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Phase fluctuations in Josephson junctions

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Thermal excitation and zero-point motion bring about fluctuations in the gauge-invariant phase coupling across Josephson junctions. These fluctuations are studied in a simple picture incorporating both the resistively shunted junction (RSJ) model and the Ambegaokar-Baratoff theory. A diagram of the parameter space of a junction delineating a boundary between the phase-coupled and phase-decoupled regimes is obtained. Estimates of the root-mean-square voltage fluctuations justify the use of the normal-state tunneling resistance rather than the temperature-dependent quasiparticle tunneling resistance in the RSJ model at the boundary.

Recent experimental studies of the onset of superconductivity in ultrathin films as a function of thickness and normal-state sheet resistance have revealed what appears to be a universal threshold condition involving only the normal-state sheet resistance.[1] Superconductivity is found at temperatures below or near the usual bulk transition temperature in films for which the normal-state resistance is below about 6000 $\Omega/\Box$. At present, there is no generally accepted theory for such a threshold. It may be possible to understand its existence in films, however, if there were a threshold for a single junction. This is a consequence of an argument by Ambegaokar, Halperin, and Langer,[2] since a film can be modeled as an array of junctions of randomly distributed coupling strengths.

In this Rapid Communication we treat a simplified picture of the threshold for phase-coherent coupling (zero resistance) across a junction where the threshold is a consequence of macroscopic quantum zero-point motion and thermal excitation. We treat the junction in terms of the resistively shunted junction (RSJ) model,[3] which has been so successful in describing the classical behavior of Josephson junctions. Previous work in the field of macroscopic quantum tunneling (MQT) has shown how to include dissipation in a quantum-mechanical treatment of this model.[4]

We consider a junction within the RSJ model in which the Ambegaokar-Baratoff[5] theory is used to take into account the temperature dependence of the Josephson effect. A diagram of the parameter space which exhibits the boundary between the phase-coupled and phase-decoupled regimes is obtained. Estimates of the root-mean-square voltage fluctuations in this model enable us to address the question of which resistance, i.e., normal-state tunneling resistance or the quasiparticle resistance, governs the dissipation in the RSJ model.

Within the RSJ model a Josephson junction is treated as a parallel combination of a capacitance C, an effective resistance R and a tunneling supercurrent channel. The current through the latter is given by $I_0 \sin\theta$, and the voltage across the junction obeys the Josephson relation $\dot{\theta} = 2eV/\hbar$. Here θ is the gauge-invariant phase difference across the junction. Current continuity gives an equation of motion of the form

$$\left(\frac{\hbar}{2e}\right)^2 C\frac{d^2\theta}{dt^2} + \left(\frac{\hbar}{2e}\right)^2\left(\frac{1}{R}\right)\frac{d\theta}{dt} + \frac{\partial V(\theta)}{\partial\theta} = 0 , \qquad (1)$$

where the potential is

$$V(\theta) = (\hbar/2e)I_0[1-\cos\theta] - (\hbar/2e)\theta I .$$

Here I_0 is the maximum Josephson current and I is the external current supplied to the junction. According to the Ambegaokar-Baratoff theory[5] for an ideal junction,

$$I_0(t) = [2\pi\Delta(0)/(4eR_N)]F(t) ,$$

where

$$F(t) = [\Delta(t)/\Delta(0)]\tanh[\Delta(t)/(2k_BT)] ,$$

$\Delta(t)$ is the energy-gap parameter, R_N is the normal-state tunneling resistance, and $t = T/T_c$. It is convenient to introduce the Josephson plasma frequency $\omega_0 = 2\times(E_cE_J)^{1/2}/\hbar$, where $E_c = e^2/C$ is the charging energy and $E_J = I_0\hbar/2e$ is the Josephson coupling energy. Equation (1) can then be written as

$$\ddot{\theta} + 2\alpha\omega_0\dot{\theta} + \omega_0^2\sin\theta = I/[C(\hbar/2e)] , \qquad (2)$$

where $\alpha = (E_c/E_J)^{1/2}/4r$ is the damping parameter and $r = Re^2/\hbar$ is the reduced resistance. Within our model we describe an ideal junction in terms of the following parameters: α, r, $r_N = R_Ne^2/\hbar$, and t. Note that in general $\alpha = \alpha(r, r_N, C, t)$ and $r = r(t)$.

The quantitative problem of the onset (or demise) of phase coupling across the junction as a function of these parameters involves the calculation of the mobility of the phase "particle" in the tilted washboard potential $V(\theta)$. When the particle is localized in a well, the junction is phase coupled and there is a zero-voltage dc current; i.e., the junction is fully superconducting. When the particle is not localized, the junction is phase decoupled and the voltage is nonzero.

There have been a number of rather different approaches which describe the transition to the finite voltage state.[6] We limit our discussion to the case of small external current I and determine the threshold criterion as follows. We first replace $1-\cos\theta$ by $\theta^2/2$ and thus approximate the potential near the bottom of the well by a parabolic potential. We then compute the mean-square phase fluctuations $\langle\theta^2\rangle$ within the harmonic approximation. Under those conditions for which $\langle\theta^2\rangle \ll 1$, this approximation is self-consistent and the junction remains phase coupled. On the other hand, under those conditions for which $\langle\theta^2\rangle > 1$, the harmonic approximation breaks down, because the phase particle then probes the anharmonic part of the potential. In the same spirit as the Lindemann criterion for the melting of solids,[7] we therefore define $\langle\theta^2\rangle = \theta_c^2$ as the threshold criterion for phase decoupling. For simplicity we assume here $\theta_c^2 = 1$; a larger value of θ_c^2 would yield a proportionally larger value of the critical resistance at the phase-decoupling boundary. The above procedure gives only approximate results when the escape is dominated by quantum-mechanical effects that depend in detail upon either the shape of the potential barrier or the periodicity of the potential (macroscopic quantum tunneling and macroscopic quantum coherence[4] or Bloch oscillations[8]). Nevertheless, our approximation is useful in revealing the general behavior of the system as a function of junction parameters.

Within our approximation Eq. (2) becomes that of a damped harmonic oscillator whose quantum-mechanical behavior in the presence of dissipation has been treated by a number of workers.[9] Using the results of these calculations, we can express mean-square fluctuations of the phase and its time derivative in the form

$$\langle\theta^2\rangle = \frac{8}{\pi} r_N \tau_{\rm eff}(\alpha, r/r_N, t) \tag{3a}$$

and

$$\langle\dot\theta^2\rangle = \frac{8}{\pi}\frac{E_c^{3/2}E_J^{1/2}}{\hbar^2}\lambda(\alpha, r/r_N, t) \ , \tag{3b}$$

where

$$\tau_{\rm eff}(\alpha, r/r_N, t) = (2\alpha^2 r/r_N)\int_0^\infty dx\, x \coth\left(\frac{\pi}{2}\frac{r}{r_N}\frac{\alpha}{\tau}x\right)/[(1-x^2)^2+4\alpha^2x^2] \tag{4a}$$

and

$$\lambda(\alpha, r/r_N, t) = 2\alpha\int_0^\infty dx\, x^3 \coth\left(\frac{\pi}{2}\frac{r}{r_N}\frac{\alpha}{\tau}x\right)/[(1-x^2)^2+4\alpha^2x^2] \ . \tag{4b}$$

Here the parameter τ is given by

$$\tau = k_BT/2\Delta(0)F(t) \ . \tag{5}$$

In the limit of high reduced temperatures ($t \to 1$) the integrals simplify, and we find the classical expressions

$$\langle\theta^2\rangle = \frac{8}{\pi} r_N \tau = k_BT/E_J \ , \tag{6a}$$

$$\langle\dot\theta^2\rangle = (4k_BT/\hbar^2)E_c \ . \tag{6b}$$

At an arbitrary reduced temperature, $\tau_{\rm eff}(\alpha, r/r_N, t)$ is a measure of the fluctuations due to both thermal and zero-point energies. We now define an effective reduced temperature $t_{\rm eff}$ by equating the mean-square fluctuations in θ due to *both* thermal and zero-point energies [Eq. (3a)] to those obtained in the classical limit [Eq. (6a)] at the reduced temperature $t_{\rm eff}$. Using Eq. (5) we obtain

$$t_{\rm eff}(\alpha, r/r_N, t) = \frac{2\Delta(0)}{k_BT_c}F(t)\tau_{\rm eff}(\alpha, r/r_N, t) \ .$$

In the limit of $t \approx 1$, $t_{\rm eff}$ reduces to the temperature t, but for $t \ll 1$ and small damping $t_{\rm eff} = \hbar\omega_0/k_BT_c$, a result found previously by a number of other workers.[8]

Since the condition $\langle\theta^2\rangle = 1$ delineates the phase-coupled region from the decoupled region, the decoupling boundary can be determined by applying this condition to Eq. (3a) with $r = r_N$ as justified below. The resulting critical reduced resistance $r_c = R_ce^2/\hbar$ is shown in Fig. 1 versus the reduced coupling energy $\tilde E_c = (e^2/C)/2\Delta(0)$ for a number of different reduced temperatures t. It is helpful to visualize the corresponding phase-decoupling boundary, $r_c(\tilde E_c, t)$ as the surface shown in Fig. 2. In this diagram a junction of normal resistance R_N and capacitance C at a temperature T is represented by a point. If that point lies below the surface, then $\langle\theta^2\rangle < 1$ and the junction is essentially phase coupled and superconducting. If, however, the phase point lies above the surface, the junction is decoupled and resistive.

Qualitatively different regions on the boundary surface of Fig. 2 can be distinguished. For small values of $\tilde E_c$ and α, the decoupling process is dominated by thermal fluctuations except for very small t. As seen from Eq. (6a) the decoupling in this region (light shading) is determined by the normal-state resistance r_N and is independent of the reduced damping resistance r. The plane defined by $t=0$ is the quantum plane. A junction in this plane is decou-

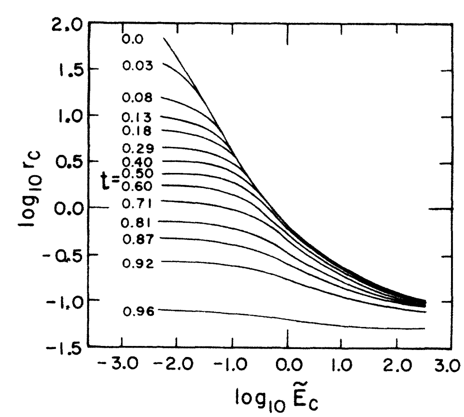

FIG. 1. Reduced critical decoupling resistance $r_c = R_ce^2/\hbar$ vs reduced charging energy $\tilde E_c = (e^2/C)/2\Delta(0)$ at various reduced temperatures $t = T/T_c$.

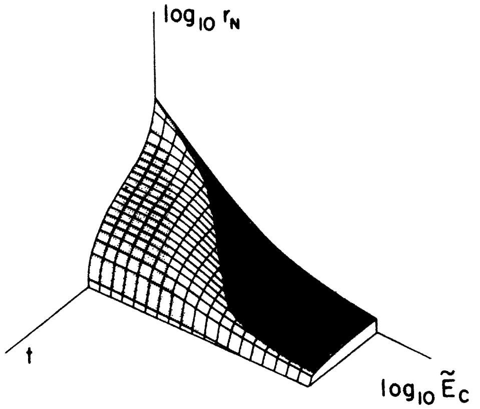

FIG. 2. Junction decoupling surface, defined by $\langle\theta^2\rangle = 1$, is plotted as a function of reduced temperature t, charging energy $\tilde{E}_c$, and resistance r_N. This surface covers the same range of r_c and $\tilde{E}_c$ as in Fig. 1. If the values of t, $\tilde{E}_c$ and r_N for a junction lie below this surface, the junction is phase coupled; if they are above, then it is phase decoupled. Quantum fluctuations dominate the destruction of phase coherence in the heavily shaded region, whereas thermal fluctuations dominate in the lightly shaded area.

pled solely by quantum-mechanical zero-point energy. The plateau emerging from this plane defines a region (dark shading) dominated by quantum-mechanical dephasing.

In the quantum regime there are two distinct types of behavior depending on damping. In the limit of weak damping ($\alpha < 1$) Eq. (4a) can be expressed as

$$\tau_{\text{eff}}(\alpha, r/r_N, t) = \frac{\pi}{2}\frac{r}{r_N}\frac{\alpha}{(1-\alpha^2)^{1/2}} \times \left[1 - \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{(1-\alpha^2)^{1/2}}\right)\right] . \quad (7)$$

For $\alpha \ll 1$ this leads to

$$\langle\theta^2\rangle = (E_c/E_J)^{1/2}(1 - 2\alpha/\pi + \tfrac{1}{2}\alpha^2 + \cdots) , \quad (8)$$

so that the condition for phase coupling essentially reduces to that of Anderson;[10] i.e., the Josephson coupling energy E_J has to be less than the charging energy E_c. On the other hand, in the limit of strong damping ($\alpha > 1$) we find

$$\tau_{\text{eff}}(\alpha, r/r_N, t) = \frac{r}{r_N}\frac{2\alpha}{(\alpha^2-1)^{1/2}}\ln\frac{\alpha + (\alpha^2-1)^{1/2}}{\alpha - (\alpha^2-1)^{1/2}} , \quad (9)$$

which for $\alpha \gg 1$ gives

$$\langle\theta^2\rangle = \frac{8}{\pi} r \ln(2\alpha) . \quad (10)$$

In using the RSJ model one faces the question of whether the damping resistance should be taken to be the quasiparticle resistance $r_{\text{qp}}(t)$ as determined from the I-V characteristic as $V \to 0$, or the normal-state resistance r_N. This question can be answered by computing the mean-square fluctuations in $\dot{\theta}$ using Eq. (3b). Since r is the effective resistance corresponding to voltage fluctuations $\langle V^2\rangle = (\hbar/2e)^2\langle\dot{\theta}^2\rangle$ across the circuit, we must have $r \approx r_N$ if $e\langle V^2\rangle^{1/2} > 2\Delta(t)$ and $r \approx r_{\text{qp}}$ if $e\langle V^2\rangle^{1/2} < 2\Delta(t)$. The reason for this is that, when voltage fluctuations are the order of or greater than the gap energy, the junction responds essentially as if it were in the normal state.

The integral in Eq. (4b) diverges logarithmically. This is a consequence of treating the damping parameter $\alpha\omega_0 = 1/(2RC)$ in Eq. (2) as frequency independent. For a real junction, however, this parameter will be greatly diminished above some frequency ω_c. We will in the following adopt the simplest possible modification[9] and set $\alpha\omega_0(\omega) = \alpha\omega_0\,\xi(\omega - \omega_c)$, where ξ is a step function and ω_c is chosen much larger than both ω_0 and $\alpha\omega_0$. Then $\lambda(\alpha, r/r_N, t)$ becomes

$$\lambda(\alpha, r/r_N, t) = \frac{\frac{1}{2} - \alpha^2}{(\alpha^2-1)^{1/2}}\ln\left(\frac{\alpha + (\alpha^2-1)^{1/2}}{\alpha - (\alpha^2-1)^{1/2}}\right) + 2\alpha\ln\left(\frac{\omega_c}{\omega_0}\right) . \quad (11)$$

For $\alpha \gg 1$ this leads to

$$\frac{e^2\langle V^2\rangle}{[2\Delta(0)]^2} = 4\pi\alpha^4(r^3/r_N^2)\ln(\omega_c/2\alpha\omega_0) . \quad (12)$$

Using Eq. (10) in this expression, we see that at the decoupling transition the rms voltage fluctuations exceed twice the gap, and, self-consistently, the appropriate resistance entering Eq. (10) should therefore be set equal to r_N. The above result, that the normal resistance for decoupling $r_c = \pi/8\ln(2\alpha)$ is only weakly dependent on capacitance, is in qualitative agreement with the results of earlier workers.[1,6]

In summary, we have presented the results of an analysis, based on the resistively shunted junction model and the Ambegaokar-Baratoff theory, of the simplest condition for the destruction of phase coherent coupling across a single Josephson junction. The results are relevant to the description of the behavior of ultrathin films which can be modeled as networks of junctions, as well as to ultrasmall junctions prepared using the current technical limits of lithography. We have obtained a map of the parameter space in resistance, capacitance, and temperature of the threshold which delineates rather clearly the various regimes. By describing the quantum fluctuations in terms of an effective temperature, we find the latter can be remarkably large for sufficiently low-capacitance junctions. Finally, by an analysis of the fluctuations of the time derivative of the phase which are related to the voltage fluctuations in this model we can understand what to use as the shunting resistance in the RSJ model at the phase boundary. Under those conditions (quantum limit, $\alpha \gg 1$) for which the damping resistance affects the placement of the phase boundary, the appropriate value is the normal-state resistance rather than the quasiparticle resistance. This result has important implications for more detailed modeling as well as for the interpretation of a number of experiments.

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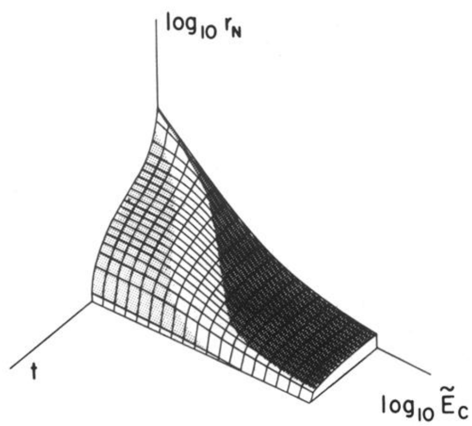

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