Strongly disordered superfluids: Quantum fluctuations and critical behavior

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Superfluidity may be destroyed by an increasing amount of disorder. The physical systems that may be candidates for this behavior are the superfluid transition of ^{4}He in Vycor and the destruction of superconductivity in "dirty" electronic systems. We introduce a model for this phenomenon which may be mapped onto a quantum spin-$\frac{1}{2}$ model with a random field in the z direction. We study the breakdown of mean-field theory and used a scaling argument to establish the existence of a zero-temperature transition between the superfluid and normal state. We then study the critical behavior of this transition by a mapping onto a $(d+1)$-dimensional field theory with correlated disorder. The critical behavior of the transition temperature and the superfluid density near the transition point are discussed and scaling relations derived.

I. INTRODUCTION

The effects of strong disorder on superfluidity have recently been investigated on two experimental fronts. The first concerns the superconductivity transition in "dirty" electronic systems[1-4] and the second superfluid transition of ^{4}He in Vycor.[5] Although the motivations for studying these two systems may differ, an important aspect that is common to them is the existence of localized single-particle states when the disorder is sufficiently strong.

In the case of disordered electronic systems, the localized states manifest themselves in the form of a metal-to-insulator transition. Experimentally, it is found that in several materials,[1-4] superconductivity persists right up to the vicinity of the metal-insulator transition itself. For the disordered boson problem, the evidence for localized states is not as direct. However, a common view seems to be that the localized states first become occupied, but Bose condensation is not possible until the chemical potential coincides with the first extended state.[6] This view was further elaborated upon by Hertz, Fleishman, and Anderson,[7] who included the effect of repulsion between the Bose particles in a Hartree-Fock scheme.

Recently, several groups have investigated the mean-field theory for the superconductivity problem in strong disorder.[8-10] In particular, Ma and Lee concluded that within this theory, superconductivity with a spatially uniform order parameter can persist into the localized side of the metal-insulator transition, if the localization length ζ is sufficiently long. The criterion for this behavior is given by $\zeta^d N_0 \Delta > 1$, where N_0 is the single-particle density of states, d is the spatial dimensionality, and Δ_0 is the zero-temperature superconducting energy gap for pure systems. One may say that the "Anderson theorem",[11] which assumes BCS pairing between time-reversed single-particle states and obtains a uniform order parameter comparable to the value in the perfect crystal, remains valid in this weakly localized regime.

When the localization length is reduced further, the order parameter presumably begins to fluctuate strongly in space. However, the mean-field theory also begins to break down. Indeed, it was pointed out[8] that mean-field theory gave the rather unphysical answer that long-range order (LRO) persists for arbitrarily short localization length, even though the order parameter can become arbitrarily small as the localization length decreases.

One expects that if one goes beyond mean field, and in particular, if quantum fluctuations are included, the LRO will be destroyed. According to the analysis of the present paper, this is indeed the case and there is a critical localization length below which superconductivity cannot exist. On the other hand, there is no indication that the superconductivity boundary coincides with the mobility edge, so that our results do not support the view that Bose condensation takes place at the first extended state, in these systems.

In both the electron and boson system, two types of theoretical questions arise. First, one may ask a quantitative question, such as how much disorder is necessary to destroy the superfluid property at zero temperature, and one may attempt to compare this with the amount of disorder necessary for localization of the one-particle wave functions in the electronic case. Clearly, the precise answer to such questions will depend on the detailed form of the interaction among the particles, as well as the nature of the disorder in the system. A second set of questions involves quantities that are expected to be "univer-

sal," i.e., independent of most details of the model. Examples are critical exponents and dimensionless ratios of critical amplitudes, in the vicinity of the point at which the superfluidity disappears.

In the present paper, we shall discuss primarily the universal critical properties of the transition, although we shall also discuss somewhat the location of the superconducting transition in a simple model. The emphasis in our discussion will be the combined effects of randomness and of quantum fluctuations in the phase of the superconducting order parameter. These elements do not appear to have been combined in previous discussions, and the results of our analysis are therefore different, in several respects, from the earlier results.

Empirically, an onset of superfluidity was observed in ^{4}He adsorbed in porous Vycor glass as the coverage of ^{4}He on the glass surface was increased. The value of superfluid density at zero temperature $\rho_s(0)$ was found to vary roughly linearly with $n-n_c$ where n is the coverage and n_c the coverage at onset. This is just the result one would obtain if one said that atoms up to the critical coverage n_c are trapped in localized states, while the excess portion $(n-n_c)$ contributes to the superfluidity as if no disorder were present. For n close to n_c, the superfluid transition temperature T_c was observed to vary with coverage in such a way that

$$T_c(n) \propto [\rho_s(0)]^x , \tag{1.1}$$

with the exponent value $x=\frac{2}{3}$. This result also coincides with the theoretical behavior of a dilute Bose gas in three dimensions, in the absence of disorder. Sufficiently close to the superfluid transition temperature $T_c(n)$, for any fixed coverage n, the superfluid density $\rho_s(T)$ has been observed to vanish as $[(T_c-T)/T_c]^\zeta$, with the exponent $\zeta \approx 0.63$ which is close to the value $\zeta \approx 0.67$ observed for bulk ^{4}He in the vicinity of the λ point. The region of validity of this asymptotic behavior was found to shrink to zero, however, as $T_c \to 0$, and outside this critical region, one found empirically the behavior of a noninteracting Bose gas:

$$\frac{\rho_s(T)}{\rho_s(0)} = \left[1-\left[\frac{T}{T_c}\right]^{3/2}\right] . \tag{1.2}$$

(This leads to an effective exponent $\zeta=1$ close to T_c, but outside the asymptotic critical region.) Rasolt *et al.*[12] have shown that the superfluid density of a dilute interacting Bose gas exhibits a similar type of crossover from the ideal-gas behavior to the asymptotic critical behavior of a conventional superfluid, and that the experimental curves are very well fitted to a renormalization-group calculation for the dilute Bose gas without disorder.

In contrast to these previous analyses, the results of the present paper do *not* coincide precisely with the properties of a simple dilute Bose gas. We find, rather, that the critical behavior must be evaluated from a more complicated model, where disorder plays an important role. One possibility is that the experimental agreement with the dilute Bose gas model is fortuitous. Alternatively, it is possible that there exists an intermediate regime where dilute Bose gas behavior is obtained, but that extremely close to n_c, crossover to our disorder-dominated behavior should occur.

The effect of quantum fluctuations on the superconducting transition was considered, some years ago, by Doniach.[13] In his paper, he had in mind a system of small superconducting grains, connected to each other by Josephson junctions. Doniach found that quantum fluctuations in the superconducting phase become large enough to destroy superconductivity if the energy to add or subtract one Cooper pair from a grain is larger, by a certain factor, than the Josephson coupling energy between grains. In Doniach's analysis, the random aspects of the problem were ignored. The model he considered could then be described loosely as a regular lattice of quantum spins, in d-space dimensions, with interactions that are invariant under uniform rotations about the z axis in spin space. The critical behavior of this model, at $T=0$, could be mapped, in turn, onto the behavior of a classical two-component ϕ^4 field theory in $d+1$ dimensions, with the extra dimension introduced to account for fluctuations along the imaginary-time axis of the quantum-mechanical problem.

In the present paper, we again map the quantum-mechanical problem onto a classical ϕ^4 field theory in $d+1$ dimensions. The quenched randomness of the original system, however, gives rise to a random spatial variation in the coefficient of the ϕ^2 term in the weight function, which is equivalent to a "random T_c" in the problem. It is important to emphasize that the random coefficients do not depend on the time coordinate, so we are dealing with randomness that is completely correlated in one of the $d+1$ dimensions. Although such models have been studied previously by renormalization-group methods based on dimensional expansions,[14–16] direct extrapolation of the published results to the dimensionalities of physical interest does not give useful answers. By combining these results with scaling laws, however, we have been able to draw some conclusions that may be meaningful. Our best theoretical estimate for the exponent x, in Eq. (1), is then,

$$x \approx \tfrac{13}{21} \approx 0.62 , \tag{1.3}$$

which is close to, but not identical with the value $x \approx \frac{2}{3}$ for the free Bose gas.

We may note that if the effects of disorder are neglected, our model can be reduced to the case considered by Doniach, in which case our analysis gives a value $x=\frac{1}{2}$, in Eq. (1), with additional corrections involving a power of $\ln[\rho_s(0)]$. Also, in that case, one finds that $\rho_s(0) \propto (n-n_c)$, again with corrections involving powers of $\ln(n-n_c)$.

The organization of this paper is as follows. In Sec. II we show that both the superconductivity and superfluid problem can be described by the same quantum spin problem: a ferromagnetic x-y spin-$\frac{1}{2}$ model with a random field in the z direction. We then present a renormalization-group argument which shows that in the limit of strong disorder, mean-field theory is invalid, and superfluid long-range order is destroyed. In Sec. III we map the quantum spin problem into a $(d+1)$-dimensional classical field theory and discuss its critical behavior. Some conclusions are drawn in Sec. IV.

II. THE MODEL AND THE QUESTION OF LONG-RANGE ORDER

In this section we introduce a Hamiltonian which is applicable to both the fermion localization superconductivity and the dirty Bose gas problem. We review how the mean-field solution always possesses LRO. We then show that this property is no longer maintained upon introduction of quantum fluctuations.

For the fermion localization-interaction problem, we assume the BCS-Gorkov Hamiltonian,

$$H=\sum_{i,\sigma}\varepsilon_i c_{i\sigma}^{\dagger}c_{i\sigma}-\lambda\sum_{i,j,k,l}\int\varphi_i^*(\mathbf{r})\varphi_j^*(\mathbf{r})\varphi_k(\mathbf{r})\varphi_l(\mathbf{r})d\mathbf{r}\times c_{i\uparrow}^{\dagger}c_{j\downarrow}^{\dagger}c_{k\downarrow}c_{l\uparrow}\ , \tag{2.1}$$

where the Latin subscripts denote eigenstate labels of the single-particle Hamiltonian, and ε_i's their eigenenergies measured from E_f. The model assumes an effective local attractive interaction between electrons, and so $\lambda>0$. The single-particle eigenstates, φ_i's, can be localized or extended, depending on whether ε_i is above or below the mobility edge. If we further restrict ourselves to only empty or doubly occupied states, we arrive at the spin ($\frac{1}{2}$) Hamiltonian

$$H_s'=\sum_i h_i S_{iz}-\sum_{i,j}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j\ . \tag{2.2}$$

In Eq. (2.2), $h_i=4\varepsilon_i$, $S_{iz}=\frac{1}{2}(n_{i\uparrow}+n_{i\downarrow}-1)$, $S_i^+=c_{i\uparrow}^{\dagger}c_{i\downarrow}$, $S_i^-=c_{i\downarrow}^{\dagger}c_{i\uparrow}$, and $J_{ij}=\lambda\int\varphi_i^2(r)\varphi_j^2(r)d^dr$. One can check that S_z, S^+, and S^- indeed satisfy spin-$\frac{1}{2}$ commutation relationships. As discussed in Ref. 8, the approximation of ignoring singly occupied states gets better as the localization length decreases. The fact that the second term in (2.2) has the Heisenberg form, proportional to $\mathbf{S}_i\cdot\mathbf{S}_j$, is an artifact of our model, which results from the point-contact interaction and the complete neglect of singly occupied states. More generally, there will be two different coupling constants for the terms $S_{iz}S_{jz}$ and $S_i^+S_j^-$. Since the Heisenberg symmetry is in any case broken by the first term in (2.2), this distinction is not important for us.

When the φ_i's are strongly localized, each spin is associated with a site position and the overlap between localized states are small. We then model the system on a perfect lattice with $J_{ij}=J$ if spins i,j are nearest neighbors, and $J_{ij}=0$. Otherwise, since the phonon-induced attraction operates only for frequency less than the Debye frequency ω_D, we restrict our attention to states with on-site energy within ω_D of the Fermi level. Thus we assume that the h_i's are distributed according to a probability distribution $P(h_i)$ which is more or less flat between $\pm h_0$ and we set h_0 to be approximately ω_D. A realistic estimate of J is very difficult, but we expect $J\ll h_0$ when the states are localized.

H_s can also be used to model a system of bosons in a disordered potential. Approximating the interactions between bosons by the hard-sphere condition, the Hamiltonian is

$$H=\sum_i(W_i-\mu)b_i^{\dagger}b_i+t\sum_{\langle i,j\rangle}b_i^{\dagger}b_j\ , \tag{2.3}$$

where the $b_i^{\dagger},b_i$'s are now bosonic creation and annihilation operators at site i. The hopping matrix element from site to site is denoted by the term t. The disorder is modeled by a random site energy W_i, and the lattice is taken to be regular. The hard-sphere condition restricts the eigenvalues of $b_i^{\dagger}b_i$ to be 0 or 1. It is simple then to see that (2.3) is identical to the following Hamiltonian:

$$H_s=\sum_i h_i S_{iz}-J\sum_{\langle i,j\rangle}(S_{ix}S_{jx}+S_{ij}S_{jy})\ , \tag{2.4}$$

where $h_i=2(W_i-\mu)$ and $J=t$. Equation (2.4) differs from (2.2) by the absence of the term $JS_{iz}S_{jz}$, which mainly serves to renormalize the distribution of h_i and is not essential to the physics. We shall focus our attention on (2.4) from now on. The chemical potential μ is varied in the experiment, and so $P(h_i)$ is not symmetric about 0. It will be seen that this is not important. What is important, rather, is that the spread in W_i be large, and that $P(0)\neq 0$.

The order parameter for superconductivity or superfluidity corresponds to an ordering of the spin in the x-y plane. We can see from Eq. (2.4) that this ordering is favored by J but opposed by the random field which tends to align spins in the z direction. Thus we expect an onset of superfluidity at some critical J/h_0. However, this is not the case if (2.4) is treated by mean-field theory, which is equivalent to treating the spins classically.[8] The difficulty is that there are sites with small h_i and it costs little energy to tilt the spins on these sites towards the x-y plane. The large x component can polarize the neighboring spins to gain an energy of order J^2/h_0, assuming that the random field on a typical neighbor is of order h_0. Thus sites with $h_i<J^2/h_0$ become nucleation centers for ferromagnetic ordering in the x-y plane. While the ordering decreases exponentially between these sites, the system nevertheless has long-range order. This feature of the mean-field theory makes it impossible to pursue the usual strategy of treating quantum fluctuation by expanding about the mean-field critical point. To proceed we must first establish the fact that the quantum Hamiltonian (2.4) indeed has a critical point.

The breakdown of the mean-field theory can best be understood by first focusing on two sites, site A with $|h_A|<J^2/h_0$ and site B with $|h_B|\approx h_0$. For concreteness, let us assume that $h_A>0$ and $h_B<0$. If we set $J=0$, the eigenstates are shown in Fig. 1(a). For $J\neq 0$, the quantum-mechanical solution of the two-site problem is shown schematically in Fig. 1(b). The ground state $|0\rangle$ consists mostly of the product of the down-spin state on site A and up-spin state on site B, denoted by $|A^{\downarrow},B^{\uparrow}\rangle$, with an admixture which can be treated by perturbation theory,

$$|0\rangle\cong|A^{\downarrow},B^{\uparrow}\rangle+\frac{J}{h_B}|A^{\uparrow},B^{\downarrow}\rangle\ . \tag{2.5}$$

The next level $|1\rangle$ is simply given by $|1\rangle=|A^{\uparrow},B^{\uparrow}\rangle$. The energy splitting between these two states is given by $h_A+2J^2/|h_B|$ which is of order J^2/h_0, the same order of magnitude as the original splitting h_A on site A in Fig. 1(a). These two doublets are split from another pair of doublets by a large energy h_B. If we had chosen both h_A

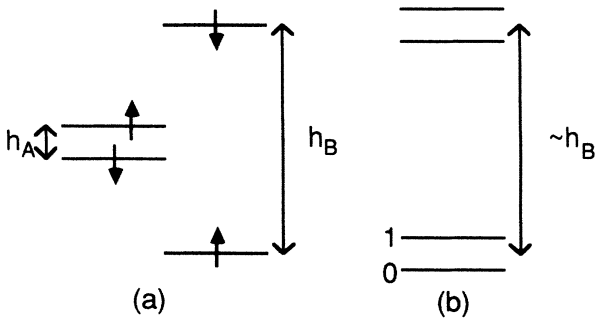

FIG. 1. (a) Level splittings between the spin-up and spin-down states on sites A and B for $J=0$, $h_A>0$, $h_B<0$. (b) Energy levels for $J \ll h_0$. Notice how the four states are split into two nearly degenerate pairs of doublets.

and h_B to be positive, the splitting of the doublet would have been $h_A - J^2/|h_B|$. While this may even be negative, its magnitude would still be of order J^2/h_0 and the basic doublet structure shown in Fig. 1(b) would remain unchanged.

From perturbation theory we see that the energy gained upon coupling the two sites is of order $J^2/|h_B|$, which is the same as that for classical spins. However, we see that the ground state $|0\rangle$ is very different from the classical solution. In the classical solution, spin A lies practically in the x-y plane, which is the classical limit of a linear superposition of $|A^{\uparrow}\rangle$ and $|A^{\downarrow}\rangle$, whereas we see from Eq. (2.5) that the ground state is mostly $|A^{\downarrow}\rangle$ with a small admixture of $|A^{\uparrow}\rangle$. At the same time, we have another state $|1\rangle$ which is nearly degenerate with it. Thus we see that mean-field theory does not reproduce the wave function of the quantum-mechanical ground state.

We can extend the above treatment of the two-site problem to the lattice by making the following renormalization-group argument. We first concentrate on spins with $|h_i| < J^2/h_0$. Let us pick one such spin and treat the coupling with its neighboring spins by perturbation theory. It is clear that the result is very similar to the two-site problem in that the low-lying states form a doublet split by J^2/h_0. The doublet is separated from the next excited states by a gap of order h_0. It is a good approximation to keep only the doublet and consider them to be renormalized spin-$\frac{1}{2}$ states.

Now consider the coupling between two such renormalized spins, at, say, sites i and j. The coupling involves going to at least nth-order perturbation in the J term, with n being the smallest number of spins between i and j. Thus we can write the renormalized Hamiltonian as

$$H' = \sum_i h_i' S_{iz} - \sum_{\langle i,j \rangle} J'(S_{ix}S_{jx} + S_{iy}S_{jy}) , \tag{2.6}$$

where

$$h_0' \sim J^2/h_0 \tag{2.7}$$

and

$$J' \sim J(J/h_0)^n . \tag{2.8}$$

Actually, an S_zS_z term is also generated, even if it is absent initially, but will be ignored as being irrelevant to the LRO. If the dimension $d>1$, then n will basically be the average spacing between the original sites, i.e.,

$$n \sim (J^2/h_0^2)^{-1/d} . \tag{2.9}$$

The procedure can now be repeated to "integrate" out spins with field h_i' whose magnitude is greater than $(J')^2/h_0'$. From (2.8) and (2.9) we see that J/h rapidly integrates to 0 for initial $J/h \ll 1$. As we perform the renormalization group (RG) we are probing the coupling between spins further away. The fact that this falls off so rapidly with the distance between them (number of iterations) indicates the lack of LRO. This is to be contrasted to the classical system, where the coupling is always "rejuvenated" by the presence of sites with small h_i in between.

Thus we conclude that the LRO found in mean-field theory in the $J/h \ll 1$ limit is an artifact of the mean-field approximation. Alternatively, we can also remark that a classical description will be closer to being correct if the distribution $P(h_i)$ has a gap around $h_i=0$. This will assure that the classical mean-field solution will not show the artificial LRO. An interpretation of our result is that if $J \ll h_0$, an effective gap is generated in the distribution $P(h_i)$ around $h_i=0$ upon renormalization. This observation will be important when we discuss a functional integral formulation of the problem in the following section.

Since the RG equations are obtained assuming $J/h_0 \ll 1$, they cannot give us the critical value of J/h_0 for LRO to occur. However, starting from the weak-disorder end, $J>h_0$, it can be shown that mean-field theory is valid and a superconducting state with a spatially uniform order parameter can be produced. According to Ref. 8, this occurs on the localized side of the mobility edge. This is because Anderson theorem[11] which states that one should pair time-reversed states, is valid if the localization length is sufficiently long. As J/h_0 is reduced, we must go over to the quantum spin regime. Thus we expect superconductivity to disappear at some critical J/h_0, which we expect to be of order unity.

III. CRITICAL PHENOMENA AND UNIVERSALITY CLASS

In Sec. II we saw how the mechanism of destruction of superconductivity and superfluidity within our model is not localization itself but enhanced quantum-mechanical fluctuations due to localization. We next discuss the critical behavior of the onset of superfluidity.

Instead of working with H_s, we will consider a modified model:

$$H = -\alpha \sum_i \left[\frac{d}{d\theta_i} - iA_i \right]^2 - \sum_{\langle i,j \rangle} J \cos(\theta_i - \theta_j) , \tag{3.1}$$

where A_i is a random variable to be specified later. The variables θ_i are defined on the interval $0 \le \theta \le 2\pi$, and we assume periodic boundary conditions for the kinetic-energy operators in (3.1). If $A_i=0$, Eq. (3.1) reduces to the model of a network of Josephson junctions studied by

Doniach.[13] In that case, J represents the Josephson coupling energy and $\alpha(d/d\theta_i)^2$ describes the charging energy due to charge fluctuations. When $\alpha \gg J$, the charging energy dominates and long-range phase coherence is destroyed. As we shall see, the introduction of A_i introduces some complications in the analysis.

In order to introduce the quantities A_i, we first set $J=0$, in which case the Hamiltonian decouples into a sum of independent terms on each site i,

$$H_i = -\alpha\left[\frac{d}{d\theta_i} - iA_i\right]^2 , \tag{3.2}$$

which has eigenfunctions $(2\pi)^{-1/2}e^{im\theta}$, and corresponding eigenvalues $\alpha(m-A_i)^2$. For $0 \le A_i \le 1$, the ground state is

$$\begin{aligned} &|0\rangle \equiv \frac{1}{\sqrt{2\pi}} \quad \text{for } A_i < \tfrac{1}{2} , \\ &|1\rangle \equiv \frac{1}{\sqrt{2\pi}} e^{i\theta} \quad \text{for } A_i > \tfrac{1}{2} , \\ &|0\rangle, |1\rangle \text{ degenerate for } A_i = \tfrac{1}{2} . \end{aligned} \tag{3.3}$$

Now consider the case with $J \neq 0$. If we ignore all states other than $m=0,1$, and also the problem of completeness, then (3.1) is equivalent to (2.4), with the identification

$$\alpha(A_i - \tfrac{1}{2}) = h_i . \tag{3.4}$$

Thus we wish to consider A_i as being random and given by a more or less flat distribution between 0 and 1, and $\alpha \sim h_0$. We recall that for an isolated site with $h_i=0$, doubly occupied and unoccupied states are degenerate in energy. In the preceding section, we saw how these sites generate an order parameter leading to LRO in the mean-field theory and that quantum fluctuations are required to destroy the LRO. In the quantum spin model a similar problem arises for sites with $A_i = \frac{1}{2}$, where the states $|0\rangle$ and $|1\rangle$ are degenerate. To get around this difficulty, we extend the model to include a probability distribution $P(A_i)$ which consists of a gap around $A_i = \frac{1}{2}$, so that sites with $|A_i - \frac{1}{2}| < G/2$ are not allowed. The original model corresponds to $G=0$. We will proceed with the analysis for $G \neq 0$ and then argue that the problems with $G \neq 0$ and $G=0$ belong to the same universality class, with the same critical exponents.

We now proceed to replace the quantum-mechanical Hamiltonian (3.1) by an effective classical field theory in $d+1$ dimensions. For the case $A_i=0$, this was done by Doniach.[13] In that problem, the classical mean-field solution corresponds to an extremum of the action in the $(d+1)$-dimensional field theory. In this model, a phase transition occurs even in the mean-field theory, and quantum effects can be treated by fluctuations about the extremum. For our present problem, $A_i \neq 0$, and if $G=0$, we cannot proceed in this way because there is no phase transition in the mean-field theory. We shall see that this difficulty shows up as divergences which we shall cut off by the introduction of a finite gap G in the probability distribution of A_i.

We consider the partition function

$$Z = \mathrm{Tr}\, e^{-\beta H} , \tag{3.5}$$

where $\beta = 1/kT$ and the trace is over the θ_i degrees of freedom. We are interested in the $T=0$ transition with changing J/α, and so $\beta \to \infty$. Equation (3.5) can be rewritten as

$$\begin{aligned} &Z = \mathrm{Tr}\left[e^{-\beta H_0} T_\tau \exp\left[-\int_0^\beta d\tau H_1(\tau)\right]\right] , \\ &H_0 = \textstyle\sum_i H_i , \; H_1(\tau) = e^{-H_0\tau} H_1 e^{H_0\tau} , \\ &H_1 = \sum_{\langle i,j\rangle} J\cos(\theta_i - \theta_j) , \end{aligned} \tag{3.6}$$

and T_τ is the "time-ordering" operator. Introducing a two-component classical vector field $\phi_i(\tau)$, we can perform a Hubbard-Stratonovich transformation[17,18] to obtain

$$Z = \int D\boldsymbol{\varphi}_i(\tau) \exp\left[-\int_0^\beta d\tau \sum_{i,j} (J^{-1})_{ij}\, \boldsymbol{\varphi}_i(\tau)\cdot\boldsymbol{\varphi}_j(\tau)\right] Z_1 , \tag{3.7}$$

where $(J^{-1})_{ij}$ is the matrix inverse of the coupling constants J_{ij},

$$Z_1 = \mathrm{Tr}\left[e^{-\beta H_0} T_\tau \exp\left[-\int_0^\beta d\tau \sum_i \boldsymbol{\varphi}_i(\tau)\cdot\mathbf{S}_i(\tau)\right]\right] \tag{3.8}$$

and $\mathbf{S}_i = (\cos\theta_i, \sin\theta_i)$.

The standard procedure is to evaluate Z_1 in a cumulant expansion to produce an effective free-energy functional in powers of $\boldsymbol{\phi}_i$. For our purpose, it is sufficient to carry this explicitly to second order in $\boldsymbol{\phi}$ so that

$$Z_1 \approx \exp\left(\sum_i S_2^i\right) , \tag{3.9}$$

where $S_2^i = S_{2x}^i + S_{2y}^i$ and

$$S_{2x}^i = \mathrm{Tr}\left[e^{-\beta H_i} T_\tau \left[\int_0^\beta d\tau \varphi_{ix}(\tau)\cos[\theta_i(\tau)]\right]^2\right] . \tag{3.10}$$

The trace can be evaluated by keeping only the ground state in the limit $\beta \to \infty$. For example, if $A_i < \frac{1}{2}$,

$$\begin{aligned} S_{2x} &= \int_0^\beta d\tau \int_0^\beta d\tau' T_\tau \langle 0|\cos[\theta_i(\tau)]|1\rangle \\ &\quad \times \langle 1|\cos[\theta_i(\tau')]|0\rangle \varphi_{ix}(\tau)\varphi_{ix}(\tau') \\ &= \int d\tau \int d\tau' e^{-\alpha|1-2A_i|\,|\tau-\tau'|} \varphi_{ix}(\tau)\varphi_{ix}(\tau') . \end{aligned} \tag{3.11}$$

We have ignored $|-1\rangle$ as a possible excited state. If $A_i > \frac{1}{2}$, then the role of $|1\rangle$ and $|0\rangle$ would be switched, but (3.11) would still hold true.

Combining (3.7), (3.8), (3.9), and (3.11), we obtain

$$Z = \int D\boldsymbol{\varphi}_i(\tau) e^{-S} , \tag{3.12}$$

where the free-energy function S is given by

$$\begin{aligned} S = &\int d\tau \sum_{i,j} (J^{-1})_{ij} \boldsymbol{\varphi}_i(\tau)\cdot\boldsymbol{\varphi}_j(\tau) \\ &- \tfrac{1}{2}\sum_i \int d\tau \int d\tau' e^{-\alpha|1-2A_i|\,|\tau-\tau'|} \boldsymbol{\varphi}_i(\tau)\cdot\boldsymbol{\varphi}_i(\tau') \\ &+ O((\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^2) . \end{aligned} \tag{3.13}$$

Writing $\varphi_i(\tau)$ in terms of its Fourier components,

$$\varphi_i(\tau)=\frac{1}{\sqrt{\beta}}\sum_{\omega}e^{i\omega\tau}\varphi_{i\omega}\ , \tag{3.14}$$

this becomes

$$S=\sum_{i,\omega,j}J_{ij}^{-1}\varphi_{i\omega}\cdot\varphi_{j,-\omega}+\tfrac{1}{2}\sum_{i,\omega}b_i(\omega)\varphi_{i\omega}\cdot\varphi_{i,-\omega}+\left[O\left[\frac{1}{\beta\alpha}\varphi^2\right]+O(\varphi^4)\right]\ , \tag{3.15}$$

where

$$b_i(\omega)=-m_i^{-1}(m_i^{-2}+\omega^2)^{-1} \tag{3.16}$$

and

$$m_i^{-1}=\alpha\,|\,1-2A_i\,|\ . \tag{3.17}$$

Now we can see that the sites with $A_i=\frac{1}{2}$ produces a divergent coefficient b_i in the $\omega=0$ limit. This simply expresses the fact that such sites have a strong tendency to nucleate long-range order in the mean-field approximation. Formally, this difficulty is avoided for the model with a gap in the distribution P, i.e., $G\neq 0$, in which case $m_i^{-1}\neq 0$ and we may expand

$$b_i(\omega)\approx -m_i+m_i^3\omega^2\ . \tag{3.18}$$

Since A_i is a random variable, so is m_i. Equations (3.12), (3.15), and (3.18) can be interpreted as a classical field theory in $d+1$ dimensions. The random nature of m_i implies a "random mass" or "random T_c" x-y model. But it is important to note that the randomness is correlated in the τ direction (m_i only depends on i).

The problem of a D-dimensional random T_c model with randomness correlated in ε_d dimensions has received considerable interest in recent years.[14-16] Although most studies deal with a nonrandom coefficient of the ω^2 term, this condition is not maintained under renormalization and we believe the presence of random coefficients in Eq. (3.18) do not introduce essential differences. To make the connection more transparent, let us rewrite (3.15) in a continuum form as

$$S=\int d\mathbf{r}\,d\tau\,[r(\mathbf{x})\phi^2(\mathbf{x},\tau)+A_\tau(\nabla_\tau\phi)^2+A_x(\nabla\phi)^2+O(\phi^4)]\ , \tag{3.19}$$

where x denotes the spatial coordinates and τ the imaginary-time coordinates. The recent advances in the understanding of the critical phenomena of this model are based on the realization that one should make a double expression in $\varepsilon=4-D$ and ε_d.[14-16] In the present case, $D=4$ (three spatial dimensions and one temperature dimension) and $\varepsilon_d=1$.

Although the ε-expansion results for the critical exponents are poorly convergent when setting $\varepsilon_d=1$,[15] one can nevertheless extract useful information by using scaling relations to relate different exponents to each other. The central modification from the usual scaling laws is that the anisotropy is relevant with the result that the τ correlation length ξ_t and the spatial correlation length ξ_x diverge with different exponents ν_t and ν_x.[14-16]

A quantity that is of interest in both the boson and electron systems is the superfluid density ρ_s. This is particularly so for the boson case where the order parameter is not a directly measurable quantity. In what follows we will relate the ρ_s exponent to other exponents.

We first identify ρ_s with respect to (3.19). ρ_s is defined, in the long-wavelength limit, as the coefficient of the $[\nabla_x\theta(r)]^2$ term in the energy change under the transformation $\phi(r,t)\rightarrow\phi(r,t)e^{i\theta(r)}$. [To be precise, the phase change should be performed on the order parameter in the ground state of (3.1), but $\langle\phi(r,t)\rangle$ is related to $\langle S(r)\rangle$ by a local linear transformation and the distinction vanishes in the long-wavelength limit.] Following the standard argument for the isotropic case,[17] we equate the fluctuation energy near T_c in a correlated volume Ω, given by $\int d^Dr\rho_s[\nabla_x\theta(r)]^2$, to the thermal energy kT_c (it must be remembered that the physical temperature is zero, and all "thermal" effects refer to the fictitious random T_c problem). Since ξ_x sets the scale for transverse fluctuations, we estimate that $\langle(\Delta\theta)^2\rangle\sim\xi_x^{-2}$. Considering the number of spins in a correlated volume $\sim\xi_t\varepsilon_d\xi_x^{D-\varepsilon_d}$, and using the fact that $D=d+\varepsilon_d$, we have

$$\rho_s\xi_t^{\varepsilon_d}\xi_x^{d-2}\sim kT_c\ . \tag{3.20}$$

Thus ρ_s scales like

$$\rho_s\sim\xi_t^{-\varepsilon_d}\xi_x^{2-d}\sim t^{(d-2)\nu_x+\varepsilon_d\nu_t}\ , \tag{3.21}$$

where $t\sim(T-T_c)/T_c$ in the fictitious thermal problem.

For the zero temperature quantum problem, we expect that t is linearly related to $(\mu-\mu_c)$, the difference between the chemical potential μ and the critical value μ_c for onset of superfluidity. We shall argue, below, that $\mu-\mu_c$ is in turn related linearly to the difference $n-n_c$ between the concentration n of He in the vycor and the onset concentration n_c. (In the superconductivity problem we may assume that t is linearly related to the alloy concentration or the degree of disorder.)

The exponents ν_x and ν_t have been calculated in a double expansion in $\varepsilon=4-D$ and ε_d. For an m-component order parameter in $D=4$ we have

$$\nu_x^{-1}=2-\frac{\varepsilon_d}{8}\frac{5m+4}{m-1}\ , \tag{3.22}$$

$$\nu_t^{-1}=2-\frac{\varepsilon_d}{4}\frac{3m+6}{m-1}\ , \tag{3.23}$$

so that the ratio z defined as ν_t/ν_x is given by

$$z=1+\frac{\varepsilon_d}{16}\frac{m-8}{m-1}\ . \tag{3.24}$$

In Eqs. (3.22) and (3.23), we have corrected some serious typographic errors in Ref. 15. Setting $m=2$, we have

$$\begin{aligned}\nu_x^{-1}&=2-\tfrac{7}{4}\varepsilon_d\ ,\\ \nu_t^{-1}&=2-3\varepsilon_d\ ,\\ z&=1+\tfrac{5}{8}\varepsilon_d\ .\end{aligned} \tag{3.25}$$

As we can see, it is not meaningful to set $\varepsilon_d=1$ for ν_t and ν_x. The expansion for z appears to be somewhat better behaved. In terms of z, we can rewrite Eq. (3.21) as

$$\rho_s = \left| \frac{\mu - \mu_c}{\mu_c} \right|^{(d-2+\epsilon_d z)\nu_t} . \tag{3.26}$$

As in the isotropic system, ρ_s can also be related to exponents other than ν.[17] For example, equating $\rho_s(\nabla_x\theta)^2$ to the singular part of the thermodynamic potential, $\rho_s \xi_x^{-2} \sim t^{2-\alpha}$, which implies

$$\rho_s \sim \left| \frac{\mu - \mu_c}{\mu_c} \right|^{2-\alpha-2\nu_x} .$$

[Incidentally, this gives an alternative derivation of the modified scaling law $\alpha = 2-(D-\epsilon_d)\nu_x - \epsilon_d \nu_t$.[15]] Unfortunately, unlike the normal Josephson's relation,[19] it is not possible to write ν_x in terms of the specific-heat exponent α alone. Also, since t is really $(\mu-\mu_c)/\mu_c$, α is not the exponent of the physical specific heat $\partial E/\partial T$.

The concentration n of helium in the vycor is given by the derivative of the thermodynamic potential with respect to the chemical potential μ. By definition, the singular part of this derivative is proportional to $(\mu-\mu_c)^{1-\alpha}$. According to the ϵ_d expansion $\alpha \approx -(7/4)\epsilon_d$, which is negative, consistent with expectations based on the Harris criterion. Therefore, the leading term in $(n-n_c)$ should be the non-singular term, proportional to $(\mu-\mu_c)$, as stated earlier.

We can also deduce how the physical transition temperature T_s goes to 0 as $n \to n_c$. Since finite temperature corresponds to a finite size $\beta = 1/T$ in the τ direction, we argue that the effect of finite temperature is felt when the correlation length equals the "sample size," i.e.,

$$\beta_s = 1/(kT_s) = \xi_t ,$$

and therefore

$$T_s \sim (\mu - \mu_c)^{\nu_t} . \tag{3.27}$$

This argument can be made more formally using the hypothesis of finite-size scaling, according to which[20]

$$\rho_s(t,\beta) = t^{(d-2)\nu_x + \epsilon_d \nu_t} f(\beta/\xi_t) , \tag{3.28}$$

where β plays the role of the finite size in the imaginary-time direction. Zero temperature corresponds to $\beta/\xi_t = \infty$, and in order to reproduce Eq. (3.21) $f(\infty)$ is a constant. At $T = T_s$, ρ_s vanishes which implies that $f(x_0) = 0$. Thus, we conclude that $(T_s \xi_t)^{-1} = x_0$ and Eq. (3.27) follows.

The above relationships are helpful in interpreting experimental data and suggesting meaningful measurements. First, ν_x and ν_t are, in principle, measurable in superconductors. The correlation function $\langle \Delta(r,t)\Delta^*(r',t') \rangle$ in the normal state can be extracted by measuring the tunneling current in a junction formed with a regular superconductor.[21] For $t = t'$, we expect

$$\langle \Delta(r,t)\Delta^*(r',t) \rangle \sim e^{-|r-r'|/\xi_x} . \tag{3.29}$$

The behavior for $t \neq t'$ will be more complicated, but there should be a characteristic time scale given by ξ_t. Thus the exponents ν_x and ν_t can in principle be determined and compared with the behavior of ρ_s and T_s given by Eq. (3.21) and (3.27).

In the case of dirty bosons, the order parameter or its correlation functions are not directly measurable. But it is still possible to use the T_s exponent and use ϵ_d-expansion value of z to calculate ρ_s exponent. Indeed, from Eq. (3.21) and (3.27) we see that

$$\ln\rho_s / \ln T_s = (d-2)/z + \epsilon_d . \tag{3.30}$$

For three-dimensional space and setting $\epsilon_d = 1$, we find

$$\ln\rho_s / \ln T_s = 1 + 1/z . \tag{3.31}$$

If we substitute the ϵ_d expansion result for z and set $\epsilon_d = 1$, we obtain $\ln\rho_s/\ln T_s = \frac{21}{13} \approx 1.62$.

Finally, we recall that the mapping onto a classical field theory is permissible only in the model with a gap G in the distribution of random site energies, whereas the physical model we wish to solve does not have such a gap. We believe that the model with or without a gap belongs to the same universality class, i.e., no new phase boundaries appear as G is continuously varied. We cite in support of this belief the discussion in Sec. II, which shows that if $J/h_0 \ll 1$, a gap is spontaneously generated in $P(h)$ upon renormalization. It is clear that anomalies associated with sites with h_i near zero are artifacts of the mean-field approximation. Once this is understood, it seems reasonable to assume that the model considered in this section adequately describes the critical behavior of the original spin problem.

IV. CONCLUSION AND DISCUSSION

In this paper we have considered the questions of superconductivity of electrons, or superfluidity of bosons in a strongly disordered system. With the assumptions that electron states are either doubly occupied or unoccupied and that interactions between bosons are modeled by the hard-sphere condition, we found that both systems can be described by the same Hamiltonian. A feature common to both of them is the existence of localized states. However, the localized states do not destroy superfluidity in the sense that a uniform mean-field solution exists near the mobility edge.

Inclusion of quantum-mechanical fluctuations will destabilize the classical ground state and destroy LRO if the localization length is short enough. We conclude that within this model, the destruction of superconductivity occurs below the mobility edge.

The applicability of our results to dirty superconductors is subject to the following qualifications. It is well known that the interplay between localization and electron-electron interactions can be very complicated in nonsuperconducting metal-insulator transitions,[22] and it is highly possible that electron-electron interactions play a major role in the destruction of superconductivity also.[23-25] For example, Fukuyama[24] has suggested that electronic spin fluctuations may play a crucial role in the suppression of superconductivity near the localization transition of a disordered electron system. The superconductivity model of the present paper assumes an attractive interaction between electrons on the same lattice site, so that singly occupied electronic states, with the attendant spin degree of freedom, do not occur. It seems likely to us, however,

that as long as there is no ordering in the spin degrees of freedom, critical exponents would be unaffected by spin fluctuations, even if such fluctuations have an important effect on the position of the onset of superconductivity.

In the Bose system, the one experimental system that our model may apply to thus far is ^{4}He in Vycor.[5] Although the Vycor certainly provides a disordered potential, the complicated geometry there casts in doubt the model of on-site disorder with well-behaved distribution. In addition, there may be complications associated with a crossover from two to three dimensions.[5,26]

Assuming that the model is adequate to describe the phase transition, we have argued that the transition should be in the same universality class as a $(d+1)$-dimensional random-T_c x-y model with the impurities being correlated in the imaginary-time direction. This anisotropy leads the spatial correlations length and the "temperature" correlation length to diverge with different exponents ν_x, ν_t. By scaling arguments, the critical exponents of the zero temperature superfluid density ρ_s and the transition temperature T_s can be related to ν_x and ν_t and, hence, also to each other. Experimental determination of these exponents will certainly be of interest.

In this paper we have addressed the critical phenomena in $n-n_c$ at $T=0$. The usual critical behavior at $n-n_c\neq 0$ and $T=T_s$ is, of course, also of interest. Far away from $n=n_c$, the critical behavior is that of the usual random-T_c d-dimensional x-y model. For superconductivity, it has, in fact, been suggested that these exponents will be observable instead of the mean-field exponents because of the decrease in the coherence length.[10] In our picture, this critical behavior crosses over to a new critical behavior of the quantum fluctuation problem. It has been suggested that the critical fluctuations crosses over to the Gaussian behavior characteristic of a weakly interacting dilute Bose gas.[5,12] However, the effect of disorder was ignored in these discussions and it is not clear to us why ^{4}He on Vycor glass can be described as a dilute Bose gas. Nevertheless, we cannot rule out the possibility of an intermediate Gaussian regime before ultimately crossing over to our quantum fluctuation regime. So far, the experimental measurement[5] of the relation between ρ_s and T_s is consistent with both the dilute Bose-gas model and the present model. Further, experimental studies of the zero-temperature quantum transition in both superfluid and superconductors will certainly be of great interest.

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