# Thermodynamic properties of an interacting two-dimensional electron gas in a strong magnetic field

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We have studied the combined effects of interaction and disorder on the entropy, the specific heat, and the magnetization of a two-dimensional electron gas in a strong perpendicular magnetic field. Explicit results have been obtained within the Hartree-Fock approximation which reflects only those anomalies associated with the energy gap between Landau levels. Some qualitative remarks, based on the hierarchy picture, are made concerning features expected in these thermodynamic properties at fractional values of the Landau-level filling factor.

## I. INTRODUCTION

The two-dimensional electron gas in a strong magnetic field, $B$, exhibits a number of interesting many-body effects including the quantum Hall effect (QHE).[1-2] The unique feature of this system responsible for its unusual behavior is the quantization of the energy spectrum, in the absence of interaction and disorder, into sharp Landau levels. Many properties of these systems are controlled by the degree to which the Landau levels have been broadened by disorder and by the energy gap which remains between the Landau levels. In principle, at least, important information about the structure of the energy spectrum can be extracted from measurements of thermodynamic properties such as the entropy, the heat capacity, and the magnetization. This possibility has attracted both experimental[3-7] and theoretical[3-9] interest. Studies to date have been carried out in the limit where the effect of disorder dominates the effect of Coulomb interactions between the electrons. In this limit, the theoretical results do account qualitatively for the measured magnetization and specific heat. Provided the electronic contribution dominates, the thermopower[10-13] provides a measure of the entropy and again qualitative agreement with theory[8,9,14] seems to exist.

Our purpose in this paper is to make predictions regarding the influence of Coulomb interactions on these equilibrium properties in the limit where many-body effects become significant. Our detailed results are based on a semiphenomenological self-consistent Born-approximation treatment of disorder scattering and a Hartree-Fock approximation (HFA) for interaction effects which is detailed in Sec. II. We discuss the entropy, the chemical potential, the heat capacity, and the magnetization resulting from this approximation in Sec. III. The use of the HFA to treat interaction effects can be justified only when the Landau-level filling factor, $\nu$, is near an integer [$\nu \equiv 2\pi l^2 n$, where $n$ is the areal density of electrons and $l \equiv (\hbar c/eB)^{1/2}$]. Thus in using the HFA we focus on the features in the thermodynamic properties which, like the integer QHE, are associated with integral values of $\nu$. The HFA completely misses the features in these thermodynamic properties which are expected near fractional values of $\nu$ and are associated with the fractional QHE. In Sec. IV we discuss the behavior of thermodynamic properties at fractional values of $\nu$. Our comments are based on the correspondence between the excitations of the incompressible fluid states occurring at fractional values of $\nu$ and those occurring for integral values of $\nu$ which are well described by the HFA. We close in Sec. V with a brief summary of our results.

## II. HARTREE-FOCK APPROXIMATION WITH A DISORDER POTENTIAL

The results to be presented in Sec. III are based on the temperature-dependent or thermal Hartree-Fock approximation (THFA). This approximation was originally reached by a variational approach[15,16] but can also be obtained from a decoupling approximation to the equation of motion for the thermal Green's function[17] and by summing a subset of Feynman diagrams in the finite-temperature perturbation theory.[18] We use the language of the last approach here.[19] The Dyson equation for the thermal Green's function may be written in the form

$$G_\sigma(n,X;n',X';i\omega_n)=G_\sigma^{(0)}(n,X;n',X';i\omega_n)+\sum_{m,Y}\sum_{m',Y'}G_\sigma^{(0)}(n,X;m,Y;i\omega_n)\Sigma_\sigma(m,Y;m',Y';i\omega_n)G_\sigma(m',Y';n',X';i\omega_n)\ , \tag{1}$$

where we have used a representation in terms of Landau-gauge kinetic-energy eigenstates for the one-particle Green's function:

$$G_\sigma(n,X;n',X';i\omega_n) = \int d\mathbf{x}\int d\mathbf{x}'\,\phi^*_{n,X}(\mathbf{x})G_\sigma(\mathbf{x},\mathbf{x}';i\omega_n)\phi_{n',X'}(\mathbf{x}')\ , \tag{2a}$$

$$G_\sigma(\mathbf{x},\mathbf{x}';i\omega_n) = \sum_{n,X}\sum_{n',X'}\phi_{n,X}(\mathbf{x})G_\sigma(n,X;n',X';i\omega_n)\phi^*_{n',X'}(\mathbf{x}')\ . \tag{2b}$$

In Eqs. (2),

$$\phi_{n,X}(\mathbf{x})=L_y^{-1/2}\exp(iXL_y/l^2)\psi_n(x-X),\quad n=0,1,2,\ldots \tag{3}$$

$$\frac{XL_y}{2\pi l^2}=0,1,2,\ldots,N_L$$

is a Landau-gauge kinetic-energy eigenstate with eigenvalue $\hbar\omega_c(n+\frac{1}{2})$ ($\omega_c=eB/m^*c$), $\psi_n(x)$ is a one-dimensional harmonic-oscillator eigenstate and $N_L=L_xL_y/2\pi l^2$ is the number of states per Landau level. Thus, in the absence of a disorder potential,

$$G_\sigma^{(0)}(n,X;n',X';i\omega_n)=\delta_{n,n'}\delta_{X,X'}[i\omega_n-\omega_c(n+\tfrac{1}{2})-g\mu_B\sigma B/2\hbar+\mu\hbar^{-1}]^{-1}\,. \tag{4}$$

(Note that we have included the Zeeman contribution to the energy.)

It will be convenient below to introduce the spectral weight of the Green's function, defining

$$\rho_\sigma(n,X;n',X';\omega)=\pi^{-1}\,\mathrm{Im}[G_\sigma(n,X;n',X';\omega+\mu\hbar^{-1}-i\eta)] \tag{5a}$$

so that

$$G_\sigma(n,X;n',X';i\omega_n)=\int_{-\infty}^{\infty}d\omega'\frac{\rho_\sigma(n,X;n',X';\omega')}{i\omega_n-\omega'+\mu\hbar^{-1}}\,, \tag{5b}$$

and the energy is

$$E=\sum_{\substack{n,X,\\ n',X'}}\sum_{\sigma}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\{\hbar[\omega'+\omega_c(n+\tfrac{1}{2}+g^*\sigma/4)]\times\delta_{n,n'}\delta_{X,X'}+\langle n'X\mid V_D\mid n,X\rangle\}\times\rho_\sigma(n,X;n',X';\omega')n_F(\hbar\omega)\,. \tag{5c}$$

In Eq. (5c) we have introduced an effective $g$ factor, $g^*=2g\mu_B/\hbar\omega_c=gm^*/m$, and $\langle n,X\mid V_D\mid n',X'\rangle$ is the one-particle matrix of the disorder potential. [$n_F(\varepsilon)=(e^{(\varepsilon-\mu)/k_BT}+1)^{-1}$ is the Fermi distribution function.] Similarly the entropy is given by

$$S=-k_B\int d\omega\{n_F(\hbar\omega)\ln[n_F(\hbar\omega)]+[1-n_F(\hbar\omega)]\ln[1-n_F(\hbar\omega)]\}\rho(\omega)\,, \tag{6}$$

where

$$\rho(\omega)=\sum_{n,X,\sigma}\rho_\sigma(n,X;n,X;\omega)\,. \tag{7}$$

It should be noted that while Eq. (5c) is a general relationship between the energy and the thermal Green's function, Eq. (6) holds only within the THFA.[20]

The approximation we use for the self-energy is

$$\Sigma_\sigma(n,X;n',X';i\omega_n)=\Sigma_\sigma^{\mathrm{HF}}(n,X;n',X')+\Sigma_\sigma^{\mathrm{SCBA}}(n,X;n',X';i\omega_n)\,, \tag{8}$$

where

$$\Sigma_\sigma^{\mathrm{HF}}(n,X;n',X')=\sum_{\sigma'}\sum_{m,Y}\sum_{m',Y'}\int d\omega\, n_F(\hbar\omega)\rho'_\sigma(m',Y';m,Y;\omega)\times(\langle n,X;m,Y\mid V_c\mid n',X';m',Y'\rangle-\delta_{\sigma',\sigma}\langle n,X;m,Y\mid V_c\mid m',Y';n',X'\rangle) \tag{9}$$

and

$$\Sigma_\sigma^{\mathrm{SCBA}}(n,X;n',X;i\omega_n)=\sum_{m,Y}\sum_{m',Y'}\int d\omega\frac{\rho_\sigma(m',Y';m,Y;\omega)}{i\omega_n-\omega}\langle\langle n,X\mid V_D\mid m',Y'\rangle\langle m,Y\mid V_D\mid n',X'\rangle\rangle\,. \tag{10}$$

This approximation is illustrated graphically in Fig. 1. The expression for $\Sigma_\sigma^{\mathrm{HF}}(n,X;n',X')$ given in Eq. (9) is the usual one, but rewritten using the definitions of the preceding paragraph. $\langle n,X;m,Y\mid V_c\mid n',X';m',Y'\rangle$ is a two-particle matrix element for the Coulomb interaction, $V_C(\mathbf{r}-\mathbf{r}')=e^2/\epsilon\,|\,\mathbf{r}-\mathbf{r}'\,|$,

$$\langle n,X;m,Y\mid V\mid m',Y';n',X'\rangle=\frac{e^2}{\epsilon}\int\frac{d\mathbf{q}}{(2\pi)^2}\frac{2\pi}{q}F_{n,m'}(\mathbf{q})F_{m,n'}(-\mathbf{q})\times\delta_{X,Y'+l^2q_y}\delta_{X',Y+l^2q_y}\exp[iq_x(X-X')]\,, \tag{11}$$

where we have used that

$$\langle n',X'\mid\exp(i\mathbf{q}\cdot\mathbf{r})\mid n,X\rangle=F_{n',n}(\mathbf{q})\delta_{X',X+q_yl^2}\exp\left[\frac{iq_x}{2}(X+X')\right] \tag{12}$$

and

$$F_{n',n}(\mathbf{q})=\left[\frac{n!}{n'!}\right]^{1/2}\left[\frac{-q_yl+iq_xl}{2}\right]^{n'-n}\times\exp\left[\frac{-q^2l^2}{4}\right]L_n^{n'-n}\left[\frac{q^2l^2}{2}\right]\,. \tag{13}$$

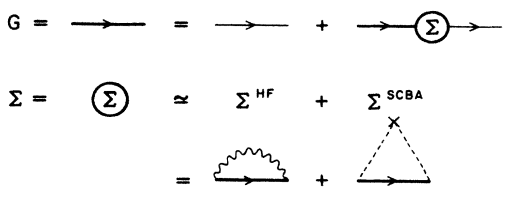


FIG. 1. Dyson equation for the Green's function showing the interaction and disorder contributions to the self-energy.

Similarly Eq. (10) expresses the usual self-consistent Born-approximation treatment of disorder scattering. We expand the disorder potential appearing in Eq. (10) as

$$V_D(\mathbf{r})=\frac{1}{L_xL_y}\sum_{\mathbf{q}}\sum_{\mathbf{R}}V_0(\mathbf{q})\exp[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{R})]\ , \tag{14}$$

where the sum over $\mathbf{R}$ is over scattering centers. The outer angle bracket in Eq. (10) represents an average over scattering centers.

Assuming a random distribution and using Eq. (12) gives

$$\langle\langle n,X\mid V_0\mid m',Y'\rangle\langle m,Y\mid V_0\mid n',X'\rangle\rangle$$
$$=n_I\int\frac{d\mathbf{q}}{(2\pi)^2}\mid V_0(\mathbf{q})\mid^2F_{n,m'}(\mathbf{q})F_{m,n'}(-\mathbf{q})$$
$$\times\delta_{X,Y'+l^2q_y}\delta_{X',Y+l^2q_y}\exp[iq_x(X-X)]\ , \tag{15}$$

where $n_I$ is the areal density of scattering centers. With both the disorder potential and the interaction potential treated perturbatively Eq. (4) gives the bare propagator for the Dyson equation. Explicitly iterating the Dyson equation and using Eqs. (8)–(11) and Eq. (15) we see that

$$G_\sigma(n,X;n',X';i\omega_n)=\delta_{n,n'}\delta_{X,X'}G_\sigma(n,i\omega_n)\ , \tag{16a}$$

where

$$G_\sigma(n,i\omega_n)=\left[i\omega_n-\hbar^{-1}(\varepsilon_{n,\sigma}-\mu)-(2\hbar)^{-2}\sum_m\Gamma^2_{n,m}G_\sigma(m,i\omega_n)\right]^{-1}, \tag{16b}$$

$$\varepsilon_{n\sigma}=\hbar\omega_c(n+\tfrac{1}{2}+g^*\sigma/4)-\frac{e^2}{\epsilon l}\sum_m\int d\omega\,\rho_\sigma(m,\omega)n_F(\hbar\omega)X_{n,m}\ , \tag{16c}$$

and

$$\Gamma^2_{n,m}=4n_I\int\frac{d\mathbf{q}}{(2\pi)^2}\mid V_0(\mathbf{q})\mid^2\mid F_{n,m}(\mathbf{q})\mid^2. \tag{16d}$$

Equation (16a) simply expresses the spatial homogeneity of the configuration averaged system which reduces the Dyson equation to an algebraic form leading to Eq. (16b). The interaction term in Eq. (16c) comes from evaluating Eq. (9) with $\rho_\sigma\propto\delta_{Y',Y}\delta_{m',m}$ in which case the direct term gives no contribution if we take $V_c(\mathbf{q}=\mathbf{0})=0$ to account for a neutralizing background. Inserting Eqs. (11) and (13) into Eq. (9) and comparing with Eq. (16c) we see that

$$X_{n,m}=X_{m,n}=\int_0^\infty d(ql)\exp\left[\frac{-q^2l^2}{2}\right]\left[\frac{q^2l^2}{2}\right]^{n-m}\times[L_m^{n-m}(q^2l^2/2)]^2$$
$$=\frac{\Gamma(n-m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(2\pi)^{1/2}n!(n-m)!}\times{}_3F_2\left[\begin{matrix}-m,n-m+\frac{1}{2},\frac{1}{2};1\\ n-m+1,-m+\frac{1}{2}\end{matrix}\right], \tag{17}$$

where the final form is for $n\geq m$ and ${}_3F_2$ is a generalized hypergeometric function. We have used the result of Glasser and Horing[21] for the integral. The exchange term reflects the Pauli exclusion principle by lowering $\varepsilon_{n,\sigma}$ when states of spin $\sigma$ are occupied and $X_{n,m}$ measures the amount by which $\varepsilon_{n,\sigma}$ is lowered as a result of the $m$th Landau level being occupied. Values of $X_{n,m}$ for some of the lower Landau levels are listed in Table I, from which we see that $X_{n,m}$ tends to the largest when $n=m$.

TABLE I. Values of $X_{n,m}\sqrt{2/\pi}$ for low-lying Landau levels. The Hartree-Fock energy for the $n$th Landau level is lowered by an amount proportional to $X_{n,m}$ as the $m$th Landau level of the same spin is occupied.

| | $m$ | | | |
|---|---|---|---|---|
| $n$ | 0 | 1 | 2 | 3 |
| 0 | 1 | $\frac{1}{2}$ | $\frac{3}{8}$ | $\frac{5}{16}$ |
| 1 | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{7}{16}$ | $\frac{11}{32}$ |
| 2 | $\frac{3}{8}$ | $\frac{7}{16}$ | $\frac{41}{64}$ | $\frac{51}{128}$ |
| 3 | $\frac{5}{16}$ | $\frac{11}{32}$ | $\frac{51}{128}$ | $\frac{147}{256}$ |

For a given model of the scattering potential Eqs. (16b) and (16c) can be solved simultaneously using an iterative procedure. It is instructive however, to consider the strong field limit where Landau-level mixing can be ignored. In that case the terms with $m\neq n$ in the sum over $m$ on the right-hand side of Eq. (16b) can be dropped and the equation can be solved to give[22]

$$\rho_\sigma(n,\omega)=\pi^{-1}\,\mathrm{Im}[G_\sigma(n,\omega-i\eta)]=\begin{cases}\dfrac{2\hbar}{\pi\Gamma_{n,n}}\left[1-\left[\dfrac{\hbar\omega-\varepsilon_{n\sigma}}{\Gamma_{n,n}}\right]^2\right]^{1/2}, & \left|\dfrac{\hbar\omega-\varepsilon_{n\sigma}}{\Gamma_{n,n}}\right|<1\\ 0, \text{ otherwise}\ .\end{cases} \tag{18}$$

Experimentally,[6,7,23,24] it has been found that the broadened Landau levels do not have the sharp edges implied by Eq. (18) and that results can be fitted more successfully with the form

$$\rho_\sigma(n,\omega)=\left[\frac{2}{\pi}\right]^{1/2}\frac{\hbar}{\Gamma_{n,n}}\exp\left[-2\left[\frac{\hbar\omega-\varepsilon_{n\sigma}}{\Gamma_{n,n}}\right]^2\right], \tag{19}$$

where $\Gamma_{n,n}$ is typically $\sim 2$ meV and usually seems to have $\sqrt{B}$ dependence on magnetic field. The $\sqrt{B}$ dependence follows from Eqs. (18) and (16c) for the case of short-range scatterers [$V_0(\mathbf{q})=V_0$] since in that case[25]

$$\Gamma^2_{n,m}=4n_I\mid V_0\mid^2\int\frac{d\mathbf{q}}{(2\pi)^2}\mid F_{n,m}(\mathbf{q})\mid^2$$
$$=4n_I\mid V_0\mid^2(2\pi l^2)^{-1}\ . \tag{20}$$

The discrepancy between Eq. (18) and experiments must reflect, in part, limitations of the self-consistent Born approximation but also probably results from the importance of Landau-level mixing and the presence of large scale inhomogeneities in the experiments. In order to make our calculations more relevant to current experiments we have chosen to use Eq. (19) for $\rho_\sigma(n,\omega)$ and solve Eq. (16c) iteratively for the Hartree-Fock energies. Results for the thermodynamic properties following from this approximation are discussed in the following section.

In closing this section we briefly describe the procedure used to solve Eq. (16c) numerically. All our calculations were performed at fixed

$$N = N_L \sum_{n,\sigma} \int d\omega \, n_F(\hbar\omega)\rho_\sigma(n,\omega) \, . \tag{21}$$

The Hartree-Fock energies were determined iteratively starting from the bare values. At each step in the procedure the chemical potential was determined by solving Eq. (21) *before* calculating the Hartree-Fock energies for the next step from Eq. (16c). With this approach both the chemical potential and the Hartree-Fock energies were found to converge rapidly to any desired accuracy. Calculations at fixed chemical potential, on the other hand, are inherently unstable since the chemical potential is not a monotonically increasing function of $N$ and there will in general be two (or more) solutions of Eq. (16c) corresponding to different values of $N$.

## III. RESULTS FOR THERMODYNAMIC PROPERTIES

### A. Entropy

For the isotropic (configuration averaged) system Eq. (6) becomes

$$S = -N_L k_B \sum_{n,\sigma} \int d\omega \{ n_F(\hbar\omega)\ln[n_F(\hbar\omega)] + [1-n_F(\hbar\omega)]\ln[1-n_F(\hbar\omega)] \} \times \rho_\sigma(n,\omega) \, . \tag{22}$$

In Fig. 2 we have plotted $S/Nk_B$ versus field for an ideal gas, an interacting gas in the absence of disorder and an interacting gas with disorder. We plot $S/Nk_B$ because in single-particle approximations (including the Hartree-Fock approximation) this quantity is proportional to the diagonal component of the electronic thermopower.[12,13,26] In Eqs. (16c), (21), and (22) the integral over the Gaussian density of states for each Landau level was performed using Hermite-Gaussian quadrature. All of the calculations we have done are based on parameters appropriate to a two-dimensional electron gas (2D EG) in GaAs so that, in temperature units, the noninteracting system Landau-level splitting is $\hbar\omega_c[\mathrm{K}] \sim 20B[\mathrm{T}]$, the spin splitting is $g^*\mu_B^* B[\mathrm{K}] \sim 0.33B[\mathrm{T}]$, and $e^2/\epsilon l[\mathrm{K}] \sim 51\sqrt{B[\mathrm{T}]}$. For the ideal system

$$\rho_\sigma(n,\omega) = \delta(\omega - \omega_c(n + \tfrac{1}{2} + g^*\sigma/4))$$

and Eq. (21) reduces to a sum over contributions from each Landau level which reaches a maximum of $N_L \ln 2$ when that Landau level is half full. At typical experimental temperatures for thermopower measurements $\hbar\omega_c \gg k_B T$ but $g^*\mu_B^* B \sim k_B T$. The main effect of interactions in the THFA is to produce a giant spin-splitting enhancement. The source of this giant enhancement is clear from Eq. (16c), where we see that the exchange lowering of the majority spin energies is proportional to $e^2/l$ which is many times larger than the bare spin splitting for typical fields. In Fig. 2 the spin splitting reduces the entropy, especially at odd integral filling factors, since only Landau levels of one spin have appreciable fractional occupation at a given field. Landau-level broadening begins to influence the entropy when $\Gamma$ becomes comparable to $k_B T$. The curve in Fig. 1 was calculated using $\Gamma/\hbar\omega_c = 0.2/\sqrt{B[\mathrm{T}]}$ which in temperature units is $\Gamma[\mathrm{K}] = 4\sqrt{B[\mathrm{T}]}$. We see that the maximum entropy value is reduced from its ideal value by increasing amounts as the field is raised. Level broadening can also influence the entropy where $\Gamma$ becomes larger than the bare spin splitting since the spin-splitting enhancements is reduced in proportion to the reduction in the difference between majority-spin and minority-spin occupations. For example, we observe that at odd integral values of $\nu$ introducing disorder increases the entropy.

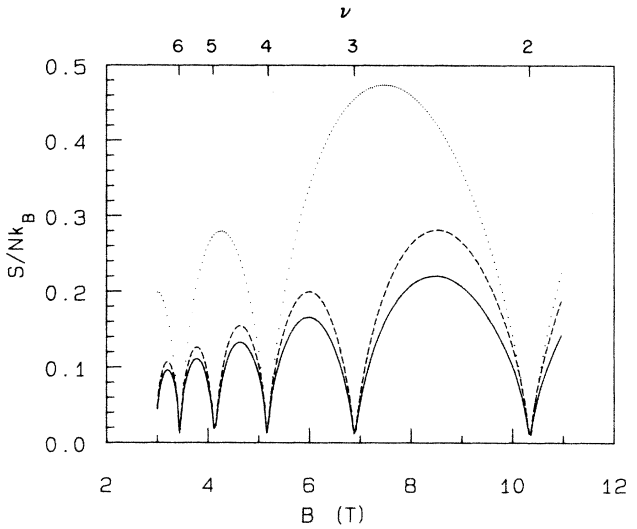


FIG. 2. Entropy versus magnetic field for ideal electron gas (dotted curve) interacting electron gas without disorder (dashed curve), and with disorder ($\Gamma[\mathrm{meV}] = 0.34\sqrt{B[\mathrm{T}]}$). $n = 5.0 \times 10^{11}$ cm$^{-2}$ and $T = 4.2$ K. The main effect of interaction is to split the peak values through the gigantic exchange enhancement as discussed in the text.

### B. Chemical potential

For a 2D EG the chemical potential at zero temperature has discontinuities as a function of magnetic field which are associated with the energy gap between Landau levels and are therefore very directly related to the unique properties of these systems. Recently the magnetic field dependence of the chemical potential has been measured for a 2D EG formed in a Si metal-oxide-semiconductor field-effect transistor by Pudalov *et al.*[27] but, as far as we are aware, the corresponding data for GaAs-$Ga_{1-x}Al_xAs$

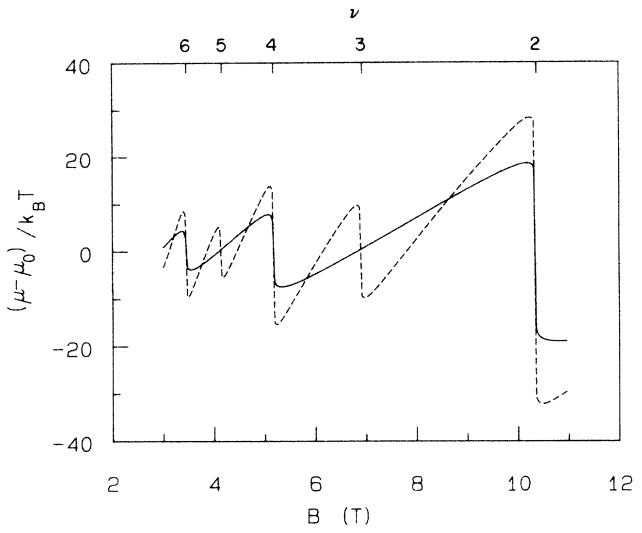


FIG. 3. Shift of the chemical potential from the zero-field value $\mu_0$. $\Gamma[\text{meV}]=0.34\sqrt{B[\text{T}]}$, $n=5\times10^{11}$ cm$^{-2}$, and $T=4.2$ K. The dashed curve shows the effect of interaction. The discontinuities in $\mu$ which occur at integer filling factors is a measure of the gaps in the quasiparticle energy spectrum (see text).

systems are not yet available. For the areal density we have chosen for the 2D gas, $5\times10^{11}$ cm$^{-2}$, the chemical potential in temperature units at $B=0$ and $T=0$ is 205 K in the absence of interactions and 60 K with interactions included in the Hartree-Fock approximation. (The exchange contribution to the chemical potential at $B=0$ and $T=0$ is $-1.8/r_s^*$ in effective Rydberg units where $r_s^*$ is the effective density parameter.[28]) In Fig. 3 we plot the difference between the chemical potentials and these values with the same value of $\Gamma$ as in Fig. 2 both with and without interactions. Without interactions the chemical potential increases in proportion to $\hbar\omega_c$ with increasing field but drops suddenly when a Landau level empties and the chemical potential moves to a lower Landau level. When interactions are included the giant spin-splitting enhancement causes a drop in the chemical potential when it moves from a minority-spin level to a majority-spin level. In addition, the rate of increase in the chemical potential with field in the middle of a Landau level is sharpened since $\hbar\omega_c$ is increasing and the exchange lowering of an energy level tends to lessen as the level empties. Correspondingly, the drop in the chemical potential after a level has completely emptied increases because exchange tends to increase the difference between the energy levels of occupied and unoccupied states.

### C. Heat capacity

The heat capacity of a 2D EG in a magnetic field already shows some interesting features in the absence of interactions which have been discussed previously.[8] In order to explain these we recall that, for a noninteracting system,

$$C_V=\left[\frac{\partial E}{\partial T}\right]_N$$

$$=k_BN_L\sum_{n,\sigma}\int d\omega\,\rho_\sigma(n,\omega)x^2(e^{x/2}+e^{-x/2})^{-2}\,, \tag{23}$$

where $x=(\hbar\omega-\mu)/k_BT$. It is useful to start by discussing the regime where $\Gamma\ll\hbar\omega_c$. Then when the chemical potential lies midway between well-separated Landau levels (assuming the spin-degeneracy is not resolved)

$$C_V\sim4k_BN_L\left[\frac{\hbar\omega_c}{2k_BT}\right]\exp\left[\frac{-\hbar\omega_c}{2k_BT}\right]. \tag{24}$$

On the other hand when the chemical potential is in the middle of a Landau level

$$C_V\sim k_B^2N_LT\int_{-\infty}^{\infty}dx\frac{x^2}{(e^{x/2}+e^{-x/2})^2}$$
$$\times\sum_\sigma\rho_{n,\sigma}(\hbar^{-1}(\varepsilon_{n,\sigma}+k_BTx))\,. \tag{25}$$

When $k_BT/\hbar\omega_c$ is not very small the prefactor in Eq. (24) causes $C_V$ to have a peak between Landau levels due to inter-Landau-level excitations. As $k_BT$ becomes much smaller than $\hbar\omega_c$, however, the contribution to $C_V$ from these excitations drops very rapidly because of the exponential factor. The contribution to $C_V$ from intra-Landau-level excitations drops more slowly with $T$ however and, as we see from Eq. (25), ultimately depends linearly on $T$ for $k_BT\ll\Gamma_{n,n}$. From these considerations we expect that (i) for fixed $T$ the specific heat will crossover with increasing field from having peaks at integral filling to having minima at integral filling; (ii) the crossover point depends primarily on $\hbar\omega_c/k_BT$ and so should

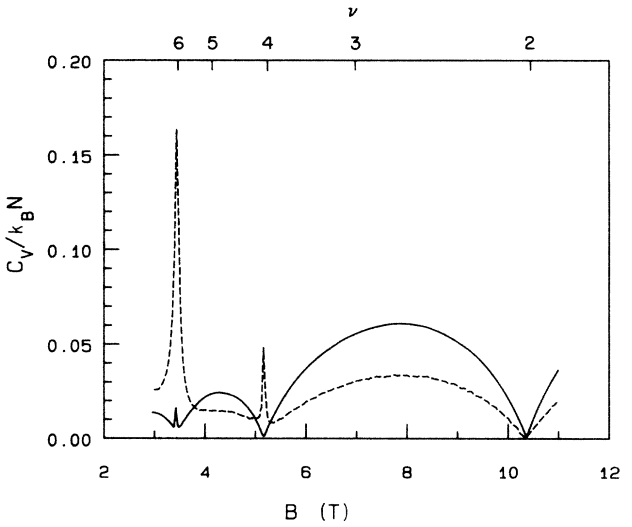


FIG. 4. Heat capacity versus magnetic field with $\Gamma[\text{meV}]=0.34\sqrt{B[\text{T}]}$ and $n=5\times10^{11}$ cm$^{-2}$. The curves show how a crossover from dominance by interLandau contributions (sharp spikes in the curves) to dominance by intraLandau level contribution (broadened peaks in the curves) occur at lower fields as the temperature is lowered from $T=8$ K (dashed curve) to $T=5$ K (solid curve).

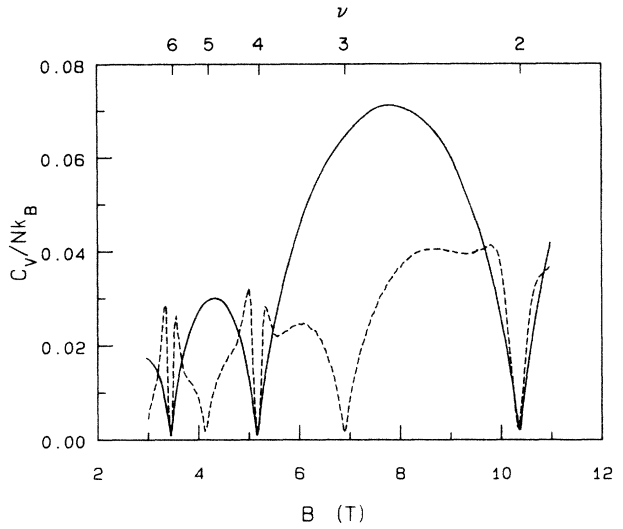


FIG. 5. Heat capacity versus magnetic field without interaction (solid curve) and with interaction (dashed curve). $\Gamma$ and $n$ are as in previous figures, $T=4.2$ K.

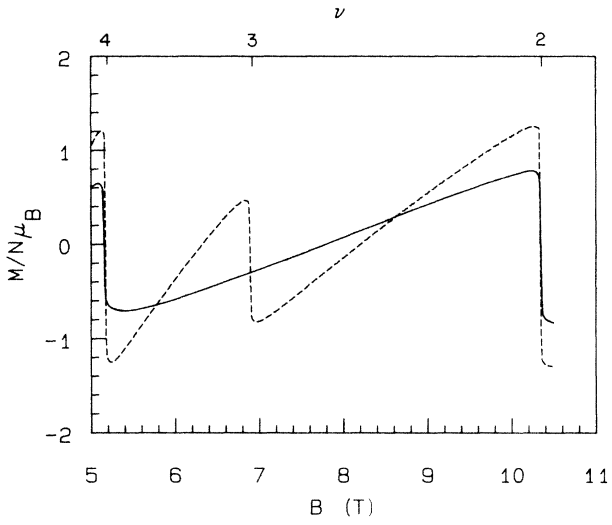


FIG. 6. Magnetization versus magnetic field for an ideal electron gas (solid curve) and with interaction and disorder (dashed curve). $\Gamma[\text{meV}]=0.34\sqrt{B[\text{T}]}$, $n=5\times10^{11}$ cm$^{-2}$, and $T=4.2$ K. The discontinuities in the magnetization is a measure of the quasiparticle energy gaps as discussed in text.

occur at lower fields for lower temperatures; (iii) since increasing $\Gamma$ reduces the gap between Landau levels it should tend to increase the inter-Landau-level peaks; (iv) since increasing $\Gamma$ reduces the density of states in a Landau level it should tend to reduce the intra-Landau-level peaks.

We have calculated the specific heat numerically to illustrate these features and discuss how they are modified by interactions. In Fig. 4 we compare the specific heats of a noninteracting system with $\Gamma[\text{K}]=4\sqrt{B[\text{T}]}$ as in Fig. 2 at two different temperatures. At the higher temperature the inter-Landau-level contribution dominates over most of the range of fields plotted but at the lower temperature the crossover to dominant intra-Landau-level contributions occurs at much lower fields. In Fig. 5 we illustrate the effect of interactions on these results. The most visible effect is the giant spin-splitting enhancement which produces minima at odd integral filling factors. We also see, however, that inter-Landau-level contributions are reduced because of the increase in the difference between Hartree-Fock energies of occupied and unoccupied Landau levels which was mentioned in the preceding section.

### D. Magnetization

At zero temperature the ground-state energy of a 2D EG has a downward-pointing cusp at integral filling factors. Thus the magnetization, which is given by

$$M=\left[-\frac{\partial(E-TS)}{\partial B}\right]_N \tag{26}$$

has a discontinuity as a function of $B$ at $T=0$ which is directly related to the discontinuity in the chemical potential discussed in Sec. III B. In fact since $\nu=2\pi l^2 N/L_x L_y$, and only the dependence on $\nu$ is singular we have for $T=0$ that

$$\frac{M_+-M_-}{N\mu_B^*}=\frac{(\mu_+-\mu_-)}{B\mu_B^*}=\frac{2(\mu_+-\mu_-)}{\hbar\omega_c}, \tag{27}$$

i.e., the discontinuity in the magnetization per electron in units of effective Bohr magnetrons ($\mu_B^*=e\hbar/2m^*c$) is twice the discontinuity of the chemical potential in units of $\hbar\omega_c$. In Fig. 6 we have illustrated our numerical results for the magnetization which were obtained from Eq. (26). The effect of interactions is to increase the discontinuity of the magnetization because of the increased Landau-level splitting, especially between different spins, as mentioned previously. Introducing disorder, of course, leads to a reduction in the Landau-level splittings and therefore causes a reduction in the magnetization discontinuity.

## IV. FEATURES AT FRACTIONAL FILLING FACTORS

In the preceding section we have studied the thermodynamic properties of a 2D EG in a strong magnetic field within the Hartree-Fock approximation. For sufficiently weak disorder and sufficiently low temperature all properties exhibit features near integral filling factors which result from the energy gap between Landau levels. In fact, as we have mentioned, both the chemical potential and the magnetization measure this energy gap directly. In the case of integer filling factors the energy gap between Landau levels is increased by electron-electron interactions. From the occurrence of the fractional quantum Hall effect,[2] it is clear that gaps open up at fractional values of the Landau-level filling factor which are *entirely* due to interactions. These gaps are absent in the Hartree-Fock approximation discussed in Sec. II and applied in Sec. III because they are due to correlations among electrons in the same Landau level. Nevertheless, we suggest below

that the results of Sec. III may be used to draw some conclusions concerning the behavior of the various thermodynamic properties at fractional filling factors.

Our argument is based on the hierarchy picture of the fractional quantum Hall effect.[29–33] In this picture the electron system has especially stable incompressible liquid[34] at a series of rational values of $\nu_0$, all of which have odd denominators ($\nu_0=p/q$, where $q$ is odd). For filling factors $\nu$ close to $\nu_0$ the ground state and the *low-lying* excited states of this system are in one-to-one correspondence with those which exist when $\nu_0$ is an integer. (See especially Ref. 33.) To be specific, there are $N_L$ possible quasielectrons and $N_L$ possible quasiholes which can be created in the ground state, the differences being that in the fractional case it does not cost an extra energy $\hbar\omega_c$ to create a quasielectron and that the quasiparticle charges have the fractional magnitude $e/q$. If $\nu$ is close to $\nu_0$ and temperatures are low so that quasiparticles populations are small, quasiparticle interactions can be neglected and the energy is given by

$$E\simeq E(\nu_0)+N_L\left[\int d\varepsilon\{\rho_+(\varepsilon)\tilde{n}_F(\varepsilon)+\rho_-(\varepsilon)[\tilde{n}_F(\varepsilon)-1]\}\varepsilon\right]+\frac{N\hbar\omega_c}{2}\ . \tag{28}$$

In Eq. (7) $\rho_\pm(\varepsilon)$ are the disorder-broadened distributions of quasiparticle energies centered about $\pm\varepsilon_\pm$ and $\tilde{n}_F(\varepsilon)=\{\exp[(\varepsilon-\tilde{\mu})/k_BT]+1\}^{-1}$ is the fermion-like distribution function for the fractionally charged quasiparticles. We note that $\tilde{\mu}=\mu/|q|$ since one electron is created or lost for each $q$ quasiparticles added to or removed from the system. The quasiparticle energies that appear

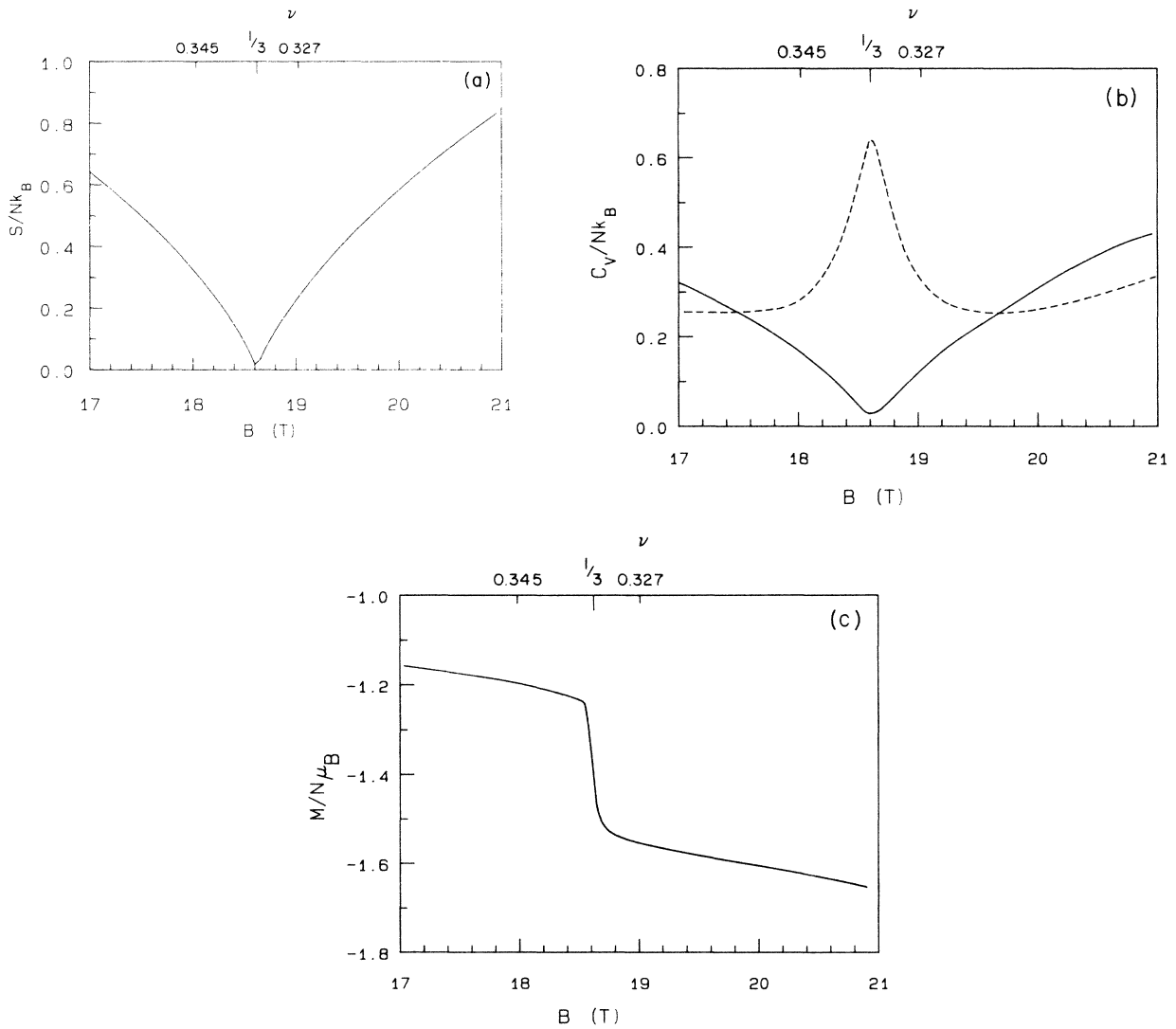


FIG. 7. (a) Contributions to the entropy from low-lying excitations at filling factor near $\nu=\frac{1}{3}$ in the fractional quantum Hall effect (FQHE) regime. $n=1.5\times10^{11}$ cm$^{-2}$, $T=0.5$ K, and $\Gamma[\text{meV}]=0.03\sqrt{B[\text{T}]}$. (b) Heat capacity in the FQHE regime [cf. Fig. 7(a)] at $\nu$ close to $\frac{1}{3}$. $\Gamma[\text{meV}]=0.09\sqrt{B[\text{T}]}$ and $n=1.5\times10^{11}$ cm$^{-2}$. At $T=1$ K (solid curve) we observe a minimum at $\nu=\frac{1}{3}$ while the reverse is the case at $T=2$ K (dashed curve), cf. Fig. 4. (c) Magnetization versus magnetic field close to $\nu=\frac{1}{3}$. The jump in the magnetization is a measure of the energy gap at $\nu=\frac{1}{3}$. $n=1.5\times10^{11}$ cm$^{-2}$, $T=0.5$ K, and $\Gamma[\text{meV}]=0.03\sqrt{B[\text{T}]}$.

here are the gross quasiparticle energies,[35,36] which for the case of $\nu$ near $\nu_0=\frac{1}{3}$ are given by $\varepsilon_+=-0.12e^2/\epsilon l$ and $-\varepsilon_-=-0.23e^2/\epsilon l$. At a given filling factor and temperature the quasiparticle chemical potential, $\tilde{\mu}$, is related to the filling factor by

$$\nu=\nu_0+q^{-1}\int d\varepsilon\{\rho_+(\varepsilon)\tilde{n}_F(\varepsilon)+\rho_-(\varepsilon)[\tilde{n}_F(\varepsilon)-1]\} . \quad (29)$$

Similarly, based on the one-to-one correspondence between the excitations for $\nu_0$ near integral values and the *low-lying* excitations in the fractional quantum Hall regime, we argue that the entropy is given by

$$S=-N_L k_B\int d\varepsilon\{\tilde{n}_F(\varepsilon)\ln[\tilde{n}_F(\varepsilon)]+[1-\tilde{n}_F(\varepsilon)] \times\ln[1-\tilde{n}_F(\varepsilon)]\}[\rho_+(\varepsilon)+\rho_-(\varepsilon')] . \quad (30)$$

In Figs. 7 we present results for the entropy, the magnetization and the specific heat for $\nu$ near $\nu_0=\frac{1}{3}$. These are discussed below but we should first emphasize that for the ideal system quasiparticle interactions may never be negligible, even for $\nu$ arbitrarily close to $\nu_0$. Our results apply in the regime where the interactions of the $\nu_0=\frac{1}{3}$ quasiparticles with the disorder potential are much stronger than their interaction with each other, at least in the range of filling factors considered. We expect this regime to exist even with the highest mobility 2D EG samples currently available. For the entropy, Eq. (30) predicts that a minimum should occur for $\nu$ near $\nu_0$, whenever the temperature is small compared to the disorder reduced energy gap, $E_g\simeq\varepsilon_+-\varepsilon_--2\Gamma$ [see Fig. 7(a)]. The heat capacity, as illustrated in Fig. 7(b), is predicted to have a maximum at $\nu=\nu_0$ for higher temperatures and a minimum at lower temperatures. As discussed in Sec. III, this behavior is closely associated with the nature of the excitation spectrum assumed in the present discussion and, if confirmed, will provide convincing support for the hierarchy picture. The magnetization is illustrated in Fig. 7(c).

In closing this section we comment on the thermodynamic properties of the completely disorder free system as $T\to 0$. In this case anomalies should be present near many fractional values of $\nu_0$ and the dependence of thermodynamic properties on magnetic field should be rich with detail. For example we show in Fig. 8 a curve for magnetization for $\nu<1$ based on the approximate expressions given by Halperin[30] for the ground-state energies at hierarchies of $\nu_0$ values. The magnetization $M=-\partial E/\partial B$ was approximated using a finite difference between states at adjacent filling factors. (States with fractional charges of magnitude greater than $|e|/39$ in the hierarchy scheme were included.) Writing $E=N[e^2f(\nu)/l+\hbar\omega/2]$, as is appropriate when Landau-level mixing can be ignored, we see that

$$\frac{M}{N\mu_B^*}=-1+\frac{e^2/l}{\hbar\omega_c}[2\nu f'(\nu)-f(\nu)] . \quad (31)$$

In the Hartree-Fock approximation (the dashed line in Fig. 8) we see from Sec. III that $f(\nu)=(-\nu/2)\sqrt{\pi/2}$ which gives

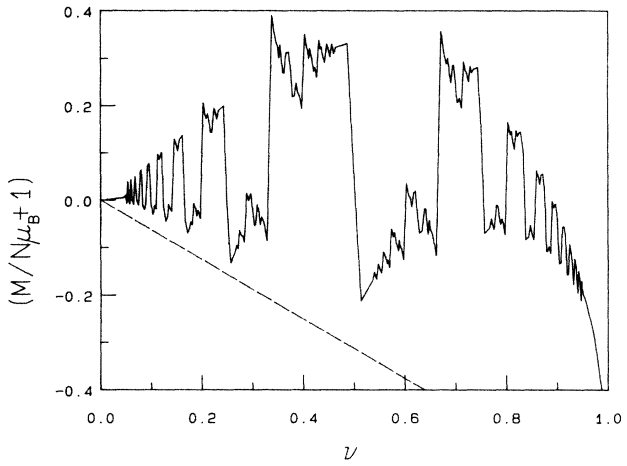


FIG. 8. Magnetization versus filling factor in the FQHE. The dashed curve is for the Hartree-Fock approximation (see text). The deviation of $M/N\mu_B^*$ from the noninteracting value, $M/N\mu_B^*=-1$, is in units of $(e^2/\epsilon l)/\hbar\omega_c$.

$$\frac{M^{\rm HF}}{N\mu_B^*}=-1-\frac{(e^2/l)\nu}{2\hbar\omega_c}\left[\frac{\pi}{2}\right]^{1/2} . \quad (32)$$

When correlations within a Landau level are included, however, the $\nu$ dependence of $f(\nu)$ at small $\nu$ is dominated by the $\nu^{1/2}$ dependence[30,32] expected for a classical Wigner crystal which is independent of field and therefore gives no contribution to the magnetization. The smaller magnitude deviations from $\nu^{1/2}$ dependence which are associated with the fractional quantum Hall effect give a magnetization with an erratic field dependence centered around the noninteracting value. Using Eq. (27) we see that the discontinuity in the magnetization per electron in units of $N\mu_B^*$ at $\nu=\nu_0$ is $2q$ times the quasiparticle energy gap in units of $\hbar\omega_c$. From this it is clear that measurements of thermodynamic properties can, in principle, test theories of the fractional quantum Hall effects in great detail. At present it appears that these experiments are extremely sensitive to sample homogeneity but as our calculations show, much will be learned if existing experimental obstacles can be surmounted.

## V. SUMMARY AND CONCLUDING REMARKS

We have studied the anomalies in the thermodynamic properties of a 2D electron gas associated with the energy gap between Landau levels in the Hartree-Fock approximation (HFA). We believe that this approximation gives a reliable description of the way in which interactions modify these anomalies. The most obvious and dramatic consequence of interactions is the giant enhancement of the spin splitting which makes anomalies at odd integral values of the Landau-level filling factor much more pronounced. The lowest-lying excitations at integral filling factors are composed of holes in the highest filled Landau level and electrons in the lowest unfilled Landau level. The approximations used here are equivalent to ignoring the interactions between these electron and holes. This is justified except when the electron and hole are close to-

gether in which case the excitation has a collective nature[37] and our description is in error. Phase space considerations, however, make the corrections to thermodynamic properties small. A more serious weakness of the HFA is its failure to include the consequences of correlations of electrons within the same Landau level. These correlations are responsible for the fractional quantum Hall effect in which the system has especially stable ground states at a series of fractional filling factors $\{\nu_0\}$. We have presented some qualitative results for the behavior of the thermodynamic properties for $\nu$ near $\nu_0$. Our predictions are based on the hierarchy picture in which there is a one-to-one correspondence between the excitations for $\nu$ near $\nu_0$ and the excitations for $\nu$ near an integer. Most interesting among these predictions, is the claim that the specific heat should change from having a maximum at $\nu=\nu_0$ to having a minimum at $\nu=\nu_0$ as the temperature is lowered. Verification of this property would provide impressive support for the hierarchy picture.

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