Configuration-interaction calculations on the fractional quantum Hall effect

G. Fano
Dipartimento di Fisica dell'Università degli Studi di Ferrara, via Paradiso 12, I-44100 Ferrara, Italy and Sezione di Bologna, Istituto Nazionale di Fisica Nucleare, via Irnerio 46, I-40126 Bologna, Italy

F. Ortolani
Dipartimento di Fisica dell'Università degli Studi di Bologna, via Irnerio 46, I-40126 Bologna, Italy and Sezione di Bologna, Istituto Nazionale di Fisica Nucleare, via Irnerio 46, I-40126 Bologna, Italy

E. Colombo
Dipartimento di Fisica dell'Università degli Studi di Ferrara, via Paradiso 12, I-44100 Ferrara, Italy and Sezione di Bologna, Istituto Nazionale di Fisica Nucleare, via Irnerio 46, I-40126 Bologna, Italy
(Received 28 January 1986)

With the use of Haldane's spherical model, the fractional quantum Hall effect is studied by means of large-scale configuration-interaction calculations. The exact wave function and the ground-state energy are computed numerically up to ten electrons for filling $\nu=\frac{1}{3}$, and up to seven electrons for filling $\nu=\frac{1}{5}$. Quasiparticle, quasihole, and exciton energies are computed. The mutual relation between wave functions on the sphere and wave functions on the disk is fully analyzed. A wave function originally proposed by Halperin for the disk geometry is tested on the sphere; the numerical results up to six electrons for $\nu=\frac{2}{5}$ and $\nu=\frac{2}{9}$ are favorable.

I. INTRODUCTION

One of the most exciting discoveries in solid-state physics in recent years is the fractional quantum Hall effect (FQHE).[1] Two-dimensional electrons in $Al_{1-x}Ga_xAs$-GaAs heterojunctions at high magnetic field B present a quantized Hall conductivity σ_{xy} at some simple rational multiples of e^2/h. Denoting by ρ the electron density, and by $\rho_0=Be/h$ ($c=1$) the density corresponding to the complete filling of the first Landau level, plateaus of σ_{xy} and anomalously small values of σ_{xx} appear at some rational value of the filling factor $\nu=\rho/\rho_0$ (a typical value is $\nu=\frac{1}{3}$). These facts suggest the existence of an energy gap in the excitation spectrum, and cusps in the correlation energy per particle $E(\nu)$ at the special values of ν. We refer the reader to the excellent review papers by Halperin[2] and by Yoshioka[3] for more details.

In all theoretical studies of the phenomenon, the electrons are placed in a positive neutralizing background. Early treatments of a two-dimensional electron system in a strong magnetic field were based on a Hartree-Fock charge-density-wave (CDW) picture.[4-6] For $\nu=\frac{1}{3}$ the computed value of $E(\nu)$ is[6] $-0.389e^2/\epsilon a_0$, where a_0 is the magnetic length $(\hbar/eB)^{1/2}$ and ϵ is the dielectric constant. This results is quite close to the variational bound,[6] which holds for any self-consistent CDW state: $E(\nu)>-0.395e^2/\epsilon a_0$. We notice that for a large system the value $E_0(\nu)$ of the energy for a normal state (a self-consistent Slater determinant without CDW) is much higher: $E_0(\frac{1}{3})=-\frac{1}{3}(\pi/8)^{1/2}e^2/\epsilon a_0$. It is also worth mentioning that the local density of a triangular CDW is very close to the local density of a Wigner crystal (WC) wave function, the latter being constructed by means of Gaussians centered on the lattice points.[6] Unfortunately, the Hartree-Fock CDW energy turns out to be a smooth function of ν, and therefore the CDW model alone does not explain the FQHE.

A real breakthrough in the subject came with the pioneering work of Laughlin.[7,8] Laughlin's wave function describes a quantum liquid, explains the plateaus of σ_{xy} at $\nu=m^{-1}$ (m odd), and gives an explicit example of a correlated many-body state. The energy $E(\nu)$ ($=-0.41e^2/\epsilon a_0$ for $\nu=\frac{1}{3}$) is lower than the CDW value, and it is not very far from the rigorous lower bound[9] $E(\nu)>-0.782\nu^{1/2}e^2/\epsilon a_0$ ($=-0.4515e^2/\epsilon a_0$ for $\nu=\frac{1}{3}$) which corresponds to a classical triangular Wigner lattice. An interesting aspect of Laughlin's treatment is the application of the idea of fractionally charged excitation,[10] as well as the connection of the quantum problem with the classical plasma theory.[7] These ideas have given rise to many interesting developments.[11-18]

However, a number of problems are left unsolved, so that the possibility of alternative approaches[19-23] is not ruled out. For instance, the experimental value of the energy gap is smaller than Laughlin's computed value and the extension of Laughlin's theory to $\nu\neq m^{-1}$ (m odd) is not straightforward. An interesting unsolved problem is whether the ground state is degenerate.[24-28] Furthermore, a recent work by Chui, Hakim, and Ma[29] proposes again a solidlike trial wave function, even for $\nu=\frac{1}{3}$. The possibility of a crystalline order is also investigated by Kivelson, Kallin, Arovas, and Schrieffer[19] using a semiclassical path integral approach. See also Baskaran.[30]

For these reasons, exact finite-size calculations with a small number N of electrons (often performed in the last few years[27,31-37]) can still be of use in order to improve

our knowledge of the structure of the wave function, of the behavior of $E(\nu)$ and of the excitation spectrum, etc. These calculations can be performed more easily using spherical geometry,[12,36] which is the only geometry without boundaries. The treatment of the neutralizing background, which is quite cumbersome on a disk, is immediate on the sphere. In Ref. 36, results of detailed numerical calculations on the sphere are exhibited for $\nu=\frac{1}{3}$ and $N\leq 8$. The effectiveness of Laughlin's wave function was confirmed; also the dispersion relation of collective excitations was found to be in quantitative agreement with the results of Girvin, MacDonald, and Platzman.[37]

The paper is divided into three parts. Section II contains essentially a review of the formalism of the spherical geometry, including some details that were omitted in the original paper of Haldane. In Sec. III we establish a correspondence between wave functions on the sphere and wave functions on the disk; this mapping allows the possibility of testing on the spherical geometry wave functions originally proposed for the plane, and clarifies some questions related to the effectiveness of these wave functions. The relation between translation invariance on the plane and rotation invariance on the sphere is fully analyzed. Section IV contains our numerical results: We have improved the accuracy of the preceding calculations, and we have added new cases; for $\nu=\frac{1}{3}$ we have computed the exact wave function up to $N=10$, and for $\nu=\frac{1}{5}$ up to $N=7$. Furthermore, we have investigated some wave functions originally proposed by Halperin[3] for the disk geometry, and corresponding, respectively, to $\nu=\frac{2}{5}$, $\nu=\frac{2}{9}$, for large values of N. A few excited states are also analyzed. In the Appendix, a compact form for the two-body matrix element is derived.

II. FORMALISM OF THE HALDANE MODEL

Let us consider a sphere of radius R with a magnetic monopole placed at the center of the sphere. Let B denote the magnitude of the (radial) magnetic field; by Dirac's quantization condition[38] the total flux $4\pi R^2 B$ must be an integral multiple $2S$ of the elementary flux quantum h/e (we take units such that $c=1$); therefore

$$B=\frac{\hbar S}{eR^2} . \tag{1}$$

Let us consider one electron of mass m constrained on the surface of the sphere. The kinetic energy H_0 is proportional to the square of the angular momentum $\mathbf{\Lambda}$:

$$H_0=\frac{|\mathbf{\Lambda}|^2}{2mR^2}=\frac{|\mathbf{\Lambda}|^2 eB}{2m\hbar S} , \tag{2}$$

$$\mathbf{\Lambda}=\mathbf{R}\times[-i\hbar\nabla+e\mathbf{A}(\mathbf{R})] . \tag{3}$$

Denoting by R,θ,φ the polar coordinates and by $\mathbf{\Omega}=\mathbf{R}/R,\theta,\varphi$ the corresponding unit vectors, the electromagnetic potential $\mathbf{A}$ can be taken to be

$$\mathbf{A}=-\frac{\hbar S}{eR}\cot\theta\,\boldsymbol{\varphi} . \tag{4}$$

Because of the presence of the magnetic field, the eigenvalues of $|\mathbf{\Lambda}|^2$ are not equal to $l(l+1)\hbar^2$ with l an integer. In order to find the eigenvalues of $|\mathbf{\Lambda}|^2$, it is useful to compute the commutation relations $[\Lambda_\alpha,\Lambda_\beta]$. Denoting by $M_\alpha=-i\hbar\varepsilon_{\alpha\beta\gamma}R_\beta\partial_\gamma$ the angular momentum components in the absence of the magnetic field, we have, using (1) and (3),

$$\begin{aligned}[\Lambda_\alpha,\Lambda_\beta]&=[M_\alpha+\varepsilon_{\alpha\mu\nu}R_\mu eA_\nu,M_\beta+\varepsilon_{\beta tu}R_t eA_u]=i\hbar\varepsilon_{\alpha\beta\gamma}M_\gamma-i\hbar e\,\varepsilon_{\alpha\mu\nu}\varepsilon_{\beta tu}\{[R_\mu A_\nu,R_t\partial u]+[R_\mu\partial_\nu,R_t A_u]\}\\&=i\hbar\varepsilon_{\alpha\beta\gamma}M_\gamma-i\hbar e\{(\delta_{\alpha\beta}\delta_{\mu u}-\delta_{\alpha u}\delta_{\mu\beta})(R_u A_\mu-R_\mu A_u)+\varepsilon_{\alpha\mu\nu}\varepsilon_{u\eta\nu}\varepsilon_{\beta tu}R_t R_\mu B_\eta\}\\&=i\hbar\varepsilon_{\alpha\beta\gamma}M_\gamma+i\hbar e\,\varepsilon_{\alpha\beta\gamma}(\mathbf{R}\times\mathbf{A})_\gamma-i\hbar e(\delta_{\alpha u}\delta_{\mu\eta}-\delta_{\alpha\eta}\delta_{\mu u})\varepsilon_{\beta tu}R_t R_\mu B_\eta=i\hbar\varepsilon_{\alpha\beta\gamma}(\Lambda_\gamma-\hbar S\Omega_\gamma) ,\end{aligned} \tag{5}$$

where $\varepsilon_{\alpha\mu\nu}$, etc., is the completely antisymmetric rank-3 Levi-Civita tensor.

Furthermore,

$$[\Lambda_\alpha,\Omega_\beta]=-i\frac{\hbar}{R}\varepsilon_{\alpha\mu\nu}[R_\mu\partial_\nu,R_\beta]=i\hbar\varepsilon_{\alpha\beta\mu}\Omega_\mu . \tag{6}$$

Therefore, defining

$$\mathbf{L}=\mathbf{\Lambda}+\hbar S\mathbf{\Omega} , \tag{7}$$

we have the commutation relations

$$\begin{aligned}[L_\alpha,L_\beta]&=i\hbar\varepsilon_{\alpha\beta\gamma}(\Lambda_\gamma-\hbar S\Omega_\gamma)+2i\hbar^2 S\varepsilon_{\alpha\beta\gamma}\Omega_\gamma\\&=i\hbar\varepsilon_{\alpha\beta\gamma}L_\gamma .\end{aligned} \tag{8}$$

The operators L_α are the infinitesimal generators of the rotation group. Since $\mathbf{\Lambda}\cdot\mathbf{\Omega}=\mathbf{\Omega}\cdot\mathbf{\Lambda}=0$, we have $\mathbf{L}\cdot\mathbf{\Omega}=\mathbf{\Omega}\cdot\mathbf{L}=\hbar S$. Therefore,

$$|\mathbf{\Lambda}|^2=|\mathbf{L}|^2-\hbar^2S^2 \tag{9}$$

and the eigenvalues of $|\mathbf{\Lambda}|^2$ are $[l(l+1)-S^2]\hbar^2$.

The first Landau level for one electron on the sphere is obtained for $l=S$. The corresponding eigenvalue of the energy is $eB\hbar/2m=\hbar\omega_c/2$, where ω_c is the cyclotron frequency.

Contrary to what happens in the plane, where there is infinite degeneracy of the first Landau level, on the sphere the degeneracy is finite; in fact, there are $2S+1$ independent degenerate eigenfunctions of $H_0=(2mR^2)^{-1}(|\mathbf{L}|^2-\hbar^2S^2)$. When all these single-particle states are occupied, the electron density is

$$(2S+1)/4\pi R^2\sim S/2\pi R^2=(2\pi a_0^2)^{-1} ,$$

where $a_0=(\hbar/eB)^{1/2}$ is the magnetic length. Therefore, for large values of S (and R) the density corresponding to full occupation of the first Landau level is the same as in the planar case.

Following Haldane, we represent the operators L_i

$(i=1,2,3)$ in the space of spinors of rank $2S$. Let us consider a rotation of angle θ about the axis Ω, let $s=(u,v)$ denote a two-component spinor, and let σ_i $(i=1,2,3)$ denote the Pauli spin matrices. An irreducible representation of the rotation in the space of homogeneous polynomials $p(u,v)$ of degrees $2S$ is given by the formula

$$(e^{-i\hbar^{-1}\mathbf{L}\cdot\Omega\theta}p)(s)=p(e^{i\sigma\cdot\Omega\theta/2}s)\ . \tag{10}$$

Differentiating both sides with respect to θ and setting $\theta=0$, we obtain

$$-\hbar^{-1}(\mathbf{L}\cdot\Omega)p=\frac{\partial p}{\partial u}\left[\frac{\sigma\cdot\Omega}{2}s\right]_u+\frac{\partial p}{\partial v}\left[\frac{\sigma\cdot\Omega}{2}s\right]_v\ , \tag{11}$$

where $[x]_u,[x]_v$ denote the two components of the spinor x. In order to be consistent with the conventions of Ref. 12, we choose the following representation of the Pauli spin matrices:

$$\sigma_1=-\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad \sigma_3=-\begin{bmatrix}1&0\\0&-1\end{bmatrix}. \tag{12}$$

Then, from (11), it follows that

$$L_1=\frac{\hbar}{2}\left[v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v}\right],$$

$$L_2=\frac{i\hbar}{2}\left[v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v}\right], \tag{13}$$

$$L_3=\frac{\hbar}{2}\left[u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}\right].$$

Of course $|\mathbf{L}|^2p=S(S+1)\hbar^2p$ for any $p(u,v)$ of degree $2S$, and the monomial u^{2S} satisfies $L_3u^{2S}=S\hbar u^{2S}$.

It is convenient to associate to each unit vector $\Omega=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$ a spinor $s=(u,v)$ satisfying the equation

$$(\sigma\cdot\Omega)s=-s\ . \tag{14}$$

Equation (14) implies

$$\frac{u}{v}=\frac{\Omega_1+i\Omega_2}{1-\Omega_3}\ . \tag{15}$$

For instance we can choose the product uv to be real, i.e.,

$$u=\cos(\theta/2)e^{i\varphi/2}\ ,\quad v=\sin(\theta/2)e^{-i\varphi/2}\ . \tag{16}$$

Putting $s'=(u',v')$, we denote by $\langle s'|s\rangle$ the scalar product $\bar{u}'u+\bar{v}'v$. The polynomials

$$p_{s'}(s)=\langle s'|s\rangle^{2S} \tag{17}$$

satisfy the eigenvalue equation

$$(\mathbf{L}\cdot\Omega)p_{s'}(s)=\hbar Sp_{s'}(s)\ . \tag{18}$$

Equation (18) can be verified by applying a general rotation to $p_{s'}$ and using (10) and (14):

$$(e^{-i\hbar^{-1}\mathbf{L}\cdot\Omega\theta}p_{s'})(s)=\langle s'|e^{i\sigma\cdot\Omega\theta/2}|s\rangle^{2S}$$
$$=\langle e^{-i\sigma\cdot\Omega\theta/2}s'|s\rangle^{2S}=e^{-iS\theta}p_{s'}(s)\ . \tag{19}$$

Differentiating both sides with respect to θ and setting $\theta=0$, we obtain the result.

Haldane calls "coherent states" the states (17); indeed they have the same physical meaning on the sphere as the coherent states on the disk geometry (see Refs. 39 and 40). The particle is localized around the direction Ω' corresponding to the spinor s' as much as is compatible with the condition that the wave function is built only from the lowest Landau level eigenfunctions of H_0. The function $p_{s'}(s)$ represents a "spin" localized at the point Ω' of the unit sphere, since $|p_{s'}(s)|^2$ is maximum for $s=s'$.

Let us denote by H_S the subspace of the Hilbert space spanned by all the homogeneous polynomials of degree $2S$, with the scalar product defined by [see Eq. (16)]

$$(p_1,p_2)=\int d\Omega\overline{p_1(u(\theta,\varphi),v(\theta,\varphi))}p_2(u(\theta,\varphi),v(\theta,\varphi))\ , \tag{20}$$

where $d\Omega=\sin\theta\,d\theta\,d\varphi$.

The following monomials, that are eigenfunctions of L_3, provide an orthonormal basis in H_S:

$$e_{s,m}=\left[\frac{2S+1}{4\pi}\begin{pmatrix}2S\\S+m\end{pmatrix}\right]^{1/2}u^{S+m}v^{S-m}$$
$$\text{for } m=S,S-1,\ldots,-S\ , \tag{21}$$

where $\binom{2S}{S+m}$ is the binomial coefficient. We have

$$L_3e_{S,m}=\hbar me_{S,m}\quad \text{for } m=S,S-1,\ldots,-S\ . \tag{22}$$

These functions represents cyclotron orbits on the sphere (parallels) with $\langle\cos\theta\rangle=m/(S+1)$.

Let us now consider the problem of N electrons on the sphere interacting with each other and with a positive background via the Coulomb potential. We assume a_0 as the unit of length and $e^2/4\pi\epsilon a_0$ as the unit of energy, where ϵ is the dielectric constant. Therefore $S=R^2$ and the Coulomb potential is simply r^{-1}.

In virtue of Gauss's theorem, the effect of the neutralizing background is equivalent to a charge $+Ne$ placed at the center of the sphere. Therefore the interaction energy electrons background is $-N^2/R=-N^2/\sqrt{S}$, and the self-energy of the background is $\frac{1}{2}N^2/\sqrt{S}$. Adding the two contributions, we obtain a shift in the energy of $-\frac{1}{2}N^2/\sqrt{S}$.

Denoting by $a_m^\dagger$ the creation operator of an electron with wave function $e_{S,m}$, the restriction of the interaction

$$\sum_{\substack{i,j\\i<j}}r_{ij}^{-1}$$

to the relevant many-particle Hilbert space is given by

$$V=\tfrac{1}{2}\sum_{m_1=-S}^{+S}\sum_{m_2=-S}^{+S}\sum_{m_3=-S}^{+S}\sum_{m_4=-S}^{+S}\left\langle m_1,m_2\left|\frac{1}{r}\right|m_3,m_4\right\rangle_S$$
$$\times a_{m_1}^\dagger a_{m_2}^\dagger a_{m_4}a_{m_3}\ . \tag{23}$$

As noticed in Ref. 36, the problem is formally equivalent to that of a partially filled nuclear shell with $j=S$, even though the interaction is quite different. We can write

$$\left\langle m_1,m_2\left|\frac{1}{r}\right|m_3,m_4\right\rangle_S=\sum_{J=0}^{2S}\sum_{M=-J}^{+J}\langle Sm_1,Sm_2\,|\,JM\rangle\langle Sm_3,Sm_4\,|\,JM\rangle V_J^{(S)}R^{-1}\,, \tag{24}$$

where the reduced matrix element $V_J^{(S)}$ is independent of M, and the remaining factors are Clebsch-Gordan coefficients.

It is possible (see the Appendix) to express the $V_J^{(S)}$ in the compact form:

$$V_J^{(S)}=2\frac{\begin{bmatrix}4S-2J\\2S-J\end{bmatrix}\begin{bmatrix}4S+2J+2\\2S+J+1\end{bmatrix}}{\begin{bmatrix}4S+2\\2S+1\end{bmatrix}^2}\,. \tag{25}$$

III. STEREOGRAPHIC MAPPING BETWEEN FUNCTION SPACES

Let us now consider the disk geometry in the planar case; the *non*-normalized single-particle wave functions for the lowest Landau level are

$$\varphi_n(z)=z^n e^{-|z|^2/4}\,,\quad \text{with } n=0,1,2,\ldots, \tag{26}$$

where $z=x-iy$. Leaving aside the exponential factor, that plays the role of a measure, let us consider the space L_S spanned by the first $2S+1$ functions (26), i.e., the space of polynomials of degree $\leq 2S$ ($2S$ integer).

We establish a mapping Γ between the spaces H_S and L_S by mapping the polynomial

$$p(u,v)=\sum_{k=0}^{2S}c_k u^{2S-k}v^k$$

defined on the sphere into the polynomial $P(z)=\sum_{k=0}^{2s}c_k z^k$ defined on the plane. See also Ref. 40. The mapping Γ is reminiscent of the stereographic mapping between points. In fact, from (15) it follows that

$$\frac{v}{u}=\frac{1-\Omega_3}{\Omega_1+i\Omega_2}=\frac{\Omega_1-i\Omega_2}{1-\Omega_3}\,. \tag{27}$$

On the other hand, from our definition of the correspondence $p\xrightarrow{\Gamma}P$, we have

$$p(u,v)=u^{2S}P\left[\frac{v}{u}\right]\,. \tag{28}$$

Since (27) represents a stereographic projection of Ω to the complex conjugate of v/u, we call Γ the "stereographic mapping between function spaces." Γ maps the cyclotron orbits on the sphere that are parallel to the plane, into cyclotron orbit on the plane. Notice that Γ is *not* isometric since the norm of $u^{2S-k}v^k$ is different from the norm of z^k.

Let us now consider a class of many-particle wave functions that are currently used in the plane geometry to describe the FQHE: These are the antisymmetric polynomials $P(z_1,z_2,\ldots,z_N)$ that are homogeneous of total degree M. Setting $z_j=\rho_j e^{i\varphi_j}$, M is the eigenvalue of the total angular momentum $M_z=-i\sum_{i=1}^N\partial/\partial\varphi_i$, which is a conserved quantity (we have set $\hbar=1$).

We can extend the mappings Γ and Γ^{-1} to the many-particle case by defining, for any P, a polynomial p by means of the relation

$$p(u_1,v_1;u_2,v_2;\ldots;u_N,v_N)=\left[\prod_i u_i\right]^{2S}P\left[\frac{v_1}{u_1},\frac{v_2}{u_2},\ldots,\frac{v_N}{u_N}\right]\,. \tag{29}$$

Denoting by $D(s_1,s_2,\ldots,s_N)$ the Slater determinant $\det||A_{ij}||$, with $A_{ij}=z_j^{s_i}$ $(i,j=1,2,\ldots,N)$, and by $\hat{D}(s_1,s_2,\ldots,s_N)$ the Slater determinant $\det||B_{ij}||$ with $B_{ij}=u_j^{2S-s_i}v_j^{s_i}$, Γ maps $\hat{D}\to D$. We notice again that D and $\hat{D}$ are not normalized.

Let us put $\hbar=1$ in (13) and let us consider the total angular momentum for a system of particles on the sphere. Since for one particle, $L_3u^{2S-k}v^k=(S-k)u^{2S-k}v^k$ (for $k=0,1,\ldots,2S$), we have

$$L_3\hat{D}(s_1,s_2,\ldots,s_N)=\left[NS-\sum_{i=1}^N s_i\right]\hat{D}(s_1,s_2,\ldots,s_N)\,,$$

and since the Slater determinants form a basis for the many-particle space, we have

$$L_3p=(NS-M)p \tag{30}$$

for any homogeneous polynomial p of total degree M.

We are now in the position to prove the following elementary theorem.

Theorem. The stereographic mapping Γ^{-1} maps homogeneous translation invariant polynomials $P(z_1,z_2,\ldots,z_N)$ with total degree $M=NS$ into fully rotationally invariant polynomials $p(u_1,v_1;u_2,v_2;\ldots;u_N,v_N)$.

Proof. The square of the total angular momentum operator applied to a homogeneous polynomial p on the sphere gives, in virtue of (30),

$$L^2p=[L^-L^++(NS-M)^2+(NS-M)]p\,, \tag{31}$$

where $L^-=\sum_{i=1}^N v_i(\partial/\partial u_i)$ and $L^+=\sum_{i=1}^N u_i(\partial/\partial v_i)$. On the other hand, from the relations $u(\partial/\partial v)(u^{2S-k}v^k)=ku^{2S-k+1}v^{k-1}$ and $(\partial/\partial z)z^k=kz^{k-1}$ (which hold for $k=0,1,\ldots,2S$), we see that

$$\Gamma^{-1}\sum_{i=1}^N\frac{\partial}{\partial z_i}\Gamma=L^+\,. \tag{32}$$

Suppose now that $P=\Gamma p$ is a translationally invariant polynomial, i.e.,

$$P(z_1+c,z_2+c,\ldots,z_N+c)=P(z_1,z_2,\ldots,z_N)\forall c\in C\ .$$

Then $(\sum_{i=1}^{N}\partial/\partial z_i)P=0$ and

$$L^+p=\Gamma^{-1}\sum_{i=1}^{N}\frac{\partial}{\partial z_i}\Gamma p=0\ . \tag{33}$$

Since we have assumed that $M=NS$, it follows from (31) that $L^2p=0$. The theorem is thus proven.

As an example of an application of the theorem, let us consider Laughlin's polynomial

$$P=\prod_{\substack{i,j=1\\ i<j}}^{N}(z_i-z_j)^m\ .$$

Since the maximum degree in one variable is $(N-1)m$, it is reasonable to assume that $2S=m(N-1)$. The function $\Gamma^{-1}P$ is just the Laughlin-Jastrow wave function on the sphere proposed by Haldane, i.e.,

$$\psi_m^L=\prod_{\substack{i,j=1\\ i<j}}(u_iv_j-v_iu_j)^m\ . \tag{34}$$

Since P is translationally invariant $M=m(N(N-1)/2)=NS$, we recover the known result $L^2\psi_m^L=0$.

Given an expansion in Slater determinants of a wave function defined on the sphere,

$$p=\sum_{s_1,s_2,\ldots,s_N}c_{s_1s_2\cdots s_N}\hat{D}(s_1,s_2,\ldots,s_N)\ , \tag{35}$$

the condition $L^+p=0$ often gives rise to useful identities. For instance, let us consider the expansion of ψ_3^L in Slater determinants for $N=3$ leaving undetermined the coefficients:

$$\psi_3^L=c_1\hat{D}(0,3,6)+c_2\hat{D}(1,2,6)+c_3\hat{D}(0,4,5)+c_4\hat{D}(1,3,5)+c_5\hat{D}(2,3,4)\ . \tag{36}$$

It is easy to verify that the condition $L^+\psi_3^L=0$ determines the coefficients c_i up to a factor (a possible solution being $c_1=1$, $c_2=c_3=-3$, $c_4=6$, and $c_5=-15$). This means that for $N=3$ Laughlin's state is the only rotationally invariant state that can be constructed in the subspace defined by (36). It follows that ψ_3^L is an exact eigenstate of the correlation Hamiltonian, as has already been pointed out by Haldane.

The following question arises: supposing that p is a good variational wave function on the sphere, does it follow that $P=\Gamma p$ is a good variational wave function on the disk (or vice versa)? We have no general results that guarantee that this is the case, but we have the feeling that this is very often true. Laughlin's wave function, for instance, is very good both on the sphere and on the disk (slightly better on the sphere, see the case $N=3$).

As we have already announced in the Introduction, we have mapped on the sphere a few interesting wave functions that have been previously proposed (see Refs. 2 and 41). In particular, we have considered the following polynomials that describe explicitly the correlation between *pairs* of electrons:

$$P=A\left[\prod_{\substack{i,j=1\\ i<j}}^{N}(z_i-z_j)^s\prod_{k=1}^{N/2}(z_{2k}-z_{2k-1})^{-t}P_B^u\right]\ , \tag{37}$$

where s, t, $u\in Z$, $s-t\geq 1$ odd, $s\geq 1$, $u\geq 0$, A is the antisymmetrization operator and P_B is given by

$$P_B=\prod_{\substack{k,\lambda=1\\ k<\lambda}}^{N/2}[z_{2k}z_{2k-1}+z_{2\lambda}z_{2\lambda-1}-\tfrac{1}{2}(z_{2k}+z_{2k-1})(z_{2\lambda}+z_{2\lambda-1})]$$
$$=\prod_{\substack{k,\lambda=1\\ k<\lambda}}^{N/2}\{\tfrac{1}{4}[(z_{2k-1}-z_{2\lambda-1})^2+(z_{2k-1}-z_{2\lambda})^2+(z_{2k}-z_{2\lambda-1})^2+(z_{2k}-z_{2\lambda})^2]-\tfrac{1}{2}[(z_{2k}-z_{2k-1})^2+(z_{2\lambda}-z_{2\lambda-1})^2]\}\ . \tag{38}$$

TABLE I. Ground-state energy per particle for $\nu=\frac{1}{3}$. ψ_0 is the Laughlin wave function.

N	Dimension of the CI matrix	E_0	$E(\frac{1}{3})$	$\langle\psi_0\mid\psi_{\text{exact}}\rangle$
3	5	−0.503 779	−0.503 779	1
4	15	−0.474 815	−0.475 024	0.998 04
5	73	−0.459 402	−0.459 510	0.999 06
6	338	−0.449 954	−0.450 173	0.996 44
7	1656	−0.443 521	−0.443 735	0.996 36
8	8512	−0.438 865	−0.439 096	0.995 40
9	45 207	−0.435 339	−0.435 591	0.994 06
10	246 448		−0.432 840	
∞		−0.4116	−0.4117	

P_B is translationally invariant. Furthermore, the total degree M is given by

$$M=\frac{N(N-1)}{2}s-\frac{N}{2}t+\frac{N}{2}\left[\frac{N}{2}-1\right]u\ , \tag{39}$$

and the maximum degree $N_{\max}$ in one variable is

$$N_{\max}=(N-1)s-t+\left[\frac{N}{2}-1\right]u\ .$$

Let us now apply the mapping Γ^{-1}; we see from the theorem that an isotropic state is obtained if

$$2S=\frac{2M}{N}=(N-1)s-t+\left[\frac{N}{2}-1\right]u\ .$$

TABLE II. Quasihole energy for $\nu=\frac{1}{3}$. $\psi_0=\tilde{\psi}_3^-$ [see Eq. (47)].

N	Dimension of the CI matrix	$\tilde{\varepsilon}(\frac{1}{3})$	$\langle\psi_0\mid\psi_{\text{exact}}\rangle$
3	6	0.042 701	1
4	23	0.037 821	0.995 78
5	98	0.035 489	0.997 14
6	464	0.033 623	0.993 49
7	2306	0.032 574	0.994 00
8	11 975	0.031 715	0.993 25
9	64 071	0.031 026	0.992 23
∞		0.0264	

TABLE III. Quasiparticle energy for $\nu=\frac{1}{3}$. $\psi_0=\tilde{\psi}_3^+$ [see Eq. (47)].

N	Dimension of the CI matrix	$\tilde{\varepsilon}_+(\frac{1}{3})$	$\langle\psi_0\mid\psi_{\text{exact}}\rangle$
3	3	0.119 676	1
4	11	0.104 692	0.995 45
5	46	0.098 959	0.996 25
6	217	0.093 635	0.992 73
7	1069	0.091 013	0.993 28
8	5529	0.088 775	0.992 33
9	29 463	0.087 000	0.991 38
∞		0.0772	

This is precisely the value of $N_{\max}$ for the polynomial P_B. It follows that the image of P on the sphere corresponds to a rotationally invariant state.

Halperin[2] and Morf and Halperin[41] consider also the following pairing factor:

$$P_B'=\prod_{\substack{k,\lambda=1\\k<\lambda}}^{N/2}[z_{2k}+z_{2k-1}-(z_{2\lambda}+z_{2\lambda-1})]^2 . \tag{40}$$

An analogous computation shows that the image on the sphere does not correspond to a rotationally invariant state, since $N_{\max}=(N-1)s-t+2(N/2-1)u>2M/N$.

The filling factor for the isotropic case (38) is given by

$$\nu^{-1}=\frac{N_{\max}}{N}=s+\frac{u}{2}-\frac{(s+t+u)}{N}\to s+\frac{u}{2}\quad\text{for large } N . \tag{41}$$

It may be verified immediately that for $t=1$, $s=2$, and $u=1$ the wave function (37) is an exact solution of the eigenvalue problem for four particles; writing

$$P=c_1\hat{D}(0,1,5,6)+c_2\hat{D}(0,2,4,6)+c_3\hat{D}(1,2,3,6)+c_4\hat{D}(1,2,4,5)+c_5\hat{D}(0,3,4,5) , \tag{42}$$

the condition $L^+P=0$ determines the coefficients c_i up to a factor (a possible solution being $c_1=1$, $c_2=-\frac{5}{2}$, $c_4=-15$, $c_3=c_5=10$). In this respect there is a strong analogy between the pair function (37) and Laughlin's function: in the subspace spanned by the functions (42) there is only one normalized rotationally invariant wave function.

In the next section numerical results for six electron are given for $s=2$ and $s=4$. The same kind of analysis can be applied to a wave function proposed by Girvin and Jach:[39]

$$\psi=\prod_{\substack{i,j=1\\i<j}}^{N}\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right]^p\prod_{\substack{k,l=1\\k<l}}^{N}(z_k-z_l)^{m+p} . \tag{43}$$

The image of ψ on the sphere turns out not to be rotationally invariant.

An obvious consequence of formula (31) is that any translation invariant homogeneous polynomial on the sphere of total degree M is an eigenstate of L^2 corresponding to the eigenvalue $NS-M$. Consequently, starting from Laughlin's polynomial $P_L=\prod_{i<j}(z_i-z_j)^m$, we can construct approximate excited states in a simple way: it is sufficient to apply to P_L any homogeneous symmetrical polynomial in the derivatives $\partial/\partial z_i$ (for $i=1,2,\ldots,N$).

Two interesting possibilities are offered by the polynomials:

$$P_{K,N}=\left[\sum_{i=1}^{N}\frac{\partial^K}{\partial z_i^K}\right]P_L , \tag{44}$$

$$Q_{K,N}=\left[\sum_{1\le i_1<i_2<\cdots<i_K\le N}\frac{\partial}{\partial z_{i_1}}\frac{\partial}{\partial z_{i_2}}\cdots\frac{\partial}{\partial z_{i_K}}\right]P_L , \tag{45}$$

TABLE IV. Energy gap $\tilde{\Delta}$ and exciton energy Δ_{exc} for $\nu=\frac{1}{3}$.

N	$\tilde{\Delta}=\tilde{\varepsilon}_++\tilde{\varepsilon}_-$	E_0	E	Δ_{exc}	L	$\langle\psi_0\mid\psi_{\text{exact}}\rangle$
3	0.162 377	−0.464 116	−0.464 116	0.118 99	2	1
4	0.142 513	−0.451 641	−0.451 641	0.093 53	3	1
5	0.134 448	−0.440 829	−0.440 887	0.093 11	3	0.998 98
6	0.127 258	−0.436 074	−0.436 500	0.082 04	4	0.989 71
7	0.123 587	−0.431 970	−0.432 188	0.080 83	4	0.995 60
8	0.120 490	−0.428 458	−0.428 896	0.081 60	5	0.987 74
9	0.118 027	−0.426 695	−0.427 113	0.076 30	5	0.987 25
∞	0.1036			0.0766		

that correspond to $L^2=K(K+1)$ and $L_3=K$.

Another important example of excited state has been proposed by Morf and Halperin:[41]

$$R_{K,N}=A\left[\frac{1}{(z_1-z_2)^2}\prod_{j=3}^{k}\frac{(z_1-z_j)+(z_2-z_j)}{(z_1-z_j)(z_2-z_j)}\right]P_L\ , \tag{46}$$

where A denotes the antisymmetrizer. However for numerical tests, $P_{K,N}$ and $Q_{K,N}$ are easier to use, since no antisymmetrization is required.

Finally, we notice that for $K=N$, $Q_{K,N}$ reduces to the quasiparticle state proposed by Laughlin, and $R_{K,N}$ reduces to the quasiparticle state proposed by Halperin.[2] However, one should remember that when the state represents a quasiparticle, the value of S must be suitably changed ($S\rightarrow S-\frac{1}{2}$).

IV. NUMERICAL RESULTS AND CONCLUSIONS

We have adapted a numerical method used for full configuration interaction (full CI) calculations in quantum chemistry[42] to the case of aligned spins. In this way it has been possible to treat large Hamiltonian matrices (for $\nu=\frac{1}{3}$ and $N=10$ the dimension of the CI matrix is 246 448). Use was made also of the Lanczos method, which consists in the diagonalization of the matrix H in the subspace generated by the vectors $H^k|\psi_0\rangle$, where $|\psi_0\rangle$ is a given starting vector. The choice of the simple sequence $H^k|\psi_0\rangle$ is convenient since the symmetry of $|\psi_0\rangle$ is preserved in the iteration procedure.

We have repeated the calculations performed in Ref. 36, adding a few more cases. For the cases of the quasiparticle and quasihole energies, we have used the Lanczos procedure, assuming, respectively, the starting vectors:

$$\tilde{\psi}_m^{+}=\prod_i\frac{\partial}{\partial v_i}\psi_{\text{exact}}^{(L=0)}\ , \tag{47}$$

$$\tilde{\psi}_m^{-}=\prod_i v_i\,\psi_{\text{exact}}^{(L=0)}\ , \tag{48}$$

where $\psi_{\text{exact}}^{(L=0)}$ is the ground state.

We have computed the quasihole and quasiparticle energies $\tilde{\varepsilon}_-$ and $\tilde{\varepsilon}_+$, respectively, by keeping *the radius R constant.* This corresponds to the prescription suggested in Refs. 11 and 41 for computing the "proper energies." In order to keep R constant, the field B has to vary when going from the ground state to the quasihole or quasiparticle state, since $S\rightarrow S\pm\frac{1}{2}$. Consequently, also the eigenvalue $\hbar\omega_c(N/2)$ of the kinetic energy H_0 has to vary. However, these variations are exactly opposite to one another, so that the relevant quantity $\tilde{\Delta}=\tilde{\varepsilon}_++\tilde{\varepsilon}_-$ does not depend on H_0. ($m\tilde{\Delta}$ is equal to the discontinuity in slope of the energy curve.[2,41])

In our calculations of $\tilde{\varepsilon}_-$ and $\tilde{\varepsilon}_+$, we have also included a correction proposed in Ref. 36, which consists of assuming a neutralizing charge of $|e|(N\pm m^{-1})$ in the background, since, physically, quasiholes and quasiparticles are charged objects. This amounts simply to replacing the energy shift $-\frac{1}{2}N^2/R$, due to the effect of the background, by the more precise value $-\frac{1}{2}[N^2-(1/m^2)]/R$.

We denote by E_0 the initial value of the energy per particle $\langle\psi_0|H|\psi_0\rangle/N$, and by ψ_{exact} the final wave function obtained by the iteration procedure. The quantity Δ_{exc} denotes the exciton energy, defined as the minimum value of the excitation gap with respect to the "wave number" L/R; L denotes the angular momentum of the excitation.

In Tables I–VIII values of E_0, $E(\nu)$, $\tilde{\varepsilon}_-$, $\tilde{\varepsilon}_+$, $\tilde{\Delta}$, Δ_{exc}, L, and of the scalar product $\langle\psi_0|\psi_{\text{exact}}\rangle$ are exhibited for $\nu=\frac{1}{3}$ ($N=3,4,\ldots,9$) and $\nu=\frac{1}{5}$ ($N=3,4,\ldots,7$). In the case of $\nu=\frac{1}{3}$, $N=10$ we have only computed the value of $E(\nu)$ due to the large dimension of the matrices involved. The results are extrapolated to $N=\infty$ by considering the various energies as functions of N^{-1}, and fitting the data with a second-order polynomial obtained by

TABLE V. Ground-state energy per particle for $\nu=\frac{1}{5}$. ψ_0 is the Laughlin wave function.

N	Dimension of the CI matrix	E_0	$E(\frac{1}{5})$	$\langle\psi_0\|\psi_{\text{exact}}\rangle$
3	13	−0.398 570	−0.398 570	1
4	86	−0.377 044	−0.377 578	0.984 05
5	649	−0.365 020	−0.365 134	0.997 43
6	5444	−0.357 993	−0.358 716	0.948 65
7	48 417	−0.353 070	−0.353 493	0.976 81
∞		−0.3282	−0.3284	

TABLE VI. Quasihole energy for $\nu=\frac{1}{5}$. $\psi_0=\tilde{\psi}_5^-$ [see Eq. (48)].

N	Dimension of the CI matrix	$\tilde{\varepsilon}_-(\frac{1}{5})$	$\langle\psi_0\|\psi_{\text{exact}}\rangle$
3	15	0.012 694	1
4	102	0.010 550	0.979 74
5	783	0.010 336	0.992 42
6	6615	0.008 494	0.946 08
7	59 105	0.009 069	0.978 45
∞		0.0071	

TABLE VII. Quasiparticle energy for $\nu=\frac{1}{5}$. $\psi_0=\tilde{\psi}_5^+$ [see Eq. (47)].

N	Dimension of the CI matrix	$\tilde{\varepsilon}_+(\frac{1}{5})$	$\langle\psi_0\|\psi_{\text{exact}}\rangle$
3	10	0.027 873	1
4	67	0.023 367	0.997 43
5	511	0.022 566	0.997 72
6	4306	0.019 699	0.981 02
7	38 375	0.020 186	0.994 38
∞		0.0173	

TABLE VIII. Energy gap $\tilde{\Delta}$ and exciton energy Δ_{exc} for $\nu=\frac{1}{5}$.

N	$\tilde{\Delta}=\tilde{\varepsilon}_+ +\tilde{\varepsilon}_-$	E_0	E	Δ_{exc}	L	$\langle\psi_0 \mid \psi_{\text{exact}}\rangle$
3	0.040 567	−0.389 336	−0.389 336	0.027 703	2	1
4	0.033 918	−0.372 708	−0.372 817	0.019 044	3	0.998 82
5	0.032 902	−0.360 907	−0.361 008	0.020 628	3	0.994 76
6	0.028 192	−0.356 214	−0.356 400	0.013 898	4	0.989 84
7	0.029 255	−0.351 064	−0.351 173	0.016 245	4	0.992 67
∞	0.0244			0.0160		

TABLE IX. Some tests of the Halperin wave function (37) for $t=1$, $u=1$, and $s=2,4$. ν denotes the asymptotic filling factor $s+u/2$.

N	Dimension of the CI matrix	D	s	ν	E_0	$E(\nu)$	$\langle\psi_0 \mid \psi_{\text{exact}}\rangle$
4	5	1	2	$\frac{2}{5}$	−0.550 098	−0.550 098	1
6	58	3	2	$\frac{2}{5}$	−0.499 378	−0.500 400	0.988 39
4	43	2	4	$\frac{2}{9}$	−0.416 932	−0.416 941	0.999 94
6	2137		4	$\frac{2}{9}$	−0.386 953	−0.387 140	0.994 07

TABLE X. Test of the quasiparticle wave function (46) ($K=N$). E_0 and E denote total energies.

N	Dimension of the CI matrix	D	E_0	E	$\langle\psi_0 \mid \psi_{\text{exact}}\rangle$
4	11	2	−1.930 110	−1.932 092	0.992 80
5	46	3	−2.319 236	−2.320 043	0.998 05

TABLE XI. Test of the exciton wave function (46) ($L=K<N$). E_0 and E denote total energies.

N	Dimension of the CI matrix	$L=K$	E_0	E	$\langle\psi_0 \mid \psi_{\text{exact}}\rangle$
5	66	3	−2.204 145	−2.204 433	0.998 98
5	63	4	−2.199 338	−2.199 566	0.999 14

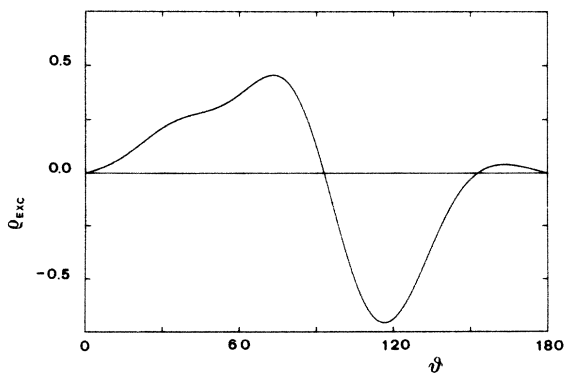

FIG. 1. Charge density ρ_{exc} versus θ for $\nu=\frac{1}{3}$, $N=9$, and $L=5$.

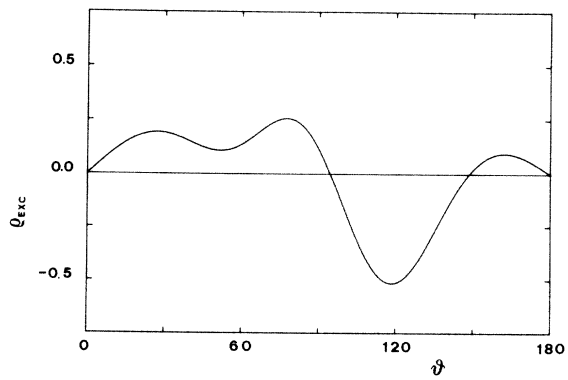

FIG. 2. Charge density ρ_{exc} versus θ for $\nu=\frac{1}{5}$, $N=7$, and $L=4$.

a least-squares method. The extrapolated values are for $\nu=\frac{1}{3}$, $E(\nu)=-0.4117\pm0.001$, $\tilde{\Delta}=0.1036\pm0.002$, $\Delta_{\rm exc}=0.0766\pm0.002$, and for $\nu=\frac{1}{5}$, $E(\nu)=-0.3284\pm0.002$, $\tilde{\Delta}=0.0244\pm0.003$, $\Delta_{\rm exc}=0.016\pm0.005$. Our value of $E(\frac{1}{5})$ is slightly larger than Laughlin's value (-0.334 ± 0.0028). However, since our calculations for $\nu=\frac{1}{5}$ have been carried out only up to $N=7$, this does not constitute a serious discrepancy. On the order hand, it should be noticed that the following second-order polynomials

$$-0.411\,67-0.184\,86N^{-1}-0.274\,289N^{-2} \text{ for } \nu=\tfrac{1}{3}$$

$$-0.328\,39-0.149\,712N^{-1}-0.182\,958N^{-2} \text{ for } \nu=\tfrac{1}{5}$$

fit the values of $E_N(\nu)$ very accurately.[43] We have repeated the calculations of $E(\frac{1}{3})$ replacing the Coulomb potential by the inverse of the arc distance between two points on the sphere. The results remain substantially stable.

Our values of Δ and $\Delta_{\rm exc}$, are in agreement with the results of Haldane and Rezayi,[15] Halperin and Morf,[41] and Girvin, MacDonald, and Platzman,[37] and lie above the results of Laughlin[7] and Chakraborty.[44]

Figure 1 shows the behavior of the charge density $\rho^{(\theta)}_{\rm exc}=\sin\theta\int_0^{2\pi}\rho(\theta,\varphi)d\varphi$ of the exciton for $\nu=\frac{1}{3}$, $N=9$, and $L=5$. The charge q of the exciton turns out to be equal to 0.399. Figure 2 shows $\rho_{\rm exc}$ versus θ for $\nu=\frac{1}{5}$, $N=7$, and $L=4$. The charge q turns out to be equal to 0.253. We have also computed some values of the "quasiexciton" dispersion relation[36] $\Delta E(L/R)$ for $\nu=\frac{1}{3}$, $N=9$ and for $\nu=\frac{1}{5}$, $N=7$. The results (see Fig. 3) are in good agreement with those of Ref. 37.

Finally, we have tested the correlated pairs wave function (37) for $t=1$, $u=1$, $s=2$, $N=4,6$ ($\nu\sim\frac{2}{5}$), and $t=1$, $u=1$, $s=4$, $N=4,6$ ($\nu\sim\frac{2}{9}$). As seen in Table IX, these wave functions appear to be quite satisfactory. However, the technical difficulties of the calculation increase very rapidly with N, since our computer program requires an explicit expansion in Slater determinants as starting vector of the iteration procedure.

We have tested the quasiparticle wave function (46) for $K=L=N$ (see Table X) and for $K=L<N$ (see Table XI). The results are again satisfactory but not conclusive due to the small number of electrons.

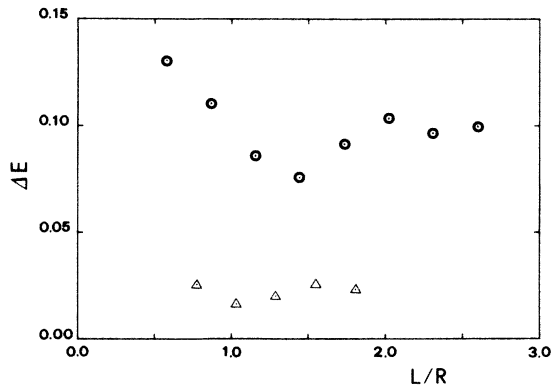

FIG. 3. Gap ΔE versus "wave vector" L/R. Circles are for $\nu=\frac{1}{3}$ and $N=9$ and triangles for $\nu=\frac{1}{5}$ and $N=7$.

We would like to add a final remark. Not only are there cases in which the $L=0$ states do not exist [see Fig. 4(d) of Ref. 36] but there are also cases where they do exist but are excited states. A simple example is provided by the case $N=4$, $S=\frac{7}{2}$, in which case the ground state corresponds to $L=2$ ($E_0=-0.507\,730$), and the only state with $L=0$ is an excited state ($E_0=-0.488\,145$). This kind of "broken symmetry" deserves further investigation.

ACKNOWLEDGMENTS

The authors thank B. I. Halperin and R. Morf for correspondence and suggestions, as well as for supplying them with results prior to publication. They are particularly grateful to E. Tosatti for constant help and encouragement, and would like to acknowledge useful conversations with G. Baskaran, R. B. Laughlin, G. F. Giuliani, and R. Tao. The authors would also like to thank I. Bonfà for assistance in writing a central routine of the package in Cray Assembly Language.

APPENDIX

We want to prove formula (25). Let us denote by $|J,J\rangle_2$ the two-particle wave function ($2S-J$ odd):

$$|J,J\rangle_2=(u_1v_2-u_2v_1)^{2S-J}u_1^Ju_2^J \tag{A1}$$

($\psi^{(S,J)}_{(1,0)}$ in Haldane's notation). The effect of the stereographic mapping applied to $|J,J\rangle_2$ is simply

$$\Gamma\,|J,J\rangle_2=(z_2-z_1)^{2S-J}\,. \tag{A2}$$

Therefore $NS-M=2S-(2S-J)=J$. From (31) it follows that the corresponding eigenvalues of L^2 and L_3 are, respectively, $J(J+1)$ and J. Since $V_J^{(S)}$ is independent of the magnetic quantum number, it can be computed by the relation

$$V_J^{(S)}=\left\langle J,J\left|\frac{1}{|\Omega_1-\Omega_2|}\right|J,J\right\rangle_2 N_0^{-1}\,, \tag{A3}$$

where the normalization factor N_0 is given by

$$N_0=\langle J,J\,|\,J,J\rangle_2 = \int d\Omega_1\int d\Omega_2\,|u_1|^{2J}|u_2|^{2J}|u_1v_2-u_2u_1|^{4S-2J}\,, \tag{A4}$$

and Ω_1 and Ω_2 are two unit vectors on the sphere.

Using polar coordinates, we can write:

$$\Omega_1\cdot\Omega_2=\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)+\cos\theta_1\cos\theta_2 =\cos\theta\,. \tag{A5}$$

Therefore defining u_1,v_1,u_2,v_2 as in (16), we have

$$|\Omega_1-\Omega_2|^2=2(1-\cos\theta)=4\,|u_1v_2-v_1u_2|^2\,. \tag{A6}$$

It follows that the numerator of the right-hand side of (A3) is given by the integral

$$I=\tfrac{1}{2}\left\langle J,J\left|\frac{1}{|u_1v_2-v_1u_2|}\right|J,J\right\rangle$$

$$=\tfrac{1}{2}\int d\Omega_1\int d\Omega_2\,|u_1|^{2J}|u_2|^{2J}|u_1v_2-u_2v_1|^{4S-2J-1}\,. \tag{A7}$$

In both expressions (A4) and (A7) it is convenient to perform the unitary transformation

$$\begin{aligned}u_2&=u_1u_2'-v_1^*v_2'\\ v_2&=v_1u_2'+u_1^*v_2'\,,\end{aligned} \tag{A8}$$

since the factor $u_1v_2-u_2v_1$ becomes simply

$$u_1v_2-u_2v_1=v_2'\,. \tag{A9}$$

Let us consider for example the integral (A7) (N_0 can be treated in the same way). Since the unitary transformation (A8) induces an orthogonal transformation $\Omega_2\rightarrow\Omega_2'$ on the points of the sphere, we can write:

$$I=\tfrac{1}{2}\int d\Omega_1\int d\Omega_2'\,|u_1|^{2J}|u_1u_2'-v_1^*v_2'|^{2J}|v_2'|^{4S-2J-1}$$

$$=\tfrac{1}{2}\int d\theta_1\int d\varphi_1\int d\theta_2\int d\varphi_2\sin\theta_1\sin\theta_2\left|\cos\frac{\theta_1}{2}\right|^{2J}\left|\sin\frac{\theta_2}{2}\right|^{4S-2J-1}\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{-i\varphi_2}\right|^{2J}\,, \tag{A10}$$

where in the last factor we have neglected the inessential phase factor $\exp[i(\varphi_1+\varphi_2)/2]$. Making use of the identity (with a and b real numbers)

$$\int_0^{2\pi}d\varphi_2\,|a-be^{-i\varphi_2}|^{2J}=2\pi\sum_{k=0}^{J}\binom{J}{k}^2a^{2k}b^{2J-2k}\,,$$

we obtain

$$I=32\pi^2\sum_{k=0}^{J}\binom{J}{k}^2\int_0^{\pi/2}d\theta_1\sin\theta_1^{2J-2k+1}\cos\theta_1^{2J+2k+1}\times\int_0^{\pi/2}d\theta_2\sin\theta_2^{4S-2k}\cos\theta_2^{2k+1}$$

$$=\frac{32\pi^2(2S+1)!}{(4S+2)!(2J+1)!}\sum_{k=0}^{J}\binom{J}{k}^2\frac{(J-k)!(J+k)!(4S-2k)!k!4^k}{(2S-k)!}\,. \tag{A11}$$

As mentioned above, the integral N_0 can be treated in the same way. The result is

$$V_J^{(S)}=\frac{I}{N_0}=\frac{1}{2}\frac{(2S+1)!^2}{(4S+2)!}\frac{\sum_{k=0}^{J}\frac{(4S-2k)!(J+k)!}{(2S-k)!(J-k)!k!}4^{k+1}}{\sum_{k=0}^{J}\frac{(2S-k)!(J+k)!}{(J-k)!k!}}\,. \tag{A12}$$

Denoting by D the differential operator d/dx, we can write

$$\sum_{k=0}^{J}\frac{(2S-k)!(J+k)!}{(J-k)!k!}=\frac{1}{J!}\sum_{k=0}^{J}\binom{J}{k}(2S-k)!(J+k)!=\frac{1}{J!}\sum_{k=0}^{J}\binom{J}{k}\left[D^{2S-k}\frac{1}{1-x}\right]_{x=0}\left[D^{J+k}\frac{1}{1-x}\right]_{x=0}$$

$$=\frac{1}{J!}\sum_{k=0}^{J}\binom{J}{k}\left[D^{J-k}D^{2S-J}\frac{1}{1-x}\right]_{x=0}\left[D^kD^J\frac{1}{1-x}\right]_{x=0}=\frac{1}{J!}D^J\left[\left[D^{2S-J}\frac{1}{1-x}\right]\left[D^J\frac{1}{1-x}\right]\right]_{x=0}$$

$$=(2S-J)!D^J\left[\frac{1}{(1-x)^{2S+2}}\right]_{x=0}=\frac{(2S-J)!(2S+J+1)!}{(2S+1)!}\,. \tag{A13}$$

In the same way, since

$$D^{2S-k}\frac{1}{(1-x)^{1/2}}\bigg|_{x=0}=\frac{(4S-2k)!}{4^{2S-k}(2S-k)!}\,,$$

we can write

$$\sum_{k=0}^{J}\frac{(4S-2k)!(J+k)!4^k}{(2S-k)!(J-k)!k!}=\frac{4^{2S}}{J!}\sum_{k=0}^{J}\binom{J}{k}\left[D^{2S-k}\frac{1}{(1-x)^{1/2}}\right]_{x=0}\left[D^{J+k}\frac{1}{1-x}\right]_{x=0}$$

$$=\frac{4^{2S}}{J!}\sum_{k=0}^{J}\binom{J}{k}\left[D^{J-k}D^{2S-J}\frac{1}{(1-x)^{1/2}}\right]_{x=0}\left[D^{k}D^{J}\frac{1}{1-x}\right]_{x=0}$$

$$=\frac{4^{2S}}{J!}D^{J}\left[\left[D^{2S-J}\frac{1}{(1-x)^{1/2}}\right]\left[D^{J}\frac{1}{1-x}\right]\right]_{x=0}=\frac{(4S-2J)!(4S+2J+2)!(2S+1)!}{(2S-J)!(2S+J+1)!(4S+2)!}\,. \tag{A14}$$

From (A12), (A13), and (A14), we obtain

$$V_J^{(S)}=2\frac{\binom{4S-2J}{2S-J}\binom{4S+2J+2}{2S+J+1}}{\binom{4S+2}{2S+1}^2}\,. \tag{A15}$$

For large values of S, J, $2S-J$, $V_J^{(S)}$ has the following asymptotic behavior:

$$V_J^{(S)}\sim\frac{1}{2}\frac{S}{[S^2-(J/2)^2]^{1/2}}\,. \tag{A16}$$

[1]D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982); A. M. Chang, P. Berglund, D. C. Tsui, H. L. Stormer, and J. C. M. Hwang, *ibid.* **53**, 997 (1984).

[2]B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983). In this paper only the form (40) of the pair wave function was considered. The more appropriate form (38) has been suggested to the authors by Dr. R. Morf (private communication).

[3]D. Yoshioka, Aspen Lectures (1984) (unpublished).

[4]H. Fukuyama, P. M. Platzman, and P. W. Anderson, Phys. Rev. B **19**, 5211 (1979).

[5]D. Yoshioka and H. Fukuyama, J. Phys. Soc. Jpn. **47**, 394 (1979).

[6]D. Yoshioka and P. A. Lee, Phys. Rev. B **27**, 4986 (1983).

[7]R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[8]R. B. Laughlin, Physica **126B**, 254 (1984).

[9]L. Bonsall and A. A. Maradudin, Phys. Rev. B **15**, 1959 (1977).

[10]W. P. Su and J. R. Schrieffer, Phys. Rev. Lett. **46**, 738 (1981).

[11]B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[12]F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[13]D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).

[14]D. J. Thouless and Yong-Shi Wu, Phys. Rev. B **31**, 1191 (1985).

[15]F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).

[16]A. H. MacDonald, G. C. Aers, and M. W. C. Dharmawardana, Phys. Rev. B **31**, 5529 (1985).

[17]S. A. Trugman and S. Kivelson, Phys. Rev. B **31**, 5280 (1985).

[18]A. Kallio, P. Pollari, J. Kinaret, and M. Puoskari (unpublished).

[19]S. Kivelson, C. Kallin, D. Arovas, and J. R. Schrieffer, Phys. Rev. Lett. **56**, 873 (1986).

[20]R. Tao and D. J. Thouless, Phys. Rev. B **28**, 1142 (1983).

[21]G. F. Giuliani and J. J. Quinn, Phys. Rev. B **31**, 3451 (1985).

[22]E. Tosatti and M. Parrinello, Lett. Nuovo Cimento **36**, 289 (1983).

[23]Francisco Claro, Solid State Commun. **53**, 27 (1985).

[24]P. W. Anderson, Phys. Rev. B **28**, 2264 (1983).

[25]R. Tao and Yong-Shi Wu, Phys. Rev. B **30**, 1097 (1984).

[26]D. J. Thouless, Phys. Rev. B **31**, 8305 (1985).

[27]W. P. Su, Phys. Rev. B **30**, 1069 (1984).

[28]R. Tao and F. D. M. Haldane, Phys. Rev. B **33**, 3844 (1986).

[29]S. T. Chui, T. M. Hakim, and K. B. Ma, Phys. Rev. B **33**, 7110 (1986).

[30]G. Baskaran (unpublished).

[31]R. B. Laughlin, Phys. Rev. B **27**, 3383 (1983).

[32]S. M. Girvin and Terrence Jach, Phys. Rev. B **28**, 4506 (1983).

[33]D. Yoshioka, B. I. Halperin, and P. A. Lee, Phys. Rev. Lett. **50**, 1219 (1983).

[34]D. Yoshioka, Phys. Rev. B **29**, 6833 (1984).

[35]W. P. Su, Phys. Rev. B **32**, 2617 (1985).

[36]F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985).

[37]S. M. Girvin, A. H. MacDonald and P. M. Platzman, Phys. Rev. Lett. **54**, 581 (1985).

[38]P. A. M. Dirac, Proc. R. Soc. London, Ser. A **133**, 60 (1931).

[39]S. M. Girvin and Terrance Jach, Phys. Rev. B **29**, 5617 (1984).

[40]V. Bargmann, Rev. Mod. Phys. **34**, 829 (1962).

[41]R. Morf and B. I. Halperin, Phys. Rev. B **33**, 2221 (1986).

[42]P. J. Knowles and N. C. Handy, Chem. Phys. Lett. **111**, 315 (1984); P. E. M. Siegbahn, *ibid.* **109**, 417 (1984).

[43]After this work was completed, Dr. R. Morf pointed out to us that a large part of the size dependence of the energy per particle $E_N(\nu)$ is due to the variation of the electron density $\rho_N=N/4\pi R^2=N/4\pi S a_0^2$. Indeed the values of $E_N(\nu)(\rho_N/\rho_\infty)^{-1/2}$ exhibit very small variations. [These values for $N=5,6,7,\ldots 10$ and $\nu=\frac{1}{3}$ are, respectively, -0.410998, -0.410950, -0.410819, -0.410737, -0.410679, and -0.410628. Notice that these values are *increasing*. This might indicate that -0.410628 is a lower bound for $E(\frac{1}{3})$.] The extrapolated value of $E(\nu)$ for $\nu=\frac{1}{3}$ turns out to be -0.410, in agreement with Monte Carlo calculations (Ref. 41).

[44]T. Chakraborty, Phys. Rev. B **31**, 4026 (1985).