ϵ expansion for transport exponents of continuum percolating systems

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(Received 4 November 1985)

Certain classes of continuum percolation problems can be mapped into lattice problems with conducting bonds whose conductivity σ is drawn from a probability density law of the form σ^{-a}. Such distributions of σ in turn can modify the conductivity exponent t when $0<a<1$. It is shown that to first order in ϵ, the continuum conductivity exponent $\bar{t}$ is given, for large values of a, by $\bar{t}=(d-2)\nu+1/(1-a)$ which agrees with a form proposed by various authors. For small values of a, a new type of crossover to the discrete lattice exponent is predicted. Numerical results are also presented.

I. INTRODUCTION

It has been known since the work of Straley and Kogut[1] that if the conducting bonds in a percolation problem have a power-law distribution, such as σ^{-a}, then the conductivity exponent can be modified in a way that depends on the distribution when $0<a<1$. This is analogous to corrections to diffusion which have been separately investigated.[2] The results of Kogut and Straley were seen as introducing "nonuniversal" aspects to the percolation problem. It has been recently pointed out, however, by Halperin, Feng, and Sen,[3] that certain classes of continuum percolation problems ("Swiss-cheese" models) have well-defined distributions of conducting "bonds" and hence should have predictable transport exponents whose values differ from the standard value obtained in diluted lattices with a single type of conducting bonds.

Kogut and Straley[1] approached the problem using the effective medium approximation (EMA), variational approximations, and exact calculations on the Cayley tree. The *corrections* to the standard lattice exponents found by these different methods all agree. If $\bar{t}$ is the conductivity exponent with continuum corrections and t its standard lattice value, the result of Kogut and Straley may be written in the form

$$\bar{t}-t=(1-a)^{-1}-1 .$$

Later, Ben-Mizrahi and Bergman[4] used the Migdal-Kadanoff approach for the same problem. If one uses the value of the correlation length exponent found by Kirkpatrick,[5] with the same approach, and replaces ϵ by $d-1$ in the calculation of Ben-Mizrahi and Bergman, their result Eq. (3.15) may be recast in the form[6]

$$\bar{t}=(d-2)\nu+(1-a)^{-1} .$$

These authors claimed that the exponent changed discontinuously at $a=0$. Later Straley[7] found exactly the above result from the "nodes-links-blobs" picture[8,9] of percolation clusters but concluded that t crosses over to its standard value with decreasing a as soon as $\bar{t}=t$, i.e., for a finite value of a. Halperin *et al.*[3] used the nodes-links-blobs picture and recovered the result $\bar{t}=(d-2)\nu+(1-a)^{-1}$ which they simply recognized as a (nonrigorous) lower bound to the continuum conductivity exponent. They also noticed that the first result of Kogut and Straley[1] is a rigorous upper bound to the continuum corrections.

In this paper, we present the predictions of the ϵ expansion around the upper critical dimension ($\epsilon=6-d$). We follow in part the argument of Halperin *et al.*[3] There are two logically distinct parts to their approach which is based on the intuition provided by the nodes-links-blobs picture of percolation clusters. First, they calculate the resistance of a segment of material the size of a coherence length (a "link"). Second, based on ideas of Ambegaokar *et al.*,[10] they argue that even though the average resistance within a coherence length diverges, the macroscopic resistance may be computed from scaling by using a "typical" value of the link resistance. Indeed, following Ref. 3, the best estimate of the resistance of the link network is obtained by considering an effective percolation problem on it: Very high resistance links can be removed, leaving links with a finite average resistance but a more dilute network of links. This defines a variational problem whose solution leads to the conclusion mentioned above. We take this argument concerning the typical value of the link resistance as correct.

More specifically, we compute the typical value of the size-dependent resistance (the link part of the problem) by field-theoretic methods which, in principle, include more than only the singly connected bonds considered by Halperin *et al.*[3] As in numerical simulations,[11] the typical value of a link resistance may be obtained by computing the average link conductance. Note, however, that by including external driving fields in the theory in the way first discussed by Stephen,[12] one could, in principle, do the whole problem without arguments concerning typical

values. This may be valuable given that these arguments do not apply to other physically observable quantities (such as $1/f$ noise[13]) but for observables which obey a variational principle, such as the resistance, there is little doubt that scaling applied to a typical link is valid.

We show from the ϵ expansion that to one-loop order there are two regimes of corrections for $0<a<1$. The first regime is for small positive values of a and the second extends over most of the allowed positive values of a. In that second regime, our results agree with those of Straley,[7] of Ben-Mizrahi and Bergman,[4] and with the lower bound of Halperin *et al.*[3] The type of crossover between continuum and discrete lattice value which we propose differs, however, from other approaches. The plan of the paper is as follows. Section II derives the field theory, Sec. III presents the mean-field results, and the results are discussed further in Sec. V. Appendix B contains the results of numerical simulations in two dimensions.

II. FIELD THEORY FOR CONTINUUM PERCOLATION

We treat the general case of a percolating network where nonconducting bonds occur with a probability $(1-p)$ while conducting bonds occur with a probability p and have a range of possible values. More specifically, the probability that a bond has a conductance σ is given by

$$P(\sigma)=(1-p)\delta(\sigma)+pf(\sigma) \tag{1a}$$

with[14]

$$f(\sigma)=(1-a)\sigma_0^{-1}\left[\frac{\sigma}{\sigma_0}\right]^{-a} . \tag{1b}$$

We refer the reader to Halperin *et al.*[3] for the relation between Eq. (1b) and continuum percolation. The conductivity exponent differs from its lattice value when the average resistance of a bond is infinite, i.e., $0<a<1$.

The field theory for the above percolation problem is derived following Stephen.[12] Let

$$Z=\int \mathscr{D}V e^{-\mathscr{H}_0} , \tag{2}$$

where

$$\mathscr{H}_0(V)\equiv\tfrac{1}{2}\sum_{\langle xx'\rangle}\sigma(x,x')[V(x)-V(x')]^2 , \tag{3}$$

$$\int \mathscr{D}V\equiv\int_{-\infty}^{\infty}\prod_x dV(x) . \tag{4}$$

The sum in Eq. (3) is over all nearest-neighbor pairs. Using the "replica trick" to compute the quenched average of $\ln Z$ from that of the $n\to 0$ limit of $(Z^n-1)/n$, we replicate Eq. (2) n times and average over the possible bond configurations. One obtains

$$\langle Z^n\rangle=\int \mathscr{D}\mathbf{V}\prod_{\langle xx'\rangle}\left[(1-p)+p\int_0^{\sigma_0}d\sigma\, f(\sigma)\exp\left[-\tfrac{1}{2}\sigma\sum_{\alpha=1}^{n}[V_\alpha(x)-V_\alpha(x')]^2\right]\right] \tag{5a}$$

$$\equiv\int \mathscr{D}\mathbf{V}e^{-\mathscr{H}_e} , \tag{5b}$$

where for each position x there is a vector $\mathbf{V}$ with n components, labeled by α, in replica space. Brackets refer to average over disorder. Neglecting an unimportant constant term, $\mathscr{H}_e$ defined in Eq. (5b) may be written

$$\mathscr{H}_e\equiv-\sum_{\langle xx'\rangle}K(\mathbf{V}(x)-\mathbf{V}(x')) \tag{6a}$$

where

$$K(\mathbf{y})=\ln\left[1+v\int_0^{\sigma_0}d\sigma\, f(\sigma)\exp\left[-\tfrac{1}{2}\sigma\sum_{\alpha=1}^{n}y_\alpha^2\right]\right] , \tag{6b}$$

with $v=p/(1-p)$. Using Fourier transforms in replica space, it is convenient to write

$$\mathscr{H}_e=-\sum_{\langle xx'\rangle}\sum_{\mathbf{k}}B_{\mathbf{k}}\psi_{\mathbf{k}}(x)\psi_{-\mathbf{k}}(x') , \tag{7}$$

with

$$\psi_{\mathbf{k}}(x)\equiv e^{i\mathbf{k}\cdot\mathbf{V}(x)} , \tag{8}$$

$$B_{\mathbf{k}}\equiv\int_{-\infty}^{\infty}d^n y\, e^{-i\mathbf{k}\cdot\mathbf{y}}K(\mathbf{y}) =\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l}v^l F_l(\mathbf{k}) , \tag{9}$$

and

$$F_l(\mathbf{k})\equiv\int_0^{\sigma_0}d\sigma_1\int_0^{\sigma_0}d\sigma_2\cdots\int_0^{\sigma_0}d\sigma_l f(\sigma_1)f(\sigma_2)\cdots f(\sigma_l)\exp\left[-\frac{1}{2}\frac{k^2}{\sigma_1+\sigma_2+\cdots+\sigma_l}\right] . \tag{10}$$

The usefulness of Eqs. (5) and (7) becomes apparent when we note that

$$\int \mathscr{D}\mathbf{V} e^{-\mathscr{H}_e}\psi_{\mathbf{k}}(x)\psi_{-\mathbf{k}}(x') = \left\langle \prod_a \int \mathscr{D}V_a e^{-\mathscr{H}_0(V_a)} e^{ik_a[V_a(x)-V_a(x')]} \Big/ \int \mathscr{D}V_a e^{-\mathscr{H}_0(V_a)} \right\rangle$$
$$= \left\langle \exp\left[-\frac{k^2}{2}R_{x-x'}\right]\right\rangle \equiv \mathscr{G}_k(x-x') , \tag{11}$$

where the first equality is a replica identity[15] and the second comes from Gaussian integration. The average resistance $R_{x-x'}$ between two connected points may then be obtained from the derivative of $\mathscr{G}_k$ with respect to k^2.

We thus need the $k\to 0$ limit of the $F_l(k)$ appearing in Eq. (10). For $k=0$, one recovers the usual result that all the F_l are equal to 1. These contributions can be resummed to give $B_0=-\ln(1-p)$. For finite but small values of k, we proceed as follows. F_1, for example, can be rewritten in the form

$$F_1(k)=1-(1-a)\left[\frac{k^2}{\sigma_0}\right]^{1-a} \times \int_0^{\sigma_0/k^2} dy\, y^{-a}(1-e^{-1/2y}) . \tag{12}$$

The integral converges if the upper bound is extended to infinity but for finite σ_0/k^2 the upper bound gives a contribution of order k^2 to $F_1(k)$. Thus,

$$F_1(k)\sim 1-v_1\left[\frac{k^2}{\sigma_0}\right]^{1-a}+w_1\left[\frac{k^2}{\sigma_0}\right]+\cdots , \tag{13}$$

with the constants v_1 and w_1 positive for $a>0$ and negative for $a<0$. Note that when $a<0$, the term of order k^2 becomes the leading contribution. In that case, as expected, one recovers the usual percolation exponents. When $0<a<1$, the other $F_l(k)$ do not modify the functional form of the small-k behavior of Eq. (13). $F_2(k)$, for example, introduces terms of order k^2 and $(k^2)^{2(1-a)}$. The origin and physical significance of these terms is discussed in Appendix A. For reasons given at the end of Sec. IV, terms of order $(k^2)^{2(1-a)}$ and similar ones coming from the other $F_l(k)$ are neglected from now on.

We apply the Hubbard-Stratonovich transformation to derive a field theory for Eqs. (5) and (7). It takes the form[16]

$$\int \mathscr{D}\mathbf{V} e^{-\mathscr{H}_e} = \int \mathscr{D}\phi \exp\left[-\tfrac{1}{2}\sum_x {\sum_{\mathbf{k}}}' [r_{\mathbf{k}}\phi_{\mathbf{k}}(x)\phi_{-\mathbf{k}}(x)+\nabla\phi_{\mathbf{k}}(x)\cdot\nabla\phi_{-k}(x)]\Delta\mathbf{k} + \frac{1}{3!}u_3\sum_x {\sum_{\mathbf{k}_1\mathbf{k}_2}}' \phi_{\mathbf{k}_1}(x)\phi_{\mathbf{k}_2}(x)\phi_{-\mathbf{k}_1-\mathbf{k}_2}(x)\Delta\mathbf{k}_1\Delta\mathbf{k}_2+\cdots\right] , \tag{14}$$

where the prime indicates that terms in which any of the k vanish are omitted. Also, if z is the number of nearest neighbors, $-r_k=1-(zB_k)^{-1}$ becomes, for p close to $p_c=1-e^{-1/z}$,

$$r_k\sim(p_c-p)+v'\left[\frac{k^2}{\sigma_0}\right]^{1-a}-w'\left[\frac{k^2}{k_0}\right] , \tag{15}$$

where v' and w' are constants whose signs correspond, respectively, to those of v_1 and w_1.

III. MEAN-FIELD THEORY AND SCALING FUNCTION

Saddle-point integration of Eq. (14) gives the mean-field result,

$$\int e^{-iq\cdot x}\langle|\psi_{\mathbf{k}}(x)|^2\rangle d^dx \sim \frac{1}{r_k+q^2}\prod_{\mathbf{p}}\left[\frac{1}{r_{\mathbf{p}}+q^2}\right]^{n/2} . \tag{16}$$

Fourier transforming back with respect to q, one finds in the limit $p=p_c$, $n=0$ that to leading order in k^2,

$$\mathscr{G}_k(x)\sim\int\frac{d^dq}{(2\pi)^d}\int_0^\infty d\lambda\, e^{iq\cdot x}e^{-\lambda(q^2+v'k^{2(1-a)})} \sim\int_0^\infty d\lambda\,\lambda^{-d/2}e^{-\lambda v'k^{2(1-a)}-x^2/4\lambda} . \tag{17}$$

This quantity is finite but $d\mathscr{G}_k/dk^2$ diverges in the limit $k\to 0$ as expected from Eq. (11) because even the average resistance of a single conducting bond diverges when $0<a<1$ in Eq. (1).

In what follows, we consider the generating function for the average conductance instead of the resistance. Indeed, the average conductance is a well-defined quantity which, for large distances, scales like the typical resistance. According to the discussion in the Introduction, this is therefore the quantity of interest. Let us define

$$\mathscr{G}_{\mathbf{W}}(x)=\int e^{i\mathbf{k}\cdot\mathbf{W}}\langle e^{-k^2R_x/2}\rangle\frac{d^nk}{(2\pi)^n} =\langle e^{-W^2/2R_x}\rangle . \tag{18}$$

The average value of the conductance for two points separated by a distance x is then

$$\langle\sigma_x\rangle=-\frac{d}{d\mathbf{W}}\cdot\frac{d}{d\mathbf{W}}\mathcal{G}_{\mathbf{W}}(x)=\int\frac{d^nk}{(2\pi)^n}k^2\mathcal{G}_{\mathbf{k}}(x)$$

$$\sim x^{-(d-2)-2/(1-a)}\int d^nz\,z^2\int_0^\infty dy\,y^{-d/2}\exp\left[-\tfrac{1}{4}yz^{2(1-a)}-\frac{1}{y}\right]. \tag{19}$$

The integrals converge so that the average conductance between two sites known to be in the same cluster, $\{\sigma_x\}$, is, in mean-field theory,

$$\{\sigma_x\}=\frac{\langle\sigma_x\rangle}{\mathcal{G}_{\mathbf{W}=0}}\sim\frac{\langle\sigma_x\rangle}{x^{-(d-2)}}\sim x^{-2/(1-a)}. \tag{20}$$

To find the conductivity exponent, we use scaling arguments. Let ξ be the correlation length. Then the macroscopic conductivity $\bar{\sigma}$ is given by

$$\bar{\sigma}\sim\xi^{-(d-2)}\{\sigma_{x=\xi}\} \tag{21a}$$

$$\sim\xi^{-(d-2)}\xi^{-2/(1-a)} \tag{21b}$$

$$\sim(p-p_c)^{[(d-2)\nu+2\nu/(1-a)]}\sim(p-p_c)^{\bar{t}}, \tag{21c}$$

where we used $\xi\approx(p-p_c)^{-\nu}$. When d is set equal to $d_c=6$ and ν is set to its mean-field value, $\frac{1}{2}$, Eq. (21c) gives the mean-field value for t valid for $d>d_c$,

$$\bar{t}=(d-2)\nu+\frac{1}{1-a}=2+\frac{1}{1-a}. \tag{22}$$

For $a<0$, the standard mean-field result $t=3$ is obtained, i.e., for $d>6$ there is a crossover at $a=0$ from the standard lattice value $t=3$ to the continuum result given by Eq. (22).

IV. ONE-LOOP CORRECTIONS

A. Recursion relations

Consider the one-loop correction to the propagator illustrated in Fig. 1. This diagram gives the corrections to order u_3^2 [Eq. (14)] to the momentum-shell recursion relations[16] for r_k,

$$\frac{dr_k}{dl}=(2-\eta)r_k-\tfrac{1}{2}u_3^2\Sigma_k, \tag{23}$$

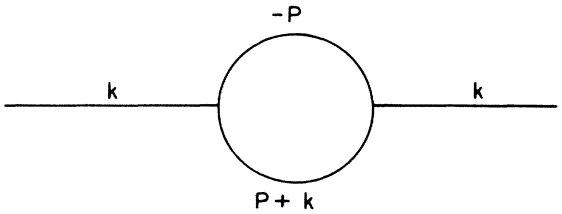

FIG. 1. One-loop contribution to Σ_k in Eq. (23).

where $\eta=-\epsilon/21$ is the anomalous dimension exponent, $(u_3^*)^2=2\epsilon/7$ at the percolation fixed point, and

$$\Sigma_{\mathbf{k}}=-2G_{\mathbf{k}}G_0+\sum_{\mathbf{p}}\Delta\mathbf{p}G_{-\mathbf{p}}G_{\mathbf{p}+\mathbf{k}}, \tag{24}$$

where $G_{\mathbf{k}}\equiv G_{\mathbf{k}}(q)=(r_k+q^2)^{-1}$ is evaluated at $q=1$. Note that the first term subtracts contributions from the lines with replica momentum index equal to zero because $\psi_{k=0}$ does not have the meaning of an order parameter. As in Ref. 16, Eq. (24) is rearranged as follows:

$$\Sigma_{\mathbf{k}}=-2G_{\mathbf{k}}G_0+\hat{G}_0^2+\delta\Sigma_{\mathbf{k}}, \tag{25a}$$

with

$$\delta\Sigma_{\mathbf{k}}=-\tfrac{1}{2}\sum_{\mathbf{p}}(G_{\mathbf{p}+\mathbf{k}}-G_{\mathbf{p}})^2\Delta p. \tag{25b}$$

The first two terms on the right-hand side of Eq. (25b) may be approximated by

$$G_0^2-2G_{\mathbf{k}}G_0=-\frac{1}{(r_0+1)^2}+\frac{2(vk^{2(1-a)}-wk^2)}{(r_0+1)^3}, \tag{26}$$

where we have redefined $v=v'/\sigma_0^{(1-a)}$ and $w=w'/\sigma_0$. Noting that $\delta\Sigma_{k=0}=0$, v and w do not enter the recursion relation for r_0 which, with the help of Eqs. (25) and (26) may be written in the same form as for the discrete lattice model,[16]

$$\frac{dr_0}{dl}=(2-\eta)r_0-\tfrac{1}{2}u_3^2\Sigma_{k=0}, \tag{27a}$$

$$\frac{dr_0}{dl}=(2-\eta)r_0+\tfrac{1}{2}u_3^2\frac{1}{(r_0+1)^2}. \tag{27b}$$

The recursion relation for u_3 can also be obtained with $k=0$, so one recovers the standard correlation length exponent $\nu=\frac{1}{2}+5\epsilon/84$.

To proceed further, we expand $\delta\Sigma_k$ in powers of k. Using the $n=0$ results,

$$\int(p\cdot k)^2f(p)d^np=\int_0^\infty dp\frac{1}{p}\int d\Omega(p\cdot k)^2f(p), \tag{28a}$$

$$\int\hat{p}_i\hat{p}_j d\Omega=\delta_{ij}, \tag{28b}$$

we find

$$\delta\Sigma_k = -\tfrac{1}{2}k^2 \int \frac{dp}{p}\left[\frac{4(1-a)^2v^2(p^2)^{1-2a}+4w^2p^2-8(1-a)vwp^{2(1-a)}}{(1+vp^{2(1-a)}-wp^2)^4}\right]. \tag{29}$$

Hence, $\delta\Sigma_k$ renormalizes only the value of w. We can thus use Eqs. (25) and (26) to write down

$$\frac{dv}{dl} = (2-\eta-u_3^2)v\ , \tag{30}$$

$$\frac{dw}{dl} = (2-\eta-u_3^2)w - \tfrac{1}{2}u_3^2\left[-\frac{d}{dk^2}\delta\Sigma_k\right]. \tag{31}$$

Note that dropping r_0 in the denominators does not influence exponents because v and w do not appear in Eq. (27b). When $a > \frac{1}{2}$, there is a term in $\delta\Sigma_k$ proportional to $k^{4(1-a)}$. This would renormalize the $k^{4(1-a)}$ in r_k but this term corresponds to an irrelevant operator, as discussed in the paragraph following Eq. (49).

Equation (30) may be solved immediately. With

$$v(l) \approx e^{\lambda_v l} v(0) \tag{32}$$

the eigenvalue λ_v is $\lambda_v = (2-\eta-u_3^2)$ which means that the crossover exponent for v is

$$\phi_v = \nu\lambda_v = 1\ . \tag{33}$$

To calculate the exponent for w, we integrate Eq. (31). We begin by introducing the field $g = wv^{-1/(1-a)}$. To lowest order in ϵ, g scales trivially,

$$g(l) = e^{-\lambda_g l} g(0),\quad \lambda_g = \frac{2a}{1-a}\ . \tag{34}$$

The recursion relation for w can now be written,

$$\frac{dw}{dl} = \frac{1}{\nu}w(l) - \tfrac{1}{2}u_3^2 w(l) J(g(l))\ , \tag{35}$$

where

$$J(g) = 2(1-a)^2 g^{-1}\int \frac{dp}{p}p^{2(1-2a)}D^4 + 2g\int \frac{dp}{p}p^2D^4 - 4(1-a)\int \frac{dp}{p}p^{2(1-a)}D^4 \tag{36}$$

with

$$D \equiv (1+p^{2(1-a)}-gp^2)^{-1}\ . \tag{37}$$

The integrals in Eq. (36) appear to be divergent because of the pole in D on the real axis. This is an artifact of our truncation of r_k at order k^2. There are presumably no poles in D if all powers in k are kept. Alternatively, a cutoff at $p \approx g(0)^{-1/2}$ in the p integrals could be introduced so that the pole is always avoided. $J(g)$ can then be expanded in powers of g,

$$J(g) = \alpha + \beta g^{-1} + gY(g)\ , \tag{38}$$

where

$$\alpha = -4(1-a)\int \frac{dp}{p}\frac{p^{2(1-a)}}{(1+p^{2(1-a)})^4} + 8(1-a)^2\int \frac{dp}{p}\frac{p^2p^{2(1-2a)}}{(1+p^{2(1-a)})^5} = -\tfrac{1}{3}(1+a)\ , \tag{39a}$$

$$\beta = 2(1-a)^2\int \frac{dp}{p}\frac{p^{2(1-2a)}}{(1+p^{2(1-a)})^4}\ , \tag{39b}$$

and $Y(g) = g^{-1}[J(g)-\alpha-\beta g^{-1}]$. Equation (35) can now be integrated. We obtain

$$w(l)\left[1+\frac{1}{2}\frac{u_3^2}{\lambda_g}\left[\frac{\beta}{g(l)}-X(g(l))\right]\right] = e^{\lambda_w l}w(0)\left[1+\frac{1}{2}\frac{u_3^2}{\lambda_g}\left[\frac{\beta}{g(0)}-X(g(0))\right]\right], \tag{40}$$

where

$$\lambda_w = \frac{1}{\nu}-\frac{1}{2}u_3^2\alpha = \frac{1}{\nu}+\frac{\epsilon}{21}(1+a) \tag{41}$$

and $X(g) = \int^g dg\, Y(g) + X_0$ where X_0 is chosen so that $X(g)$ is finite. The crossover exponent for w is thus

$$\phi_w = \nu\lambda_w = 1+\frac{\epsilon}{42}(1+a)\ . \tag{42}$$

Therefore are two important things to note about Eq. (40). First, the exponent λ_w is determined by the part of $J(g)$ which is independent of g. This part does not depend on the details of cutoffs placed on the k integrals to ensure convergence. All other parts of $J(g)$ contribute to the nonlinear scaling field but do not affect λ_w. Second, the nonlinear scaling field is proportional to wg^{-1} (for $a < \frac{1}{2}$). Since $g(0)$ cannot be zero, this presents no problems. The coefficient β diverges for $a > \frac{1}{2}$, signaling that the expansion of Σ_k in powers of k^2 breaks down. In fact, in this case, there are terms in the expansion of Σ_k which are proportional to $k^{4(1-a)}$, leading to a renormalization of the exponent associated with the coefficient of this $k^{4(1-a)}$ term. This coefficient remains irrelevant as discussed after Eq. (49).

B. Scaling behavior

Close to the fixed point, interaction parameters in the generating function Eq. (11) scale as just described so that we can write, with b the scale factor,

$$\mathcal{G}_k(x,p-p_c,v'(k^2/\sigma_0)^{1-a},w'(k^2/\sigma_0),w_2'(k^2/\sigma_0)^2,\ldots)$$
$$=\mathcal{G}_k(x,p-p_c,(v')^{1/(1-a)}(k^2/\sigma_0),w'k^2/\sigma_0,(w_2')^{1/2}(k^2/\sigma_0)k^2,\ldots)$$
$$=b^{-(d-2+\eta)}\mathcal{G}_k(b^{-1}x,b^{1/\nu}(p-p_c),[(b^{\lambda_v}v')^{1/(1-a)}(k^2/\sigma_0),b^{\lambda_w}w'(k^2/\sigma_0),(b^{\lambda_{w_2}}w_2')^{1/2}(k^2/\sigma_0)k^2,\ldots) . \quad (43)$$

As noted in Ref. 16, there is an infinity of scaling fields w_m', respectively, related to k^{2m}. Their scaling exponents may be modified by continuum corrections, but we expect them to reduce to their standard lattice values as $a\to 0$. For clarity, in Eq. (43) we have not noted all scaling fields and we have written the linear instead of the nonlinear scaling fields. As in Eq. (18), we Fourier transform to find the generating function for the conductivity, assuming that this function is controlled by the same set of scaling fields. Hence,

$$\mathcal{G}_w=b^{-(d-2+\eta)}\mathcal{G}_w(b^{-1}x,b^{1/\nu}(p-p_c),(v'b^{\lambda_v})^{-1/(1-a)}\sigma_0W^2,b^{-\lambda_w}(w')^{-1}\sigma_0W^2,b^{-\lambda_{w_2}/2}(w_2')^{-1/2}\sigma_0W^2,\ldots) . \quad (44)$$

In other words, we basically replace k^{-2} by W^2. Choosing $b^{1/\nu}(p-p_c)\approx 1$, and taking the derivative with respect to W^2, this can be rewritten,

$$\frac{d\mathcal{G}_W}{dW^2}=b^{-(d-2+\eta)}\frac{d\mathcal{G}_W}{dW^2}(x/\xi,1,(p-p_c)^{\phi_v/(1-a)}\sigma_0W^2,(p-p_c)^{\phi_w}\sigma_0W^2,(p-p_c)^{\phi_{w_2}/2}\sigma_0W^2,\ldots) \quad (45)$$

where for clarity we dropped factors such as $(v')^{-1/(1-a)}$. If we proceed as in Eqs. (20)—(21c), assuming that the largest eigenvalue dominates the scaling behavior, we obtain our final result,

$$\bar{t}=(d-2)\nu+\phi(a,d) , \quad (46)$$

where

$$\phi(a,d)=\max\left[\frac{\phi_v}{1-a},\phi_w(a,d),\ldots\right] . \quad (47)$$

For $a>0$, this implies

$$\bar{t}=(d-2)\nu+\phi_w=(d-2)\nu+1+\frac{\epsilon}{42}(1+a) \quad \text{for } 0<a<\epsilon/42 , \quad (48)$$

$$\bar{t}=(d-2)\nu+\frac{\phi_v}{1-a}=(d-2)\nu+\frac{1}{1-a} \quad \text{for } \epsilon/42<a<1 . \quad (49)$$

Note that our exponent $\bar{t}$ varies continuously. For $a<0$, one recovers the standard discrete lattice result[16,17] $t=(d-2)\nu+1+\epsilon/42$.

We can now justify the neglect of $F_2(k)$ mentioned after Eq. (13) by noting that apart from trivial changes of the bare values of the interaction parameters (e.g., an additional contribution to the coefficient of k^2) $F_2(k)$ would simply add another scaling field controlling the terms of order $(k^2)^{2(1-a)}$. On dimensional grounds, this would lead to an extra dependence on a term of the form $\sigma_0W^2(p-p_c)^{\phi_x/2(1-a)}$ in Eq. (45). The crossover exponent ϕ_x should be of order unity as all those encountered in this problem so that this extra contribution would always be negligible compared with that of $(p-p_c)^{\phi_v/(1-a)}$. A more detailed calculation confirms these expectations. Physically, $F_2(k)$ represents the resistance of pairs of resistors in parallel (see Appendix A). Since the singular behavior induced by continuum corrections to the resistance of these pairs is much less than that of singly connected resistors, the multiply connected resistors modify the scaling behavior of the overall conductivity only through their contribution to the scaling field w, which also appears in the standard discrete lattice problem.

V. CONCLUSION

Our results for the continuum transport exponent $\bar{t}$ are given by Eqs. (46)—(49). We expect Eq. (46) to be valid with

$$\phi(a,d)=\begin{cases}\phi_w(d), & a<0\\ \phi_w(a,d), & 0<a<a_c\\ 1/(1-a), & a>a_c\end{cases} \quad (50)$$

where $\phi_w(a_c,d)=1/(1-a_c)$. a_c is to be distinguished from the crossover value $a_w=1-\phi_w^{-1}(d)$ first proposed by Straley.[7] At a_w, $\bar{t}=(d-2)\nu+(1-a)^{-1}$ becomes equal to $t=(d-2)\nu+\phi_w(d)$. To first order in ϵ, $a_c=a_w=\epsilon/42$. To estimate a_c and a_w to second order in ϵ, we expand $\phi_w(d)$ and $\phi_w(a,d)$ as follows.

$$\phi_w(d)=1+\frac{1}{42}\epsilon+\frac{4}{7^33^2}\epsilon^2 ,$$
$$\phi_w(a,d)=1+\frac{1}{42}\epsilon(1+a)+\left[\frac{4}{7^33^2}+h(a)\right]\epsilon^2 . \quad (51)$$

The second-order coefficient in $\phi_w(d)$ was calculated in Ref. 18. The second-order coefficient in $\phi_w(a,d)$ has not been calculated. Hence $h(a)$ is an unknown function of a but to obtain a_c, it suffices to require, as was true to first order in ϵ, that $\phi_w(a,d)$ tends to $\phi_w(d)$ as $a\to 0$. Indeed, in this case one finds

$$a_c=\frac{\epsilon}{42}+\frac{4}{7^33^2}\epsilon^2+O(\epsilon^3) ,$$
$$a_w=\frac{\epsilon}{42}+\left[\frac{4}{7^33^2}-\frac{1}{(42)^2}\right]\epsilon^2+O(\epsilon^3) \quad (52)$$

independent of $h(a)$. Thus to this order, $a_c>a_w$. We expect this result to be valid for all $d<6$.

Our results are summarized in Fig. 2. The behavior for large values of the parameter a [Eqs. (49) and (50)] is the same as that of mean-field theory, i.e., the physics is controlled by the singly connected bonds. As discussed at the end of the preceding section, parallel bonds are less singular so that for sufficiently large values of a they do not contribute to continuum corrections. When the parameter a is sufficiently small, then the functional form of the continuum corrections changes to finally take a value equal to the standard discrete lattice exponent in the limit where a tends to 0. Our main assumption is that the largest exponent of the set $(\phi_v/(1-a),\phi_w,\phi_{w_2}/2)$ determines the conductivity exponent. This assumption is valid for $a<0$ and we believe it to be generally valid. To establish completely its validity, one would need to carry out a detailed calculation of the scaling function itself, $\mathcal{G}_w$, at least to first order in ϵ.

For large values of the parameter a, our result Eq. (49) is the same as that of Straley,[7] of Ben Mizrahi and Bergman,[4] and of Machta, Guyer, and Moore,[19] all of whom predict that the exponent should be exactly equal to the lower bound of Halperin *et al.*[3] The various approaches disagree in their prediction for the crossover to the standard discrete lattice value. Ben-Mizrahi and Bergman, using a $1+\epsilon$ Migdal-Kadanoff approach, argue that the continuum problem is described by a different fixed point so that the limit of small a is not smooth. What distinguishes the two universality classes according to them is whether $f(\sigma)$ in Eq. (1b) takes a finite or vanishing value at $\sigma=0$. We have not considered the marginal cases in detail, but we find that continuum and discrete lattice problems are described by the percolation fixed point but with different crossover fields and exponents. In analogy with the standard problem, crossover fields are associated with allowing resistances of occupied bonds to be nonzero. On the other hand, Straley and Machta *et al.* propose that the behavior, Eq. (49), holds until one reaches the value of a where t is equal to the standard discrete lattice exponent. This would occur, to first order in ϵ, at $a=\epsilon/42$. The difference between the results of Straley and Machta *et al.*, compared with our results, shows up in the crossover value of a only to second order in ϵ. The recent results of Machta *et al.*[19] are based on hierarchical lattices which mimic percolation clusters.

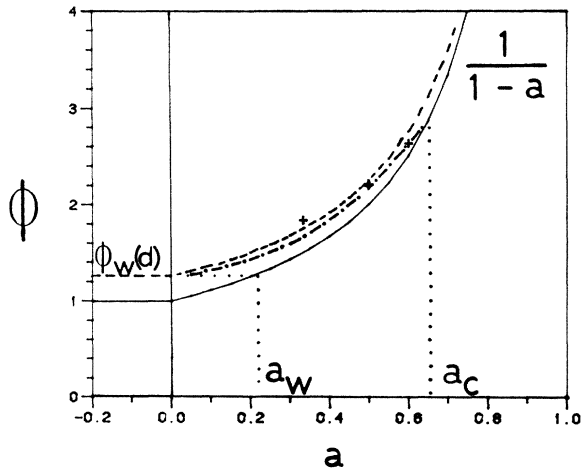

FIG. 2. Conjectured behavior of the crossover exponent ϕ as a function of a. The solid line gives the correct asymptotic large-a result $\phi=1/(1-a)$. $a_w=1-\phi_w^{-1}(d)$ is the crossover point predicted by Straley[7] and Machta *et al.* (Ref. 19), a_c defined by $\phi_w(a_c,d)=1/(1-a_c)$, is the new crossover point predicted in this paper. Our proposed curve starts from the lattice value $\phi_w(d)$ through the dashed-dotted line $\phi_w(a,d)$ and then crosses over to the asymptotic large-a value $1/(1-a)$. Note that the dashed-dotted line and the numerical results of Ref. 11 (crosses) are, within numerical accuracy in the latter case, between the upper bound $\phi_w(d)+[1/(1-a)-1]$ (dashed line) and the nonrigorous lower bound $1/(1-a)$ (solid line). To first order in ϵ, $a_c=a_w$, but this is not necessarily true to higher order.

Finally, we note that the numerical calculations of Sen *et al.*[11] in two dimensions give results which even for large values of a (e.g., $a=0.5$) are above the asymptotic value $\phi=1/(1-a)$. For values of a larger than the crossover point $a_w\approx 0.23$ of Straley[7] and Machta *et al.*,[19] the data with quoted uncertainties are thus larger than the predictions of all the models (nodes-links-blobs,[7,3] hierarchical lattices,[19] $1+\epsilon$ expansion[4]) in the regime where they agree. The numerical results[11] and the present $6-\epsilon$ expansion suggest that the value of a (a_c), where the crossover to the asymptotic regime occurs, is larger than 0.5 in two dimensions. The results of Machta *et al.*[19] on the other hand, suggest that the true asymptotic regime for numerical calculations can be reached only for system sizes far bigger than those of Ref. 11. Appendix B contains the results of numerical calculations which improve the statistical accuracy of some of the results of Ref. 11 and also contain new results on the form of fixed distributions for the present problem. For the value of a considered, we have not been able to detect signs that the asymptotic regime had not been attained but we cannot rule out the suggestion of Machta *et al.*

ACKNOWLEDGMENTS

We are indebted to B. I. Halperin, J. Machta, and R. Rammal for extremely useful conversations. We should also like to thank P. Sen for a discussion and for sending us a copy of Ref. 11 prior to publication. We also thank J. Machta for pointing out Ref. 7 to us and for discussing his results prior to publication. P. Bénard and M.-A. Lemieux have helped us perform the numerical calculations. Time was provided by the Centre de Calcul de l'Université de Sherbrooke. One of us, T.C.L., acknowledges support from the Office of Naval Research under Grant No. 0158. Another of us, A.-M.S.T., is indebted to the France-Canada exchange program and to the Centre de Recherche sur les Très Basses Températures where part of this work was completed. A.-M.S.T. is supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada through an individual operating grant.

APPENDIX A: CONTRIBUTION FROM PARALLEL PATHS

In this appendix we clarify the relation between $F_2(k)$ and the contribution from parallel resistors and find the

functional dependence of $F_2(k)$ on k. Let $P_2(R)$ be the probability that two resistances in parallel add up to a value R. Then, with the probability distribution for conductivity of the individual elements given by Eq. (1b) we obtain

$$P_2(R)=\int_0^{\sigma_0} d\sigma_1 \int_0^{\sigma_0} d\sigma_2 \delta\left[R-\frac{1}{\sigma_1+\sigma_2}\right] \times \frac{(1-a)^2}{\sigma_0^2}\left[\frac{\sigma_1}{\sigma_0}\frac{\sigma_2}{\sigma_0}\right]^{-a} . \quad \text{(A1)}$$

Changing variables to $u_1=\sigma_1/\sigma_0$ and Fourier transforming Eq. (A1) to obtain the characteristic function, we find

$$P_2(k)=(1-a)^2 \int_0^1 du_1 \int_0^1 du_2 (u_1u_2)^{-a} \times e^{-ik/[\sigma_0(u_1+u_2)]} . \quad \text{(A2)}$$

Replacing $2ik$ with k^2, this quantity is $F_2(k)$. Changing variables to $p^2=k^2/2\sigma_0$ and $u_1=p^2y_1$ and using the symmetry $y_1 \leftrightarrow y_2$ of the integrand, $F_2(k)$ may be rewritten in the form

$$F_2(k)=1+(1-a)^2(p^2)^{2(1-a)}(2\mathscr{D}) , \quad \text{(A3)}$$

where

$$\mathscr{D}=\int_0^{p^{-2}} dy_1 \int_0^{y_1} dy_2 (y_1y_2)^{-a}(e^{-1/(y_1+y_2)}-1) . \quad \text{(A4)}$$

To evaluate $\mathscr{D}$, we use

$$\exp\left[-\frac{1}{y_1}\right] \le \exp\left[-\frac{1}{y_1+y_2}\right] \le \exp\left[-\frac{1}{2y_1}\right] , \quad \text{(A5)}$$

which is valid in the range of integration of Eq. (A4). All we need to consider then is the integral,

$$I(c)\equiv \int_0^{p^{-2}} dy_1 \int_0^{y_1} dy_2 (y_1y_2)^{-a}(e^{-1/cy_1}-1) . \quad \text{(A6)}$$

Performing the y_2 integral, the y_1 integral can be written with the help of the incomplete gamma function whose series expansion may be found in Ref. 20, Eq. (8.354.2). We are finally left with

$$I(c)=\frac{c^{2(a-1)}}{1-a}\left[\Gamma(2(a-1))+\frac{1}{2a-1}\left[\frac{p^2}{c}\right]^{2a-1} -\frac{1}{4a}\left[\frac{p^2}{c}\right]^{2a}+\cdots\right] , \quad \text{(A7)}$$

where Γ is the gamma function. Recall that

$$I(c=1)\le \mathscr{D} \le I(c=2) . \quad \text{(A8)}$$

There are two cases to distinguish. If $a<\frac{1}{2}$, then for small p, $(p^2)^{2a-1}$ dominates and Eq. (A8) yields

$$\mathscr{D} \sim -A(p^2)^{2a-1} , \quad \text{(A9)}$$

where A is a positive constant. For $a>\frac{1}{2}$ we have

$$\mathscr{D} \sim -B , \quad \text{(A10)}$$

where B is a positive constant. Substituting in Eq. (A3), and returning to the original variables, we see that

$$F_2(k)=1-B'\left[\frac{k^2}{2\sigma_0}\right]^{2(1-a)}-A'\left[\frac{k^2}{2\sigma_0}\right] , \quad \text{(A11)}$$

where A' and B' are undetermined positive constants. As above, the crossover between the dominance of one term or the other is at $a=\frac{1}{2}$. Physically, for $0<a<\frac{1}{2}$ the average resistance of two parallel resistors is finite while for $\frac{1}{2}<a<1$ it diverges. This should be contrasted with the case of a single resistor whose average is infinite for the whole range $0<a<1$.

APPENDIX B: NUMERICAL CALCULATIONS

In this appendix, we present the results of numerical calculations which improve the statistical accuracy of some of the results of Ref. 11. More specifically, we consider a square lattice of size $L\times L$ using the method of Refs. 21 and 22. The conductivity of the bonds are chosen with a probability described in Eqs. (1). Note that the correspondence between the notation of Ref. 11 and ours is $\alpha=a$ and $a=1-m^{-1}$. One-hundred samples per lattice size L were used in Ref. 11. We have increased that number by at least a factor of 75. Our largest system has $L=23$ which is smaller than the one used in Ref. 11 by a factor of about 2, but, as discussed below, the emphasis of our analysis is different.

We note that the scaling regime for the standard lattice problem is reached for relatively small sizes, namely[23,22] $L\approx 7$. We calculated the exponent for the continuum problem with $a=\frac{1}{3}$ and found $t/\nu=1.374\pm0.01$ from the results displayed in Table I for $L=7,8,9,10$ and then $t/\nu=1.42\pm0.04$ from those for $L=20,21,22,23$. There does not seem to be a significant difference in the effective exponent for these two sets of data. We would have hoped to see such a change if, as suggested by Machta *et al.*,[19] the asymptotic regime had not been reached. Assuming no systematic error, the exponent estimated from all eight values of L is $t/\nu=1.383\pm0.004$ in agreement

TABLE I. Summary of the numerical data for finite size scaling. Average conductivity σ_L of blocks of size $L\times L$ at p_c for $a=\frac{1}{3}$. N_s is the number of samples generated. Only the percolating samples, on the average $\frac{1}{2}$ of those generated, are kept for the statistics. Uncertain digits are in parentheses: Estimated error on these digits is ±30 for $L=7-10$ and ±50 for $L=20-23$.

L	N_s	$\sigma_L\times10^2$
7	35 000	2.6(58)
8	35 000	2.2(08)
9	35 000	1.8(82)
10	35 000	1.6(27)
20	17 250	0.62(59)
21	15 250	0.58(13)
22	15 250	0.54(49)
23	15 250	0.51(28)

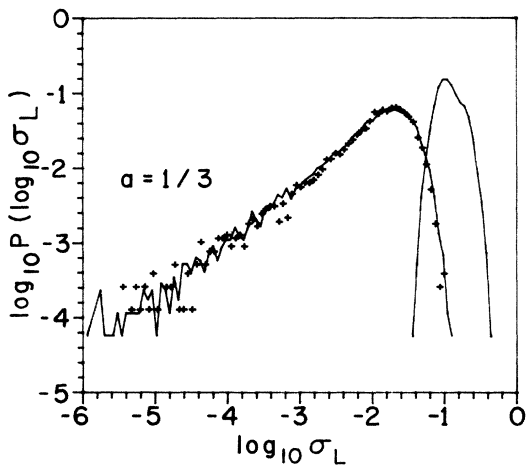

FIG. 3. Histograms for the conductivity σ_L of $L\times L$ blocks at $p_c=0.5$. There are 100 identical logarithmic intervals $\Delta\log\sigma_L=0.06$ on the horizontal axis. The probability density $P(\log\sigma_L)$ is normalized to unity in all cases. Logarithms base 10 are used. Solid lines are for $L=10$. The curve to the right is for the standard lattice problem (Ref. 22) while the other one is for the continuum problem with $a=\frac{1}{3}$. The crosses are for $L=23$. In the latter case the curve has been translated to the right by $1.38\times\log(\frac{23}{10})$ to illustrate scaling. More than 1.7×10^4 conducting samples were calculated for $L=10$ and more than 7.5×10^3 for $L=23$.

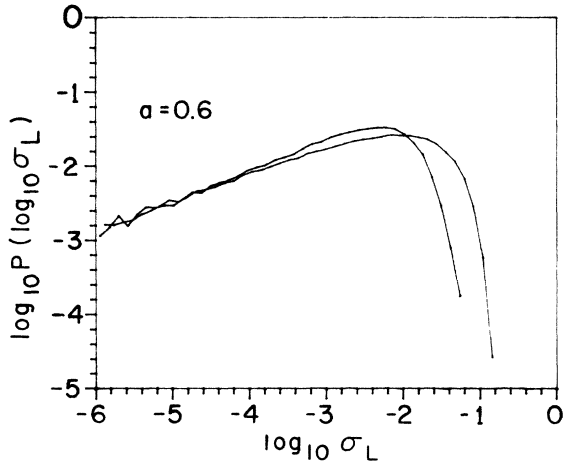

FIG. 4. Left curve is a histogram (1.7×10^4 samples) for the conductivity σ_L of 10×10 blocks at $p=p_c$ for $a=0.6$ while the right curve is the corresponding histogram (1.5×10^5 samples) for six singly connected bonds. There are 50 identical logarithmic intervals $\Delta(\log\sigma_L=0.12$ on the horizontal axis. The probability density $P(\log\sigma_L)$ is normalized to unity.

with the result $t/\nu=1.38\pm0.04$ of Ref. 11 and inconsistent with either the standard lattice result[24] $t/\nu=0.97$ and the large a result $t/\nu=1.125$.

Figure 3 illustrates the probability density for the conductivity σ_L of blocks of different size L. The curve to the right is for the standard lattice problem with $L=10$. The distribution for $L=23$, $a=\frac{1}{3}$ (crosses) ($m=1.5$) is identical, within statistical uncertainty, to that for $L=10$, $a=\frac{1}{3}$ when the conductivity of the former is scaled by $(\frac{23}{10})^{t/\nu}$. There is no sign that the distribution of conductances illustrated in Fig. 3 is not the fixed distribution. We should note however that Machta *et al.*[19] obtained convergence to a fixed distribution for the hierarchical lattice[26] only when the number of singly connected bonds in a box of size L was of order 2^5. On real percolation clusters, where the number of singly connected bonds in a box of size L is about[26] $1.1\times L^{1/\nu}$, this would correspond to sizes $L\approx100$. Note also that a further sign that the asymptotic regime may not have been reached in our calculation is that the value $t/\nu=1.383\pm0.004$ quoted above seems inconsistent with the upper bound $t/\nu\approx1.35$.

We have also plotted in Fig. 4 the probability density for the conductivity of blocks of size L when $a=0.6$ ($m=2.5$). The curve to the right on this same figure is the probability density for the conductivity of six singly connected bonds in series (i.e., disregarding all parallel paths). That number of bonds is the average number contained[26] in a box of size $L=10$ at $p=p_c$. Clearly, for conductivities smaller than 10^{-4} the two probability densities are identical within numerical uncertainty. The expected disagreement is at high conductivities where parallel paths are likely to contribute. In the case of the singly connected bonds, one knows from the theory of stable distributions[14] that there is a power-law tail of the form σ_L^{-a}, i.e., in Fig. 4, a straight line of slope $1-a$. Since in a theory with only singly connected bonds one knows[3] that the conductivity exponent is $t=m=(1-a)^{-1}$, one can extract the conductivity exponent from that tail when this model is valid. It turns out that for this value of a, the exponent found by Sen *et al.* is not appreciably different from the singly connected bonds results and this should not be too surprising given that the two curves in Fig. 4 are so close.

On the other hand, the probability densities in Fig. 3 differ somewhat more from the singly connected bonds results. In particular, coming from the low conductivity side the curve bends upward before reaching a maximum, forming a hump at high conductivities. That hump disappears when a gets larger, as in Fig. 4. Curiously, if one extracts an effective value of a from the tail of the distribution in Fig. 3, the value of t obtained from $t=m=(1-a)^{-1}$ is close to the one actually calculated even though we know of no reason to expect this to be so.

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