuniversal parameters	χ^{-1}	C_{sing}	RG behavior
$\tilde{v}_0 > -2\Delta_0$	$\lvert t\rvert \exp[-(\frac{6}{53}\lvert\ln\lvert t\rvert\rvert)^{1/2}]$	$\exp[-2(\frac{6}{53}\lvert\ln\lvert t\rvert\rvert)^{1/2}]$	Rigid
$\tilde{v}_0 = -2\Delta_0$	$\lvert t\rvert \exp[(\frac{6}{53}\lvert\ln\lvert t\rvert\rvert)^{1/2}]$	$\exp[2(\frac{6}{53}\lvert\ln\lvert t\rvert\rvert)^{1/2}]$	Renormalized
$\tilde{v}_0 < -2\Delta_0$	First-order transition		Runaway

higher-order approximation.

One-loop trajectories end at a point on one of the two fixed lines. The point is determined by the initial (bare) values of the parameters in H_{eff}. If one-loop equations were exact, this would imply nonuniversal asymptotic behavior. Nevertheless, in the vicinity of a fixed line, higher-order corrections make the trajectories turn toward the trivial fixed point at the origin ($u^*=\Delta^*=0$), the only real one in the model. The renormalization-group flow bears some recollection of familiar crossover phenomena at multicritical points: the initial enhancement of critical fluctuations are governed, to a good approximation, by the fixed points of the one-loop equations instead of the trivial one. Note, however, an essential difference: in the present case, one-loop fixed points stop to act as a fixed point as soon as trajectories come close to them, and so there exists no scaling region dominated by their exponents.

Now we are going to calculate fluctuation corrections in the region, where one-loop equations provide a good approximation. At first the isotropic case is considered. The recursion relations for u and Δ, Eqs. (3.4) and (3.5), are the same as those in a rigid system. Solving them, we follow Vause and Bruno.[18] The projection of the trajectory on the (u,Δ) plane is determined by

$$\frac{d\Delta}{du}=\frac{2\Delta}{3u}\,, \tag{4.1}$$

which yields

$$\left[\frac{\Delta}{\Delta_0}\right]^3=\left[\frac{u}{u_0}\right]^2, \tag{4.2}$$

where u_0 and Δ_0 are the initial (bare) values of the parameters. The fixed line is intersected at the point

$$u^*=\tfrac{2}{3}\Delta^*=\frac{1}{u_0^2}\left[\frac{2\Delta_0}{3}\right]^3\equiv a^3\,. \tag{4.3}$$

Inserting (4.2) into (3.4), we obtain

$$\frac{du}{dl}=-9u(u-au^{2/3})K_4\,. \tag{4.4}$$

Introducing the new variable

$$x=u^{1/3}\,, \tag{4.5}$$

the fractional power can be eliminated:

$$\frac{dx}{dl}=3x^3(a-x)K_4\,. \tag{4.6}$$

Elementary integration gives

$$l=\frac{1}{3a^3K_4}\left[\ln\left[\frac{x(x_0-a)}{x_0(x-a)}\right]+a\left[\frac{1}{x_0}-\frac{1}{x}\right]+\frac{a^2}{2}\left[\frac{1}{x_0^2}-\frac{1}{x^2}\right]\right]. \tag{4.7}$$

Here, $x_0\equiv u_0^{1/3}$ is the initial value of x. As l increases, x approaches the value of a monotonously.

Instead of constructing interpolating formulas and thus introducing further approximations, as done by Vause and Bruno[18] in the rigid case, we proceed with the exact solution of the one-loop equations. From now on we shall parametrize the trajectories by x, instead of l. In view of (4.2) and (4.5), we have

$$u(x)=x^3,\quad \Delta(x)=\tfrac{3}{2}ax^2\,. \tag{4.8}$$

Combining (3.11) and (4.6) we obtain

$$\frac{d\tilde{v}}{dx}=\frac{2x-a}{x(x-a)}\tilde{v}+\frac{\tilde{v}^2}{3x^3(x-a)}\,. \tag{4.9}$$

The solution is given by

$$\frac{1}{\tilde{v}(x)}=\frac{x_0(x_0-a)}{x(x-a)}\left[\frac{1}{\tilde{v}_0}+\frac{x_0-x}{3x_0^2x(x_0-a)}\right], \tag{4.10}$$

where $\tilde{v}_0$ is the initial value of $\tilde{v}$. It can easily be verified that this solution for $\tilde{v}(x)$, indeed, is consistent with three different regions of RG flow (Fig. 2) as was discussed in the preceding section following Eq. (3.15).

The solution of (3.3) can also be expressed as a function of x:

$$\ln[t(x)/t_0]=2l(x)-K_4\int_{x_0}^{x}[3u(x')-\Delta(x')+\tilde{v}(x')]\frac{dl(x')}{dx'}dx'\,, \tag{4.11}$$

where $l(x)$ is defined by (4.7). Using Eqs. (4.8) and (4.10) the integration on the right-hand side of (4.11) can be carried out and we obtain

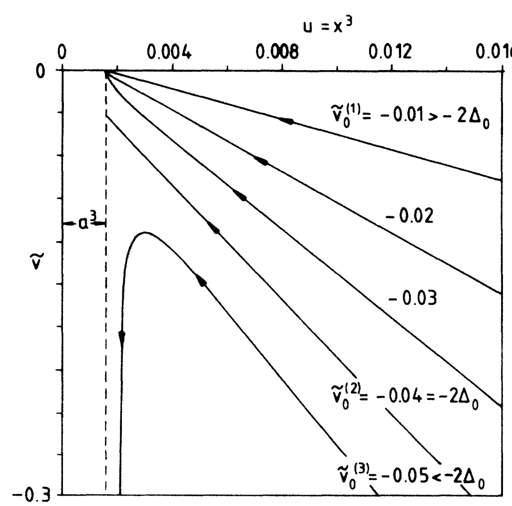

FIG. 2. RG flow in the $(u,\tilde{v})$ plane as obtained in one-loop approximation for fixed values $u_0=0.2$ and $\Delta_0=0.02$ together with various values of $\tilde{v}_0$, i.e., $\tilde{v}_0^{(1)}>-2\Delta_0$ and attraction by the rigid fixed line, $\tilde{v}_0^{(2)}=-2\Delta_0$ and attraction by the renormalized fixed line, and $\tilde{v}_0^{(3)}<-2\Delta_0$ and runaway behavior.

$$\ln\left[\frac{t(x)}{t_0}\right]=2l(x)+\frac{1}{2}\ln\left[\frac{x(x-a)}{x_0(x_0-a)}\right]-\ln\left[1+\frac{\widetilde{v}_0(x-x_0)}{3x_0^2(a-x_0)x}\right]. \quad (4.12)$$

The effect of the coupling to elastic deformations is represented by the last term of (4.12) which vanishes in rigid systems where $\widetilde{v}_0=0$. Hence, for $\widetilde{v}_0=0$ and $a\to 0$, Eqs. (4.7) and (4.12) reproduce the formulas describing a rigid pure system.

The results derived for isotropic systems remain valid when weak anisotropy is included. The only modification is that Δ and v in this case represent the angular averages $\langle\Delta\rangle$ and $\langle v\rangle$, respectively. This follows from the fact that in the case of weak anisotropy angular averages are transformed by the isotropic equations. However, it remains to be checked that anisotropy will not become strong due to the renormalization even if it is weak at the beginning. We shall characterize the degree of anisotropy by the ratio

$$\frac{\delta v(\hat{k})}{\langle v\rangle}=\frac{v(\hat{k})-\langle v\rangle}{\langle v\rangle}. \quad (4.13)$$

Using the expressions given by (4.8) for u and $\langle\Delta\rangle$, we obtain from (3.25)

$$\delta v(\hat{k})=\delta v_0(\hat{k})\frac{x(x-a)}{x_0(x_0-a)}, \quad (4.14)$$

and $\langle v\rangle$ is transformed by (3.6), which coincides with (3.12), the equation for $\widetilde{v}$. Hence, $\langle v\rangle$ is given by (4.10) on inserting the appropriate initial value. With that initial value, the ratio (4.13) then takes the form

$$\frac{\delta v(\hat{k})}{\langle v\rangle}-\frac{\delta v_0(\hat{k})}{\langle v_0\rangle}\left[1+\frac{\langle v_0\rangle}{3x_0^2x}\frac{x_0-x}{x_0-a}\right]. \quad (4.15)$$

This ratio remains finite as x varies from x_0 to a and $\delta v(\hat{k})$ decreases about as fast as $\langle v\rangle$. The value of that ratio at $x=a$ diverges, however, in the limit of vanishing disorder ($a\to 0$): the region where linearization with respect to anisotropy is allowed shrinks with decreasing strength of disorder. In a pure system any amount of anisotropy drives the system away from isotropic behavior.[22,24]

The temperature dependence of the susceptibility is now derived in using the solutions of the renormalization-group equations, as outlined in Sec. II already. Under the condition $t(x^*)=1$ the susceptibility is thus given by

$$\chi=e^{2l(x^*)}, \quad (4.16)$$

and x^* is related to the reduced temperature $t\equiv t_0\propto(T-T_c)/T_c$ via (4.12). Due to the complicated structure of the resulting expressions, however, here the remaining analysis has to be carried out numerically. In practice, it is convenient, first to choose the value of x^* in the interval $a<x^*<x_0$ and then to calculate the corresponding values of t from (4.12) and of χ from (4.16). There are two limitations to this approach:

(i) The one-loop approximation breaks down when $3u-2\Delta\sim O(u^2)$. For computational purposes we fix the limit by

$$3u(x^*)-2\Delta(x^*)=10u^2(x^*), \quad (4.17)$$

which, supplemented by the condition $t(x^*)=1$, defines a characteristic temperature t_*. The one-loop description is reliable only for $t>t_*$. Therefore, t_* has been computed for several sets of the bare parameters. The results are summarized in Fig. 3. In accord with related previous estimates[13,18] of a characteristic temperature t_x we find in weakly disordered systems ($\Delta_0<u_0$) t_* being separated from the noncritical region ($t\approx 1$) by orders of magnitude. Consequently, in these systems only the one-loop region $t>t_*$ seems to be accessible experimentally.

(ii) H_{eff} has a stable minimum only in the region where $u+\widetilde{v}>0$. Trajectories leaving this region are followed up to the plane of instability. Our approach works only for $t>t_+$, where the characteristic temperature t_+ is to be determined by the condition

$$u(x^*)+\widetilde{v}(x^*)=0. \quad (4.18)$$

One should note, however, that this second limitation becomes effective only if $\widetilde{v}_0$ is sufficiently large, whereas in the opposite case, trajectories do not leave the region of stability.

We may now characterize the effect of fluctuations on the order-parameter susceptibility by the effective exponent

$$\gamma_{\text{eff}}=-\frac{\partial\ln\chi}{\partial\ln t}. \quad (4.19)$$

Using (4.7), (4.12), and (4.16) we can evaluate (4.19) explicitly to obtain

$$\gamma_{\text{eff}}=-2\frac{\partial l(x^*)}{\partial x^*}\Big/\frac{\partial\ln t}{\partial x^*}=1+\tfrac{1}{2}[3u(x^*)-\Delta(x^*)+\widetilde{v}(x^*)]K_4, \quad (4.20)$$

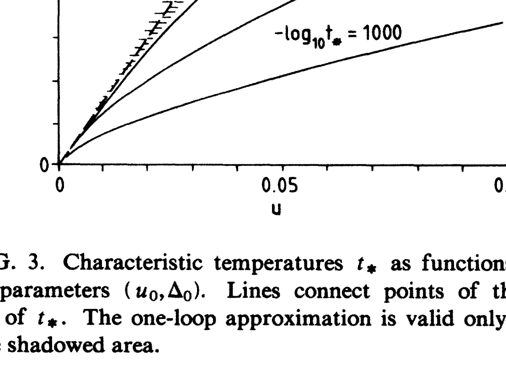

FIG. 3. Characteristic temperatures t_* as functions of the bare parameters (u_0,Δ_0). Lines connect points of the same value of t_*. The one-loop approximation is valid only outside of the shadowed area.

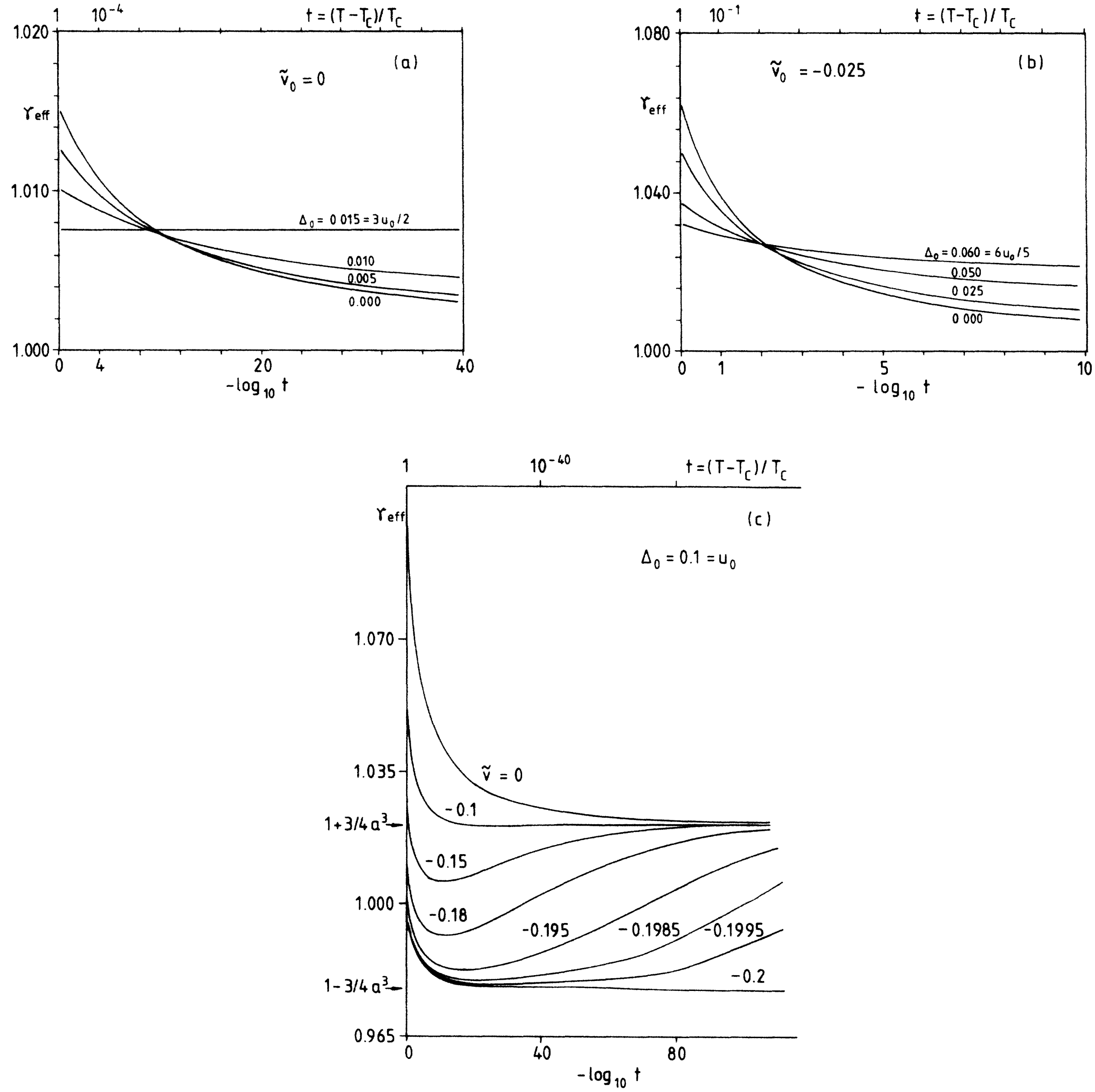

FIG. 4. Effective exponent γ_{eff} of the order-parameter susceptibility versus $-\log_{10}t$ where $t=(T-T_c)/T_c$: (a) for $\tilde{v}_0=0$ and various values of (u_0,Δ_0); (b) for $\tilde{v}_0=-0.025$ and various values of (u_0,Δ_0); (c) for fixed values $\Delta_0=u_0=0.1$ and various values of $\tilde{v}_0$.

where on the right-hand side of (4.20) only terms linear in the coupling parameters have been retained. For sufficiently small values of Δ_0, no striking deviations in γ_{eff} take place as compared to the case of pure ($\Delta_0=0$) logarithmic corrections [Figs. 4(a) and 4(b)]. It is also interesting to note that in the limit $x^*\to a$, i.e., close to the fixed line, we would obtain from (4.20):

$$\gamma_{\text{eff}}\to\gamma=1\pm\tfrac{3}{4}a^3\,, \tag{4.21}$$

where the upper (lower) sign refers to the rigid (renormalized) case [Fig. 4(c)]. The same result could have been obtained for the exponent of the order-parameter susceptibility from (3.9) using the scaling law $\gamma=2\nu$. These limits, however, have no physical relevance in the present context, since the temperature regime $t\ll t_x$ where they could be observed is far beyond the validity of the one-loop approximation they are based on.

V. ELASTIC PROPERTIES

Applying the renormalization-group transformation to the original full Hamiltonian (2.1) provides the recursion relations for the elastic moduli $\lambda_{\alpha\beta\gamma\delta}$ and for the coupling constants $g_{\alpha\beta}$ between the field of the order-parameter $\sigma(x)$ and the elastic deformations $e_{\alpha\beta}(x)$. In addition to the order-parameter field, the short-wavelength fluctuations of the displacement field are successively eliminated, too, and the remaining components of the displacement field are then rescaled like lengths, i.e., $u_\alpha(k) \to u_\alpha(bk)/b$, so that elastic moduli are not affected by rescaling. It is convenient to introduce Voigt's notation and express $\lambda_{\alpha\beta\gamma\delta}$ and $g_{\alpha\beta}$ as a matrix C_{ij} and a vector g_i, respectively, where $i,j=1,2,\ldots,d(d+1)/2=10$. The particular way in which pairs of subscripts (α,β) are grouped together to subscripts i, is irrelevant in the following considerations.

In one-loop approximation the recursion relations can be written as

$$\frac{dC_{ij}}{dl} = -2K_4 g_i g_j \,, \tag{5.1}$$

$$\frac{dg_i}{dl} = -g_i(3u - \langle \Delta \rangle + \langle v \rangle)K_4 \,. \tag{5.2}$$

Close to the trivial fixed point ($u^*=\Delta^*=0$), along the fixed lines, the integral of (5.2) $\ln(g_i/g_i^0) = -K_4\int_0^l (3u - \langle \Delta \rangle + \langle v \rangle)dl'$ can easily be evaluated using the asymptotic solutions (3.16) and (3.20) for u, $\langle \Delta \rangle$, and $\langle v \rangle$. We then find

$$\ln(g_i/g_i^0) \simeq -2\left[\frac{3l}{53}\right]^{1/2}, \tag{5.3}$$

and thus from (5.1)

$$C_{ij} - C_{ij}^\infty \propto g_i^0 g_j^0 \exp[-4(3l/53)^{1/2}] \,, \tag{5.4}$$

where g_i^0 is the initial (bare) value of the coupling constant g_i and $C_{ij}^\infty = \lim_{l\to\infty} C_{ij}$ are the values at the critical point. The physical values of the elastic constants are given by

$$C_{ij}^{\text{phys}} = C_{ij}(l^*) \,, \tag{5.5}$$

where l^* is determined by the matching condition $t(l^*)=1$. According to (3.22), $l^* \simeq \frac{1}{2}\,|\ln t\,|$ close to the critical point. Thus, the singular parts of the elastic constants behave as

$$(C_{ij}^{\text{phys}})_{\text{sing}} \propto g_i^0 g_j^0 \exp\{-2[(6/53)\,|\ln t\,|\,]^{1/2}\} \,. \tag{5.6}$$

This is the same cusplike singularity as the one of the specific heat in the case of rigid behavior. Note, however, that (5.6) remains valid also in the case of renormalized behavior.

The system is stable against homogeneous deformations if the eigenvalues $C_{(i)}$ of the matrix of elastic constants C_{ij} are all positive. The eigenvalues can be classified according to the irreducible representations of the symmetry group of the system. Since g_i has the full symmetry, it is clear from (5.1) that only eigenvalues associated with the identity representation are changed by the renormalization-group transformation.

An elastic instability manifests itself also in the behavior of v_m, the effective coupling due to homogeneous deformations. In Voigt's notation, expression (2.6) for v_m simplifies to

$$v_m = 2\sum_{i,j} g_i g_j (C^{-1})_{ij} \,. \tag{5.7}$$

Using the identity[31]

$$\frac{d}{dl}\det|\,C_{ij}\,| = \det|\,C_{ij}\,|\sum_{i,j}(C^{-1})_{ij}\frac{dC_{ij}}{dl} \,, \tag{5.8}$$

we find

$$\frac{d}{dl}\ln(\det|\,C_{ij}\,|) = -K_4 v_m \,. \tag{5.9}$$

Now, we are in a position to investigate elastic stability using the results obtained in Sec. III. Depending on the bare values of the parameters, three possibilities are found.

(i) Rigid behavior. Anisotropy in $v(\hat{k})$ and $\Delta(\hat{k})$ rapidly decays and the trivial fixed point is approached along the rigid fixed line. Combining (3.11), (3.18), and (3.20) we get, for $l\to\infty$,

$$v_m \propto \exp[-4(3l/53)^{1/2}] \,. \tag{5.10}$$

The integral of the right-hand side of (5.9) thus converges when $l\to\infty$. Consequently, $\det|\,C_{ij}\,|$ does not vanish along the trajectory, elastic stability is preserved.

(ii) Renormalized behavior. The trivial fixed point is approached along the renormalized fixed line, where, in view of (3.19) and (3.20),

$$K_4 v_m \simeq -K_4\tilde{v} \simeq -2\left[\frac{3}{53l}\right]^{1/2} \,. \tag{5.11}$$

The integral of the right-hand side of (5.9) diverges when $l\to\infty$, hence

$$\det|\,C_{ij}\,| \propto \exp[-4(3l/53)^{1/2}] \,. \tag{5.12}$$

An eigenvalue of the elastic constants matrix, say, $C_{(1)}$, vanishes at the fixed point. The temperature dependence of the physical eigenvalue is obtained, close to the critical point, by inserting $l=l^*=\frac{1}{2}\,|\ln t\,|$ into (5.12)

$$C_{(1)}^{\text{phys}} \propto \exp\{-2[(6/53)\,|\ln t\,|\,]^{1/2}\} \,. \tag{5.13}$$

(iii) Runaway. The renormalization-group transformation generates large values for v_m. The recursion relation (3.7) reduces to

$$\frac{dv_m}{dl} \simeq K_4 v_m^2 \,, \tag{5.14}$$

which implies, that v_m diverges at some finite value $l=l_*$ as $v_m \propto (l_* - l)^{-1}$. It follows from (5.9) that an eigenvalue of C_{ij} changes sign at $l=l_*$ and the system becomes unstable against the associated homogeneous deformation. Since this deformation transforms as the identity representation, the elastic instability is not accompanied by the softening of an acoustic mode.[32] In the physical system the elastic instability leads to a first-order transition.

In the isotropic case, systems with parameters on the invariant plane, $\tilde{v} \equiv v - v_m = -2\Delta$, exhibit renormalized behavior, while rigid behavior and first-order phase transitions are observed above and below this plane, respectively. The eigenvalue, changing sign when the instability appears, is proportional to the bulk modulus B. Our recursion relations (5.1) and (5.2) can be integrated in the whole region of l. As a result, the elastic constants are given by

$$C_{ij}(l) = C_{ij}^0 - 2K_4 g_i^0 g_j^0 Q(l) \,, \tag{5.15}$$

where C_{ij}^0 denotes initial (bare) values and

$$Q(l) \equiv \int_0^l dl' \exp\left[-K_4 \int_0^{l'} (6u - 2\langle\Delta\rangle + 2\langle v\rangle) dl''\right] . \quad (5.16)$$

If the anisotropy is weak, this expression can be evaluated explicitly using the solutions of the isotropic one-loop equations, which provide a good approximation far from the fixed line. Proceeding just as in Sec. IV, we eliminate l in favor of $x \equiv u^{1/3}$. The solutions for u, Δ, and v are given by (4.8) and (4.10), in the latter $\tilde{v}_0$ replaced by v_0, the bare value of v. Thus, we finally obtain

$$Q[l(x)] = \frac{x_0 - x}{x_0 v_0 - x[v_0 - 3x_0^2(x_0 - a)]} . \quad (5.17)$$

On the basis of (5.15) and (5.17) one may now calculate the physical elastic constants numerically just as it was done in Sec. IV in the case of the susceptibility.

The elastic instability cannot be observed if we exclude homogeneous deformations by choosing periodic boundary conditions instead of free boundary conditions assumed up to now. In this case the term proportional to v_m is absent in H_{eff}. As a consequence, v_m does not appear in the recursion relations for r and t, i.e., Eqs. (2.10) and (2.16). Apart from this change, recursion relations (including those for the elastic constants, too) remain valid. Since v_m is no longer an expansion parameter, its runaway does not influence the physical behavior at periodic boundary conditions. It follows immediately from the discussion in Sec. III that a continuous transition and rigid critical behavior can be observed for any set of the bare parameters which is consistent with ϵ-expansion results obtained before.[26]

Periodic and free boundary conditions simulate systems at fixed volume and fixed pressure, respectively. The phenomenological theory of constrained systems as developed by Fisher[25] can be used to relate the critical singularities observed in the two cases. For the sake of simplicity only isotropic systems will be considered. The first step is to derive a relation between the reduced temperatures $t_V = [T - T_c(V)]/T_c(V)$ and $t_P = [T - T_c(P)]/T_c(P)$, where V and P denote the volume and the pressure of the system. Following the particularly simple formulation of Achiam,[33] we find

$$t_P \propto B(t_V) t_V , \quad (5.18)$$

where $B(t_V)$ is the bulk modulus.

(i) Above the invariant plane, $B(t_V)$ is finite for $t_V \to 0$ and

$$t_P \propto t_V . \quad (5.19)$$

The critical behavior is insensitive to a change of external conditions.

(ii) On the invariant plane the temperature dependence of $B(t_V)$ is given by (5.13). The reduced temperatures are related by

$$t_P \propto \exp\{-2[(6/53)\,|\ln t_V|\,]^{1/2}\} t_V . \quad (5.20)$$

This can be inverted by iteration, i.e.,

$$t_V \propto \exp\{2[(6/53)\,|\ln t_P|\,]^{1/2}\} t_P . \quad (5.21)$$

At periodic boundary conditions (fixed V) the susceptibility behaves as

$$\chi \propto \frac{1}{t_V} \exp[(6/53)\,|\ln t_V|\,]^{1/2} \quad (5.22)$$

(see Table I). The susceptibility at free boundary conditions (fixed P) is obtained by substituting (5.21),

$$\chi \propto \frac{1}{t_P} \exp\{-[(6/53)\,|\ln t_P|\,]^{1/2}\} , \quad (5.23)$$

in agreement with the results listed in Table I. In order to get the modification of the specific heat, this substitution has to be performed in the entropy

$$S_{\text{sing}} \simeq a t_V + b t_V \exp\{-2[(6/53)\,|\ln t_V|\,]^{1/2}\} , \quad (5.24)$$

where a and b are constants. Derivations before and after the substitution reproduce C_{sing} in the rigid and renormalized cases, respectively, again in agreement with Table I.

This transformation is an example of Fisher renormalization in a system with finite specific heat.[33,34] A further peculiar feature of this example is the appearance of exponential singularities.

VI. DISCUSSION AND CONCLUSIONS

We have investigated phase transitions in a random compressible uniaxial system at marginal dimensionality $d^* = d = 4$ by analyzing solutions of the RG recursion relations. Effects of a competition between quenched disorder and elastic fluctuations are shown to affect significantly the marginal critical behavior of these systems. Depending on the relative strength of disorder and of elastic fluctuations being measured by nonuniversal parameters Δ_0 and $\tilde{v}_0$, respectively, a first-order or second-order transition turns out to take place, being separated by a tricritical point. We found that the lattice compressibility gradually vanishes as $T \to T_c$, if $\tilde{v}_0/\Delta_0 > -2 + O(\Delta_0)$, leading to the same critical behavior as in the rigid lattice case. For $\tilde{v}_0/\Delta_0 = -2 + O(\Delta_0)$, however, a new critical behavior occurs due to lattice compressibility. In the asymptotic vicinity of the transition including the tricritical point, we have determined the universal singularities, i.e., of the order-parameter susceptibility and of the specific heat. The temperature range where these universal singularities might be observable, however, seems to be extremely narrow.

Therefore, we have focused our attention on calculating fluctuation corrections by using the one-loop approximation in a temperature range $t_x \ll t \ll 1$ sufficiently far from the asymptotic critical region $t \ll t_x$ with t_x being related to the peculiar degeneracy of the RG recursion relations to leading order. In the course of these investigations we succeeded in improving a recent approach by Vause and Bruno[18] for rigid random systems by solving the recursion relations exactly. It has been shown how a convenient and adequate representation of the results can be obtained in terms of nonuniversal effective exponents varying continuously as functions of $\tilde{v}_0$ and Δ_0. Due to the close correspondence existing between the lowest-order fluctuation corrections of uniaxial $d=4$ systems and of uniaxial dipolar $d=3$ systems,[2,5] this nonasymptotic critical behavior should also be important for analyzing and understanding experiments in random-diluted uniaxial dipolar ferromagnets such as, e.g., $LiTb_pY_{1-p}F_4$, or uniaxial ferroelectrics such as, e.g., partially deuterated triglycine sulfate.

We have also investigated the elastic properties of anisotropic solids in the vicinity of these phase transitions. Specifically, the first-order transition which occurs for $\tilde{v}_0/\Delta_0 < -2$ has been associated with a macroscopic elastic instability. The influence of boundary conditions on critical properties has been revealed, in particular at the tricritical point using the concept of Fisher renormalization. A microscopic instability due to weak elastic anisotropy which develops in pure compressible systems leading to a first-order phase transition,[22,24] however, does not occur in random compressible systems. Our results suggest effects of weak lattice anisotropy to become negligible as the critical point is approached in random systems.

In conclusion, our unified theoretical approach provides the calculational tool for a sensible assessment of the important role of fluctuations at phase transitions in random compressible systems of marginal dimensionality. In addition, it should also simplify the experimental verification of the role of these fluctuations in a much wider class of uniaxial dipolar materials exhibiting such transitions, i.e., in various real anisotropic compressible solids of some sort of random dilution. In that context, experiments on such materials for estimating our nonuniversal parameters $\tilde{v}_0$ and Δ_0 measuring elastic fluctuations and quenched disorder, respectively, would be invaluable.

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