Distribution of shortest path lengths in percolation on a hierarchical lattice

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Some characteristics of the shortest paths connecting distant points on a percolation network are studied. Attention is focused on the probability distribution of the lengths of shortest paths, especially on the manner in which this distribution changes as the distance between the two points is increased. Calculations are performed on a bond-diluted hierarchical lattice of the Wheatstone-bridge type. The evolution of the probability distribution is followed numerically. At the critical percolation concentration, the distribution is seen to approach a nontrivial function under proper rescaling of its argument. Away from criticality, the scaled distribution approaches a δ function whose location $1/v$ is a measure of how tortuous the shortest paths are. Here v is the wetting velocity. As the bond occupation probability is increased, there is a second phase transition when the shortest paths coincide with directed paths whose lengths are the smallest possible ($v=1$). This occurs at the directed percolation concentration. It is conjectured that the variation of v near the directed threshold is governed by the exponent $\nu_{\parallel}$ which describes the divergence of the parallel correlation length in directed percolation.

I. INTRODUCTION

In bond percolation, a certain fraction $(1-p)$ of the bonds of a lattice are removed at random, and two points are said to be connected if there is at least one unbroken chain of bonds between them. It is then of interest[1] to ask for the length (measured by the number of links l) of the shortest connections which bridge, say, the opposite faces of the cube enclosed by the planes $(\pm x, \pm y, \ldots)=\frac{1}{2}R$. The Cartesian separation of opposite planes is R and the limiting ratio

$$v=\lim_{R\to\infty} R/l \quad (1)$$

provides a quantitative measure of how tortuous these shortest paths are. v is called wetting velocity.[2] v is unity for the most direct possible connections, whereas $v=0$ corresponds to extremely tortuous connections.

Besides the limiting ratio v, it is important to know how the *probability distribution* $P(l)$ of shortest path lengths l behaves as a function of the Cartesian separation R. For instance, at the critical percolation concentration $p=p_c$, the scaled distribution approaches a characteristic function $\widetilde{P}$ which is universal in the same sense that critical exponents are. In this paper we follow the evolution of P as R increases. Our calculations are performed on a diluted Wheatstone-bridge hierarchical lattice defined below. We study the probability distribution $P(l)$ numerically both for $p=p_c$ and for $p\neq p_c$. We find that, although the percolation recursion relations exhibit perfectly smooth flows as p is varied between p_c and 1, there is a *second* phase transition associated with the distribution $P(l)$ alone. For $p>p_c$, the distribution for l/R evolves, as $R\to\infty$, into a δ function located at $1/v$. The second transition is associated with the p dependence of the position of the δ function—it reaches $v=1$ as $p\to p_d^-$, and does not move for larger values of p. Here, p_d is the directed percolation threshold,[3] and the transition signals the proliferation of the most direct possible ($v=1$) paths for $p>p_d$.

The wetting velocity v is singular near both p_c and p_d, but while the vicinity of p_c has been studied in detail[4–16] for various lattices, not many studies exist of the full range of p.[2,15] We find that it is crucial to keep track of the *full* probability distribution not only in order to obtain reliable values of critical exponents, but also to correctly obtain even such qualitative features as the saturation of v at 1 for values of p exceeding p_d; approximations which do not keep the full distribution can lead to $v<1$ for all p between p_c and 1. Finally, we discuss the range $0<p<p_c$. In this case, the probability of connections drops exponentially as R increases, and v describes the nature of the most probable of these infrequent connections.

II. THE MODEL AND PROCEDURE

The hierarchical lattice we consider is of the Wheatstone-bridge type and is defined recursively (outward) through the prescription indicated in Fig. 1. At the nth stage, we obtain a construct whose top and bottom nodes T_n and B_n are separated by $R=2^n$ bonds. These nodes are analogous to *faces* of a cube on a regular lattice. Now consider the percolation problem on the hierarchical lattice where each bond of the lattice is present with probability p. Let p_n be the effective probability that nodes T_n and B_n are connected. Then p_{n+1} can be determined in terms of p_n,

$$p_{n+1}=2p_n^5-5p_n^4+2p_n^3+2p_n^2 . \quad (2)$$

This is the same recursion relation as the approximate position-space renormalization-group transformation of Reynolds, Stanley, and Klein[17] for the square lattice, illustrating the point that such approximate recursions are often exact on hierarchical lattices.[18] Equation (2) has an unstable fixed point at $p_c=\frac{1}{2}$. The correlation length exponent ν is found on linearizing Eq. (2) around p_c. Explicitly, we have

$$\nu=\ln 2/\ln(\partial p_{n+1}/\partial p_n)\,|_c ,$$

which yields $\nu \simeq 1.428$.

In a particular realization of the lattice, let l be the length of the shortest path between nodes T_n and B_n; it lies in the range $2^n \le l \le 3^n$ if the nodes are connected. If they are not connected, we say l is infinite. Sampling over all configurations of the bonds will generate a probability distribution $P_n(l)$ describing the relative frequency of occurrence of different shortest path lengths l. The sum over $P_n(l)$ for all finite l yields the connection probability p_n. For a given realization, it is clear that the shortest path length l between B_{n+1} and T_{n+1} is a function $\Delta(l_1,l_2,l_3,l_4,l_5)$ only of the shortest path lengths $l_1,l_2,\dots,l_5$ (Fig. 1) of the nth order connections. Explicitly we have

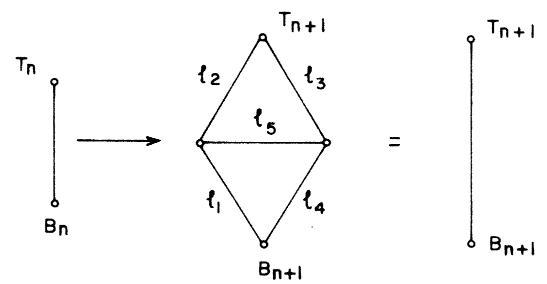

FIG. 1. The construction of an $(n+1)$th level bond of the hierarchical lattice from nth level bonds is indicated. The length of the shortest path between B_{n+1} and T_{n+1} depends only on the five level-n shortest path lengths $l_1,l_2,\dots,l_5$.

$$\Delta(l_1,\dots,l_5)=\min[l_1+l_2,\ l_3+l_4,\ l_1+l_5+l_3,\ l_2+l_5+l_4] . \quad (3)$$

Consequently, the probability distribution $P_{n+1}(l)$ is determined by $P_n(l)$. We have

$$P_{n+1}(l)=\int \left[\prod_{i=1}^{5} dl_i P_n(l_i)\right]\delta(l-\Delta(l_1,\dots,l_5)) . \quad (4)$$

For any finite value of n, the distribution P consists of a series of δ functions at integers between 2^n and 3^n, and one at ∞ corresponding to no connection. Consequently, integrals on the right-hand side of Eq. (4) reduce to finite sums.

At the start ($n=0$) we have either $l=1$ (with probability p) or no connection (with probability $1-p$). At the first stage ($n=1$), l may be equal to 2 or 3 (with probabilities $2p^2-p^4$ and $2p^3-4p^4+2p^5$, respectively) or the bond may be absent (corresponding to $l=\infty$ with probability $1-p_1=1-2p^5+5p^4-2p^3-2p^2$). We have carried out the calculation by hand for $n=2$ (when l ranges from 4 to 9, or is infinite), but as n increases, it becomes progressively more tedious to generate recursions analytically.

For the next two orders ($n=3,4$) we carried out exact calculations on the computer, in accordance with Eq. (3). For instance, to find $P_3(l)$ we generated all 7^5 configurations $\{l_i\}\equiv\{l_1,l_2,l_3,l_4,l_5\}$ of level-2 bonds. With each such configuration was associated a weight $[\prod_{i=1}^{5} P_2(l_i)]$ and a shortest path length $l\equiv\Delta(\{l_i\})$. Summing over every configuration $\{l_i\}$ of level-2 bonds finally yields $P_3(l)$, as in Eq. (4).

For larger n (up to $n=11$ for $p=p_c$ and 13 for $p\neq p_c$) we used a Monte Carlo method to study the evolution of $P_n(l)$. Rather than generating each nth level length configuration $\{l_i\}$ and weighting it appropriately, we used a random number generator to select each l_i ($i=1\dots5$) with relative frequency $P_n(l_i)$. For each such try, the $(n+1)$th level l was determined from Eq. (3), and a histogram representing $P_{n+1}(l)$ was built up by repeating the procedure many times (between 2 million and 20 million times for each $n\to n+1$). We checked our Monte Carlo procedure against exact results for $n=1$ to 4 and found good agreement [with differences in the fourth significant figure for individual probabilities $P(l)$, and in the fifth figure for averaged quantities like the mean length].

III. $p=p_c$: SCALING FORM

At the critical percolation concentration $p=p_c$, the probability distribution $P_n(l)$ is expected to approach a scale-invariant form for large enough n:

$$P_n(l)\approx \Lambda_n^{-1}\tilde{P}(l/\Lambda_n) , \quad (5)$$

where the scale factor Λ_n is given by

$$\Lambda_n=\lambda^n . \quad (6)$$

In Eq. (5), $\tilde{P}$ is a function characterizing the distribution of shortest paths at criticality, and is expected to be universal, e.g., independent of many details of the initial ($n=0$) distribution $P_0(l)$, provided only that the system is at the critical point. Similar scaled probability distributions for other quantities have been studied before in related contexts, e.g., the conductivity distribution in a random resistor network[19] and the bond-strength distribution of diluted Potts models.[20,21] $\tilde{P}$ is also analogous to the universal effective-block-spin weight functions studied in spin systems at criticality.[22,23]

Our numerical results for the distributions of shortest paths verify the scaling form Eq. (5). The function $\tilde{P}$, as determined from the data for $n=7,8,9,10$ is shown in Fig. 2.

In order to extract an estimate of the scale factor λ we

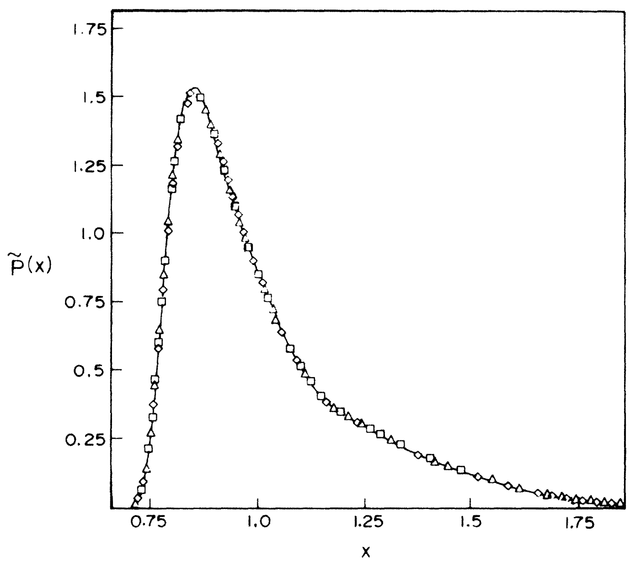

FIG. 2. Scaling function describing the distributions of shortest path lengths at criticality is shown. Data was obtained from a run in which 20 million Monte Carlo trials were performed at every iterative step. The fact that data for $n=7$ (solid line), 8 (squares), 9 (triangles), and 10 (diamonds) all fall on the same curve for a particular value of λ ($=2.1037$) provides evidence for the scaling form Eq. (5).

proceed as follows. From Eq. (5), the mean and root-mean-squared shortest path length[24] at the nth level obey

$$\langle l \rangle_n \approx C_1 \Lambda_n \ , \tag{7a}$$

$$(\langle l^2 \rangle_n)^{1/2} \approx C_2 \Lambda_n \ , \tag{7b}$$

where the constants C_1 and C_2 involve moments of the

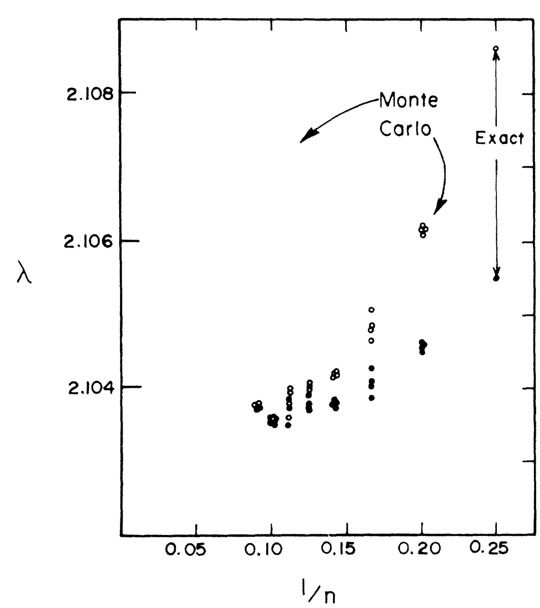

FIG. 3. An estimate of the scale factor λ is obtained from the ratios of mean- and root-mean-squared shortest path lengths at $p=p_c$. These are shown as solid and open circles, respectively. Results of four different runs are shown, in each of which 20 million Monte Carlo trials were performed at every stage in the iteration.

scaled function $\tilde{P}$ defined in Eq. (5). We have

$$C_1 = \int dx\, x\tilde{P}(x) \ , \tag{8a}$$

$$C_2^2 = \int dx\, x^2\tilde{P}(x) \ . \tag{8b}$$

In view of Eqs. (6) and (7), the ratios $\langle l_{n+1} \rangle / \langle l_n \rangle$ and $(\langle l_{n+1}^2 \rangle / \langle l_n^2 \rangle)^{1/2}$ should both approach λ as $n \to \infty$. We have determined $\langle l \rangle_n$ and $(\langle l^2 \rangle_n)^{1/2}$ from our numerically generated distributions $P_n(l)$ and have plotted the ratios versus $1/n$ in Fig. 3. We estimate $\lambda = 2.103 \pm 0.001$.

The dependence of the mean shortest path length $\langle l \rangle_n$ on n at $p=p_c$ is

$$\langle l \rangle_n \sim R^{d_s} \ , \tag{9}$$

where $R=2^n$ on the hierarchical lattice and d_s is a critical exponent. From Eqs. (6) and (7) we find

$$d_s = (\ln\lambda)/\ln 2 = 1.072 \pm 0.001 \ . \tag{10}$$

Note that the dependence of $\langle l \rangle$ on R at $p=p_c$ [Eq. (9)] is consistent with $v=0$.

IV. THE REGION $p_c < p < 1$: WETTING VELOCITY

For $p > p_c$, the appropriate scale to measure shortest path lengths l at the nth stage is set by the internodal separation $R=2^n$. The probability distribution for the variable l/R is found, numerically, to approach a δ function as $R \to \infty$. Identifying the location of the δ function with $1/v$ ensures consistency with the definition of v [Eq. (1)]. Defining a scaled distribution $P'(x)$ by

$$P'(x) = RP(Rx) \ , \tag{11}$$

we find the asymptotic form

$$P'_n(l/R) \approx p_n \delta(l/R - 1/v) \ . \tag{12}$$

The probability p_n of connections between nodes B_n and T_n (Fig. 1) approaches unity as $n \to \infty$.

The wetting velocity v obtained from the numerically determined asymptotic form of P'_n is plotted as a function of p in Fig. 4. It varies between 0 and 1 as p moves from $p_c = 0.5$ to $p_d = (\sqrt{5}-1)/2 \simeq 0.618$, and remains saturated at $v=1$ for $p > p_d$. The wetting velocity is singular near both p_c and p_d

$$v \sim (p - p_c)^{\theta} \ , \tag{13}$$

$$1 - v \sim (p_d - p)^{\theta'} \ . \tag{14}$$

It is difficult to estimate either θ or θ' directly from the data plotted in Fig. 4, but θ can be related to the exponent d_s by arguing that the crossover from Eq. (5) to Eq. (12) occurs as R crosses the correlation length ξ. Arguments presented in detail elsewhere[7,10,15] then lead to $\theta = \nu(d_s - 1)$. For the dilute hierarchical lattice, we thus find $\theta \simeq 0.103$.

Turning now to the behavior of v near p_d [Eq. (14)], we conjecture that the critical exponent θ' is equal to $\nu_{\parallel}$, the exponent which characterizes the divergence of the parallel correlation length $\xi_{\parallel}$ as the directed percolation concentration p_d is approached. Consider the shortest paths

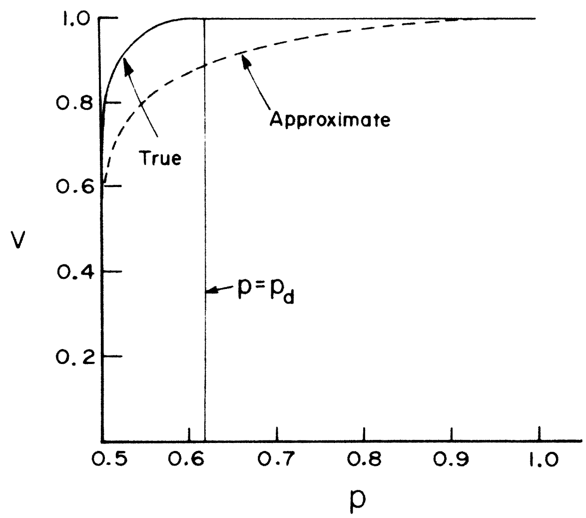

FIG. 4. The wetting velocity v is plotted versus p (solid curve). Results were obtained by evolving the full probability distribution. An approximate treatment of the probability distribution (discussed in Sec. IV) leads to an estimate (dashed curve) which misses the transition at p_d.

between two faces a distance R apart, when p is just below p_d and R is much greater than $\xi_{\parallel}$. We imagine that the shortest path will have many directed segments of mean length $\xi_{\parallel}$ and that these segments are connected by bridges of mean length b. An estimate of v is then $\sim\xi_{\parallel}/(\xi_{\parallel}+b)$, which leads to

$$1-v\sim b/\xi_{\parallel}\sim b(p_d-p)^{\nu_{\parallel}} . \tag{15}$$

Assuming that b remains finite as $p\rightarrow p_d$, we may read off $\theta'=\nu_{\parallel}$ from Eqs. (14) and (15).

Now, on the Wheatstone-bridge hierarchical lattice, "directed" ($v=1$) paths are just those which involve only the outer bonds 1,2,3,4 (Fig. 1) of the diamond at every stage. From the recursion

$$p'_{n+1}=(2p'_n)^2-(p'_n)^4 ,$$

which describes probabilities of directed connections, it is straightforward to find

$$\nu_{\parallel}=\ln 2/\ln(\partial p'_{n+1}/\partial p'_n)\,|_{p_d}\simeq 1.63 .$$

On the other hand, our numerical data for v versus p yields an estimate $\theta'\approx 1.73$ with uncertain but large error bars. We do not regard the difference as very serious; a possible reason for the discrepancy is as follows. On the Bethe lattice where one can compute v exactly,[15] the behavior

$$(1-v)\sim(\xi_{\parallel}\ln\xi_{\parallel})^{-1} \text{ as } p\rightarrow p_d^- \tag{16}$$

was found.[25] This differs from Eq. (15) by a multiplicative logarithm. If this is a general feature, present on all lattices, it would account for the larger effective exponent observed on the hierarchical lattice.

In concluding this section, we wish to emphasize that the phase transition at $p=p_d$ is obtained in the present calculation, from a study of the asymptotic form of the probability distribution of the shortest connecting paths. The "percolation recursion" Eq. (2) exhibits a smooth flow from $p=p_c=0.5$ to $p=1$, with no inkling of the anomaly at p_d. It is essential to track the evolution of $P_n(l)$ in order to obtain the singularity of v at $p=p_d$. In this connection it is of interest to ask how an approximate treatment of $P_n(l)$ would fare. A commonly used approximation[6,16] (in this and related contexts) is to replace $P_n(l)$ (for $l\neq\infty$) by a δ function. The weight of the δ function is the probability of connection p_n and its location is fixed by matching first moments (excluding $l=\infty$). At $p=p_c$, this approximation yields[16] $d_s\simeq 1.16$, which is not too far from the correct value $\simeq 1.07$. However, it misses the second transition (at $p=p_d$) altogether. v, as predicted by this approximate theory, is plotted in Fig. 4. It varies smoothly from 0 to 1 as p changes from p_c to 1, in contrast to the correct behavior in which $v=1$ for all $p>p_d$.

V. THE REGION $p<p_c$

While customary discussions of the wetting velocity are confined to the region $p>p_c$, it should be noted that the evolution of the probability distribution can be followed, and v found, in the region $0<p<p_c$ as well. For the asymptotic form of the probability distribution, Eq. (12) is still valid, but p_n approaches 0 as $n\rightarrow\infty$, in contrast to the situation discussed in Sec. IV. After a few iterations, most of the weight is pumped out from finite l to $l=\infty$ (corresponding to no connections). Consequently, it is numerically more difficult to obtain reliable results for the finite-l distribution. We find that v increases smoothly from 0 to 1 as p decreases from p_c to 0. As p approaches 0, the demand that two distant nodes be connected singles out directed paths between the two, as these have the largest weight in the limit $p\rightarrow 0$.

VI. CONCLUSION

Our study of the probability distribution $P(l)$ of the lengths of shortest paths in percolation shows that it is sensitive to phase transitions at both the ordinary and directed percolation concentrations p_c and p_d. At $p=p_c$, the distribution approaches an invariant form under proper rescaling of arguments, while as p approaches p_d the wetting velocity characteristic of the distribution approaches its saturation value $v=1$. Our study underscores the need to keep track of the full distribution at every iteration. Approximations which replace the distribution by a δ function which preserves only low-order moments are apt to miss the phase transition at $p=p_d$ altogether. Moreover, such approximations do not yield the exact value of critical exponents near and at p_c. This comment applies equally to approximations for other physical properties such as the conductivity of a percolating network; so far the critical behavior of the conductivity of a dilute hierarchical lattice has been determined only approximately.[26] Finally, while the present study of the shortest path length distribution has been made only for percolation on a hierarchical lattice, we expect that the corresponding distribution would sense the phase transitions at p_c and p_d in percolation on regular lattices as well.

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