Absorption and emission of radiation by plasmons in two-dimensional electron-gas disks

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The theoretical formalism of Ritchie *et al.* is used to characterize the electromagnetic properties of plasma oscillations in two-dimensional electron-gas disks, such as those produced lithographically in the GaAs-$Al_xGa_{1-x}As$ semiconductor heterojunction system. The disks are treated as very thin, oblate spheroids of free-electron gas on the surface of a high-dielectric-constant substrate, and the resonant frequencies are obtained for modes symmetric about the plane of the disk, particularly the lowest-order (dipole) mode. The plasmon fields in this quasi-two-dimensional system are quantized in order to calculate the interaction between the plasmons and the photon field. Closed-form expressions are obtained for the photon absorption cross section and the radiative contribution to the decay width, and the results are compared with those from recent relevant experimental work.

I. INTRODUCTION

The observation of two-dimensional behavior for electrons in modulation-doped GaAs-$Al_xGa_{1-x}As$ structures grown by molecular-beam epitaxy[1,2] has encouraged extensive theoretical and experimental research on these materials. Because of their high electron mobilities, these structures should exhibit well-defined, collective electronic excitations (plasmons). In fact, such plasmons have been observed in heterojunction superlattices[3] and in single modulation-doped heterojunctions;[4] their observed characteristics correspond to theoretical predictions.[5,6]

Much less work has been done on plasmons in confined geometries than in infinite geometries in these high-mobility structures. Plasmons in confined geometries are of practical interest because they can couple directly to the radiation field. In contrast, a periodic perturbation[4,6] must be imposed on plasmons in infinite geometries to satisfy wave-vector conservation in radiative processes. Allen *et al.*[7] used far-infrared absorption measurements to observe plasmons in two-dimensional electron-gas disks (Fig. 1) formed by etching circular mesas into the surface of a single modulation-doped GaAs-Al_xGa_{1-x} heterojunction. They calculated the oscillation frequency of the lowest symmetric mode in these disks by taking the disk limit of the depolarization factor for oblate spheroids.[8]

Ritchie *et al.*[9] gave a general theoretical formalism for treating surface plasmons in any coordinate system in which Laplace's equation is separable and the surface of interest is described by a constant value of one of the coordinates. This formalism includes quantization of the plasmon field as well as other fields (e.g., photons) with which the plasmons interact. Recently, this formalism has been applied to studying the interaction of light with plasmons in ellipsoidal silver particles.[10-12]

In this paper, we describe our application of the formalism of Ritchie *et al.*[9] to plasmons in two-dimensional electron-gas disks of the type described by Allen *et al.*[7] To treat disks, we consider the limit of oblate spheroidal particles as the semiminor-to-semimajor axis ratio of the particles becomes very small. Effects of a polarizable substrate are taken into account with an image-charge method that gives a simple analytic correction in the disk limit to the resonant frequencies of modes that are symmetric about the plane containing the disks. In addition

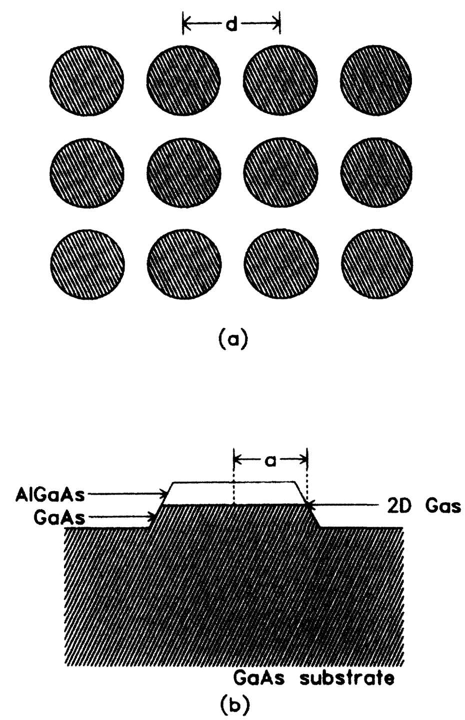

FIG. 1. Geometry for the experiment reported by Allen *et al.* (Ref. 7) (a) Top view showing periodic array of disks with period *d*. (b) Side view of a single disk of radius *a*, composed of a GaAs/$Al_xGa_{1-x}As$ heterojunction on a GaAs substrate. A two-dimensional electron gas is formed at the GaAs/$Al_xGa_{1-x}As$ interface.

to calculating the resonant frequencies, we consider the interaction of plasmons with the radiation field in a dielectric medium and describe both plasmon and photon fields by a second quantization formalism. We also calculate explicitly the absorption cross section and the radiative contribution to the resonant linewidth for the lowest frequency symmetric mode in the electric dipole approximation and compare our results with previous theoretical results and available experimental data.

II. RESONANT FREQUENCIES

In the following discussion of plasma resonances, we assume that the spatial extent of a disk is small compared with the wavelength of light it radiates. In this way, the electrostatic approximation can be used to determine the resonant frequency (i.e., retardation is neglected), and perturbation theory can be used to describe radiative processes. The wave-vector dependence of the dielectric functions used in the calculations is ignored also.

A. General formulation for oblate spheroidal particles

Figure 2 shows the system under consideration. The two-dimensional electron-gas disk is approximated by an oblate spheroidal particle, which is characterized by the semimajor axis length a and the semiminor axis length b. The particle rests on the surface of a semi-infinite substrate whose frequency-dependent dielectric function is $\epsilon_1(\omega)$. The upper half space outside the spheroid is assumed to have a dielectric function $\epsilon_2(\omega)$. A second spheroid (dotted curve), shown below the substrate surface, represents the image of any charge densities that appear on the particle. Fields in the upper-half region due to polarization charges on the surface of the substrate will appear to arise from this image.[13] In real materials, the dielectric function is complex, and oscillations of charge

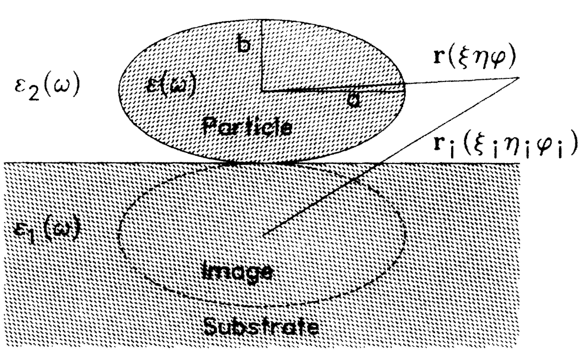

FIG. 2. Geometry used in the calculations. An oblate spheroidal particle of dielectric function $\epsilon(\omega)$ rests on the surface of a semi-infinite substrate of dielectric function $\epsilon_1(\omega)$. The medium surrounding the particle has a dielectric function $\epsilon_2(\omega)$. At a point outside the particle, the electric potential is a superposition of contributions arising from charges on the particle and from image charges in the substrate. The two-dimensional electron-gas disk is obtained in the limit as the ratio of the semiminor axis b to the semimajor axis a approaches zero.

in the particle will lead those of the image by a finite phase. If the imaginary parts of ϵ_1 and ϵ_2 are negligible compared with their real parts, this phase angle is negligible, and we shall ignore it here. This approximation is appropriate for the materials and frequencies considered here. The particle itself is described by a complex, frequency-dependent dielectric function $\epsilon(\omega)$.

Inside the particle, the electrostatic potential is given by[14]

$$\Phi^{\text{in}} = \sum_{l,m,p} A_{lmp}(t) P_l^m(i\xi) Q_l^m(i\xi_0) Y_{lm}^p(\eta,\phi) , \tag{1}$$

where $P_l^m(i\xi)$ and $Q_l^m(i\xi)$ are Legendre functions of imaginary argument, $Y_{lm}^p(\eta,\phi)$ is the real spherical harmonic,[15] and ξ, η, and ϕ are generalized radial, polar, and azimuthal oblate spheroidal coordinates. The particle surface is represented by ξ_0, which is related to the ratio of the semiminor and semiminor axes by

$$\xi_0 = \frac{b/a}{[1-(b/a)^2]^{1/2}} . \tag{2}$$

The expression in Eq. (1) is summed over the integers $l=1,2,3,\ldots$, $m=0,\pm1,\pm2,\ldots,\pm l$, and $p=\pm1$, the various components of multipole moments of the charge distribution. Outside the particle (and above the substrate), the potential can be written as a superposition of the fields[14] due to the particle and its image:

$$\Phi^{\text{out}} = \sum_{l,m,p} B_{lmp}(t) P_l^m(i\xi_0) Q_l^m(i\xi) Y_{lm}^p(\eta,\phi) + \sum_{l',m',p'} C_{l'm'p'}(t) P_{l'}^{m'}(i\xi_0) Q_{l'}^{m'}(i\xi_i) Y_{l'm'}^{p'}(\eta_i,\phi_i) , \tag{3a}$$

whereas within the substrate itself the potential is

$$\Phi^{\text{sub}} = \sum_{l,m,p} D_{lmp}(t) P_l^m(i\xi_0) Q_l^m(i\xi) Y_{lm}^p(\eta,\phi) , \tag{3b}$$

where the coordinates with subscript i belong to an oblate spheroidal coordinate system whose origin is at the center of the image spheroid (see Fig. 2).

In considering the boundary conditions, we anticipate the fact that solutions to the equations of motion are sinusoidal in time, which allows us to use the frequency-dependent dielectric functions. If we require the potential and the normal component of the displacement vector to be continuous at the substrate boundary, we obtain

$$C_{lmp}(t) = (-1)^{l+m} \left[\frac{\epsilon_2(\omega)-\epsilon_1(\omega)}{\epsilon_2(\omega)+\epsilon_1(\omega)} \right] B_{lmp}(t) \equiv g_{lm} B_{lmp}(t) , \tag{4a}$$

$$D_{lmp}(t) = \frac{2\epsilon_2(\omega)}{\epsilon_2(\omega)+\epsilon_1(\omega)} B_{lmp}(t) , \tag{4b}$$

which are the appropriate generalizations of the usual point-image expressions to an extended charge distribution. In deriving Eqs. (4a) and (4b), we have used the reflection symmetry of Y_{lm}^p as well as the fact that $\xi_i=\xi$, $\eta_i=-\eta$, $d\xi_i/dz=-d\xi/dz$, and $d\eta_i/dz=d\eta/dz$ at any point on the substrate boundary.

The potentials in Eqs. (1) and (3) are subject to the boundary conditions that, for $\xi=\xi_0$, the potential and the normal component of the displacement vector must be continuous. Applying these boundary conditions—noting that the two sets of functions $Y_{lm}^p(\eta_i,\phi_i)$ and $Y_{lm}^p(\eta,\phi)$ are not orthogonal to each other—we obtain a condition for the dielectric function of the particle that must be satisfied to obtain normal-mode oscillations. This condition can be written (after much tedious algebra) as

$$\epsilon(\omega_{lm})=\epsilon_2\left[\frac{P_l^m(i\xi_0)[Q_l^m(i\xi_0)]'}{Q_l^m(i\xi_0)[P_l^m(i\xi_0)]'}\right]\frac{N}{D}, \tag{5}$$

where

$$N=Q_l^m(i\xi_0)\left[B_{lmp}(t)P_l^m(i\xi_0)[Q_l^m(i\xi_0)]' + \sum_{l',m',p'} g_{l'm'}B_{l'm'p'}(t)P_{l'}^{m'}(i\xi_0)J_{lmp}^{l'm'p'}\right], \tag{6}$$

and

$$D=[Q_l^m(i\xi_0)]'\left[B_{lmp}(t)P_l^m(i\xi_0)Q_l^m(i\xi_0) + \sum_{l',m',p'} g_{l'm'}B_{l'm'p'}(t)P_{l'}^{m'}(i\xi_0)I_{lmp}^{l'm'p'}\right], \tag{7}$$

and where

$$I_{lmp}^{l'm'p'}\equiv\int_{-1}^{1}d\eta\int_0^{2\pi}d\phi\, Y_{lm}^p(\eta,\phi) \times[Q_{l'}^{m'}(i\xi_i)Y_{l'm'}^{p'}(\eta_i,\phi_i)]_{\xi=\xi_0}, \tag{8}$$

and

$$J_{lmp}^{l'm'p'}\equiv\int_{-1}^{1}d\eta\int_0^{2\pi}d\phi\, Y_{lm}^p(\eta,\phi)\frac{d}{d\xi} \times[Q_{l'}^{m'}(i\xi_i)Y_{l'm'}^{p'}(\eta_i,\phi_i)]_{\xi=\xi_0}. \tag{9}$$

A term of the form $[Q_l^m(i\xi_0)]'$ indicates the derivative of $Q_l^m(i\xi)$ with respect to ξ evaluated at $\xi=\xi_0$. The expression N/D in Eq. (5) is the correction factor for the resonant value[10] of the dielectric function due to polarization charges induced on the surface of the substrate. In general, the integrals I and J in Eqs. (8) and (9) must be evaluated numerically.

The potentials given by Eqs. (1) and (3), together with the conditions (4) and (5) determined by the boundary conditions, satisfy Laplace's equation everywhere in space and *all* boundary conditions. Therefore, they constitute a *unique* solution to the problem. Because of the summation over all different multipole modes B_{lmp} in Eqs. (6) and (7), solutions to the resonant condition, Eq. (5), are generally superpositions of these modes. In the disk limit, however (as we shall see), the symmetric ($l+m$ even) modes decouple completely.

B. Resonant frequencies in the disk limit

It can be shown[16] that in the limit of a disk-shaped particle ($\xi_0\to 0$), the modes decouple, i.e., the off-diagonal terms in the sums in Eqs. (6) and (7) vanish. Thus, the factor N/D in Eq. (5) approaches the simple expression

$$\frac{N}{D}=\frac{\epsilon_1(\omega)+\epsilon_2(\omega)}{2\epsilon_2(\omega)}, \tag{10}$$

for even values of $l+m$. The modes with $l+m$ even represent charge-density distributions on the particle that are symmetric about its plane of symmetry. These distributions do not tend to cancel out as the two surfaces of the particle come together, i.e., as the particle becomes very flat. Because the antisymmetric ($l+m$ odd) modes have much higher resonant frequencies than the symmetric modes in the disk limit, they are not of interest in this work. Since N/D represents the correction factor due to the substrate, we see that a disk lying on the surface of a substrate responds as if it were embedded in a medium of effective dielectric function

$$\epsilon_{\text{eff}}=\frac{\epsilon_1(\omega)+\epsilon_2(\omega)}{2}, \tag{11}$$

which is the average of the two dielectric functions. It is easy to show that this simple result also holds for the case of an infinite, two-dimensional electron gas bounded by two different dielectric media.

We can now find the resonant frequencies for the symmetric ($l+m$ even) modes on a disk by combining Eq. (5) with Eq. (10). For a free-electron gas of volume density n_v embedded in a polarizable medium whose dielectric function is $\epsilon_m(\omega)$, the dielectric function of the particle has the form[17]

$$\epsilon(\omega)=\epsilon_m(\omega)-\frac{\omega_p^2}{\omega(\omega+i\Gamma)}, \tag{12}$$

where ω_p is the bulk plasma frequency of the electron gas in a vacuum:

$$\omega_p=\left[\frac{4\pi n_v e^2}{m^*}\right]^{1/2}, \tag{13}$$

with m^* the effective mass, e the electronic charge, and Γ the relaxation rate. If we define

$$\Lambda_{lm}=-\lim_{\xi_0\to 0}\left[\xi_0\frac{P_l^m(i\xi_0)[Q_l^m(i\xi_0)]'}{Q_l^m(i\xi_0)[P_l^m(i\xi_0)]'}\right], \tag{14}$$

then the relation that determines the resonant frequencies becomes, for small Γ,

$$\omega_{lm}^2=\frac{\xi_0\omega_p^2}{\epsilon_{\text{eff}}\Lambda_{lm}}, \tag{15}$$

for the symmetric modes. If ϵ_{eff} is independent of frequency, Eq. (15) is a direct solution for the resonant frequencies; otherwise, it must be solved as a transcendental equation. For the symmetric dipole mode ($l=1$, $m=1$), $\Lambda_{11}=4/\pi$, and substituting Eq. (13) into Eq. (15) yields the result

$$\omega_{11}^2 = \frac{\pi^2 e^2 n_v b}{m^* \epsilon_{\text{eff}} a} , \tag{16}$$

where we have used the limiting value of $\xi_0 = b/a$ from Eq. (2) as $b/a \to 0$. In this limit, a is the radius of the disk.

The two-dimensional limit of Eq. (16) is obtained by relating the factor $n_v b$ (as $b \to 0$) to a sheet charge density n_s. In the case of a planar electron-gas slab, we define $n_s = n_v t$, where t is the thickness of the slab [which approaches zero in the (two-dimensional) 2D limit]. However, the sheet charge density obtained by projecting a uniformly charged oblate spheroid onto a disk varies as a function of distance from the center of the disk. (In this respect, the model chosen, that of a thin oblate spheroid, does not accurately represent a uniformly charged disk.) Therefore, we must define an average sheet charge density, given by

$$n_s = \frac{n_v V}{A} , \tag{17}$$

where V is the volume of the ellipsoid $(4\pi a^2 b/3)$ and A is the disk area (πa^2). The quantity n_s would be the surface charge density if the charge were uniformly distributed over the disk. Thus, in Eq. (16) we can replace $n_v b$ by $3n_s/4$. For the lowest symmetric mode, we then obtain

$$\omega_{11}^2 = \frac{3\pi^2 n_s e^2}{4m^* \epsilon_{\text{eff}} a} . \tag{18}$$

III. INTERACTION OF PLASMONS WITH THE RADIATION FIELD

To characterize the interaction of the collective oscillations in the two-dimensional electron-gas disks with the radiation field, we first describe the second quantization of both the plasmon field and the photon field and then calculate the matrix element for absorption or emission of a photon for the lowest symmetric disk mode in the electric dipole approximation. Finally, we determine the absorption cross section and the radiative decay rate for this mode. In considering plasmon interactions with the photon field, we treat the photons as if they were immersed in a medium whose dielectric function is ϵ_{eff}, as given in Eq. (11). While this approximation is quite crude [it does not follow naturally, as does Eq. (11) in the context of resonant frequencies], it allows much greater simplification than a rigorous treatment of the photon field in two joined dielectric media.

A. Second quantization formalism

The formalism that we use has been described in detail elsewhere;[10,18] we present only a summary here. We can introduce the plasmon creation and annihilation operators, b_{lmp}^{+} and b_{lmp}, which satisfy the usual boson commutation rules:

$$[b_{lmp}, b_{l'm'p'}^{\dagger}] = \delta_{ll'}\delta_{mm'}\delta_{pp'} ;$$
$$[b_{lmp}, b_{l'm'p'}] = [b_{lmp}^{\dagger}, b_{l'm'p'}^{\dagger}] = 0 . \tag{19}$$

In terms of these, the plasmon Hamiltonian becomes

$$H_{\text{pl}} = \sum_{l,m,p} \tfrac{1}{2}\hbar\omega_{lm}(b_{lmp}b_{lmp}^{\dagger} + b_{lmp}^{\dagger}b_{lmp}) , \tag{20}$$

where $\hbar$ is Planck's constant divided by 2π. The vector potential for the photon field in a medium whose dielectric function is $\epsilon_{\text{eff}}(\omega)$ can be written[19] as follows:

$$\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{k},j} \left[\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}} V \epsilon_{\text{eff}}(\omega_{\mathbf{k}})} \right]^{1/2} (c_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{x}} + c_{\mathbf{k}j}^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{x}})\hat{\epsilon}_{\mathbf{k}j} , \tag{21}$$

where the sum is taken over wave vectors $\mathbf{k}$ and polarizations $j = 1$ and 2, $\hat{\epsilon}_{\mathbf{k}j}\cdot\mathbf{k} = 0$, $\omega_{\mathbf{k}}$ is the frequency of a photon with wave vector $\mathbf{k}$, V is the quantization volume, and $c_{\mathbf{k}j}^{\dagger}$ and $c_{\mathbf{k}j}$ are photon creation and annihilation operators obeying boson commutation relations similar to those in Eq. (19). The photon Hamiltonian is given by

$$H_{\text{ph}} = \sum_{\mathbf{k},j} \tfrac{1}{2}\hbar\omega_{\mathbf{k}}(c_{\mathbf{k}j}c_{\mathbf{k}j}^{\dagger} + c_{\mathbf{k}j}^{\dagger}c_{\mathbf{k}j}) , \tag{22}$$

The interaction Hamiltonian for coupling between the plasmons and the radiation field is given by[10]

$$H_{\text{int}} = \frac{1}{c}\int \mathbf{J}\cdot\mathbf{A}\, d^3x , \tag{23}$$

where $\mathbf{J}$ is the plasmon current density operator. In Eq. (23), the integration extends over all space; i.e., $\mathbf{J}$ arises not only from charge fluctuations on the particle itself but also from the time-varying polarization charge induced in the medium in which the particle is immersed. Equation (23) can be converted into a surface integral and expressed in terms of the creation and annihilation operators, i.e.:

$$H_{\text{int}} = \sum_{\mathbf{k},j}\sum_{l,m,p} \frac{\alpha_{lm}[\epsilon(\omega_{lm}) - \epsilon_{\text{eff}}(\omega_{lm})]}{8\pi c} P_l^m(i\xi_0)Q_l^m(i\xi_0) \left[\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}} V \epsilon_{\text{eff}}} \right]^{1/2}$$
$$\times (b_{lmp} - b_{lmp}^{\dagger}) \int d\mathbf{S}\cdot\hat{\epsilon}_{\mathbf{k}j} Y_{lm}^p(\eta,\phi)(c_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{x}} + c_{\mathbf{k}j}^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{x}}) , \tag{24}$$

where the integral is taken over the surface of the particle ($d\mathbf{S}$ points along the outward normal to the surface), and where[10]

$$\alpha_{lm}^2 = (-1)^m \frac{8\pi(l-m)!\hbar\omega_{lm}^3}{i(l+m)!(a^2-b^2)^{1/2}P_l^m(i\xi_0)Q_l^m(i\xi_0)} . \tag{25}$$

B. Matrix element for photon emission

The state consisting of one (lmp)-mode plasmon and no photons is given by $b_{lmp}^{\dagger}|0\rangle$ and that for a single photon with wave vector $\mathbf{k}$ and polarization j by $c_{\mathbf{k}j}^{\dagger}|0\rangle$. Therefore, the matrix element for decay of a plasmon into a photon is given by

$$\langle 0|c_{\mathbf{k}j}H_{\text{int}}b_{lmp}^{\dagger}|0\rangle \equiv M_{\text{em}} = \frac{\alpha_{lm}[\epsilon(\omega_{lm})-\epsilon_{\text{eff}}(\omega_{lm})]}{8\pi c} P_l^m(i\xi_0)Q_l^m(i\xi_0)\left[\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}V\epsilon_{\text{eff}}}\right]^{1/2} Z_{lmp}^{\mathbf{k}j} , \tag{26}$$

where

$$Z_{lmp}^{\mathbf{k}j} = \int d\mathbf{S}\cdot\hat{\boldsymbol{\epsilon}}_{\mathbf{k}j} Y_{lm}^p(\eta,\phi)e^{-i\mathbf{k}\cdot\mathbf{x}} . \tag{27}$$

The matrix element, Eq. (26), occurs in the expressions for the absorption cross section and the radiative-decay rate, and it has been calculated previously in closed form for a general oblate spheroid without further approximation.[10,18] For our purposes, it is easier to evaluate it explicitly for the lowest symmetric mode in the dipole approximation. We assume that the wave vector of the light $\mathbf{k}$ has angular coordinates (ψ,χ) in a spherical coordinate system centered about the particle and then consider the two orthogonal polarizations s and p, for which the unit polarization vectors are given by

$$\begin{aligned}\boldsymbol{\epsilon}_{\mathbf{k}s} &= \hat{\mathbf{x}}\sin\chi - \hat{\mathbf{y}}\cos\chi ,\\ \boldsymbol{\epsilon}_{\mathbf{k}p} &= -\hat{\mathbf{x}}\cos\psi\cos\chi - \hat{\mathbf{y}}\cos\psi\sin\chi + \hat{\mathbf{z}}\sin\psi .\end{aligned} \tag{28}$$

If we express $d\mathbf{S}$ in oblate spheroidal coordinates and approximate $e^{-i\mathbf{k}\cdot\mathbf{x}}$ by unity, the integral given in Eq. (27) can be evaluated easily; in this limit, it is nonvanishing only for dipole ($l=1$) modes. The results for these modes are as follows:

$$Z_{111}^{\mathbf{k}j} = \left[\frac{4\pi}{3}\right]^{1/2} ab \times \begin{cases} -\sin\chi, & s\text{-polarization} \\ \cos\psi\cos\chi, & p\text{-polarization}\end{cases}$$

$$Z_{11-1}^{\mathbf{k}j} = \left[\frac{4\pi}{3}\right]^{1/2} ab \times \begin{cases} \cos\chi, & s\text{-polarization} \\ \cos\psi\sin\chi, & p\text{-polarization}\end{cases} \tag{29a}$$

$$\tag{29b}$$

$$Z_{10}^{\mathbf{k}j} = \left[\frac{4\pi}{3}\right]^{1/2} a^2 \times \begin{cases} 0, & s\text{-polarization} \\ \sin\psi, & p\text{-polarization}\end{cases} \tag{29c}$$

where a and b are the semimajor and semiminor axes of the ellipsoid.

As discussed earlier, the ($l=1$, $m=0$) antisymmetric mode is not of interest here. For the lowest symmetric mode ($l=1$, $m=1$), we obtain

$$(\epsilon-\epsilon_{\text{eff}})P_1^1(i\xi_0)Q_1^1(i\xi_0) = -\frac{2i}{\xi_0}\epsilon_{\text{eff}}, \tag{30}$$

using Eqs. (5) and (10) and the explicit forms[20] for the Legendre functions. Furthermore, from Eq. (25) we obtain

$$\alpha_{11}^2 = \frac{4\pi\hbar\omega_{11}^3(1+\xi_0^2)^{1/2}}{a[(1+\xi_0^2)\cot^{-1}\xi_0-\xi_0]} . \tag{31}$$

Thus, Eq. (26) produces the following result for the matrix element for emission from the lowest symmetric modes:

$$M_{\text{em}} = -\frac{i}{\xi_0}\left[\frac{\epsilon_{\text{eff}}\omega_{11}^3\hbar^2(1+\xi_0^2)^{1/2}}{2a\omega_{\mathbf{k}}V[(1+\xi_0^2)\cot^{-1}\xi_0-\xi_0]}\right]^{1/2} Z_{11p}^{\mathbf{k}j} . \tag{32}$$

where $Z_{11p}^{\mathbf{k}j}$ is defined in Eq. (29).

C. Absorption cross section

Suppose we have incident on the particle a monochromatic beam of photons of frequency $\omega_{\mathbf{k}}$. It may be easily shown that the matrix element M_{abs} for absorption of one of the photons and creation of a plasmon is related to the matrix element M_{em} described above by

$$M_{\text{abs}} = -\sqrt{n}\,M_{\text{em}}^* , \tag{33}$$

where n is the number of photons in the incident beam. The transition probability per unit time is given[21] by Fermi's golden rule number 2, or

$$w_{\text{abs}} = \frac{2\pi}{\hbar^2}\sum_{l,m,p}|M_{\text{abs}}|^2 g(\omega_{\mathbf{k}}-\omega_{lm}) , \tag{34}$$

where we have replaced the usual δ function by a line-shape function $g(\omega)$, normalized so that

$$\int_{-\infty}^{\infty} g(\omega)d\omega = 1 . \tag{35}$$

The absorption cross section is given by the transition probability per unit time divided by the incident flux of photons ($=cn/V\epsilon_2^{1/2}$ for photons incident from the top in Fig. 2). We assume the linewidth is sufficiently narrow that the sum in Eq. (34) includes only the terms $l=1$, $m=1$, $p=\pm1$. We thus obtain

$$\sigma_{11} = \frac{2\pi V\epsilon_2^{1/2}}{\hbar c}\sum_{p=\pm1}|M_{\text{em}}|^2 g(\omega_{\mathbf{k}}-\omega_{11}) . \tag{36}$$

Applying the results of Eqs. (29) and (32) to Eq. (36), we find

$$\sigma_{11} = \frac{2\pi^2\epsilon_{\text{eff}}\epsilon_2^{1/2}\omega_{11}^3 a^3 g(\omega_{\mathbf{k}}-\omega_{11})}{c\omega_{\mathbf{k}}f(\xi_0)} \times \begin{cases} 1, & s\text{-polarization} \\ \cos^2\psi, & p\text{-polarization}\end{cases} \tag{37}$$

where

$$f(x) \equiv \tfrac{3}{2}(1+x^2)^{1/2}[(1+x^2)\cot^{-1}x - x] . \tag{38}$$

The result of Eq. (37) (with $\epsilon_{\text{eff}}=1$ and $\epsilon_2=1$) agrees with that obtained by a different method by Kennerly *et al.*[12] The disk limit is represented by $x=0$ $[f(0)=3\pi/4]$, and the sphere limit is represented by $x=\infty$ $[f(\infty)=1]$. If we assume a Lorentzian line profile:

$$g(\omega)=\frac{\Gamma}{2\pi[\omega^2+(\Gamma/2)^2]} , \tag{39}$$

Eq. (37) gives the following result for the peak cross section for normal incidence (independent of polarization):

$$\sigma_{11}^{\max}=\frac{4\pi\epsilon_{\text{eff}}\epsilon_2^{1/2}\omega_{11}^2a^3}{c\Gamma f(\xi_0)} , \tag{40}$$

where we have assumed that the linewidth Γ is small compared with ω_{11}. Using Eq. (18) for ω_{11}, we find in the disk limit

$$U^{\max}\equiv\frac{\sigma_{11}^{\max}}{\sigma^{\text{geom}}}=\frac{4\pi n_s e^2\epsilon_2^{1/2}}{m^*c\Gamma} , \tag{41}$$

where $\sigma^{\text{geom}}=\pi a^2$ is the geometrical cross section of the disk. This expression (with $\epsilon_2=1$ for vacuum) leads to the result for the peak sheet conductivity given in Ref. 7.

D. Radiative lifetime

According to Fermi's golden rule number 2, the radiative lifetime of a plasmon is given by[21]

$$\gamma_{lm}^r=\frac{2\pi}{\hbar^2}\sum_{\mathbf{k}j}|M_{\text{if}}|^2\delta(\omega_{\mathbf{k}}-\omega_{lm}) , \tag{42}$$

where the sum is taken over all modes of the radiation field. To be consistent, the δ function in Eq. (42) should be replaced[22] by the normalized line-shape function g, as in Eq. (34); however, for sufficiently sharp resonances ($\Gamma\ll\omega_{lm}$), Eq. (42) is sufficient. We can convert the sum over $\mathbf{k}$ into an integral in the usual fashion (keeping in mind that $\omega_{\mathbf{k}}=ck/\epsilon_{\text{eff}}^{1/2}$):

$$\begin{aligned}\sum_{\mathbf{k}}&=\frac{V}{(2\pi)^3}\int d^3k\\&=\frac{V\epsilon_{\text{eff}}^{3/2}}{(2\pi c)^3}\int d(\cos\psi)d\chi\,\omega_{\mathbf{k}}^2d\omega_{\mathbf{k}} .\end{aligned} \tag{43}$$

We may now use Eqs. (29), (32), and (43) in Eq. (42) and perform the integrations to obtain the limiting result for the radiative lifetime of the lowest symmetric mode:

$$\gamma_{11}^r=\frac{2\epsilon_{\text{eff}}^{5/2}a^3\omega_{11}^4}{3c^3f(\xi_0)} , \tag{44}$$

which, in the limiting case of a disk, becomes

$$\gamma_{11}^r=\frac{8\epsilon_{\text{eff}}^{5/2}a^3\omega_{11}^4}{9\pi c^3}=\frac{\pi^3\epsilon_{\text{eff}}^{1/2}n_s^2e^4a}{2m^{*2}c^3} , \tag{45}$$

where we have used Eq. (18) for the resonant frequency.

IV. DISCUSSION

The preceding results have been examined for their application to collective excitations of the two-dimensional electron gas formed at a GaAs-$Al_xGa_{1-x}As$ heterojunction. Our purpose is to determine, among other things, whether the $l=1$, $m=1$ plasmon can be coupled strongly to the radiation field, given present technological limitations on the radii of the disks and the density of the 2D electron gas.

Figure 3 shows a plot of the frequency ($=\omega/2\pi$) of this mode versus disk radius a for several values of the sheet density n_s. These were calculated from Eq. (18) by assuming a substrate dielectric constant ϵ_1 of 12.86 (appropriate[23] for GaAs in the millimeter-wave region) and an exterior dielectric constant ϵ_2 of 1 (vacuum). The effective mass for electrons in GaAs is taken as 0.0665. We see that the same frequency can be obtained by changing the sheet density or the disk radius. (This could be done also by changing the dielectric constant ϵ_2 of the exterior medium.) In the experiment of Allen *et al.*,[7] $n_s=5.5\times10^{11}/\text{cm}^2$ and $a=1.5$ μm, yielding a frequency $\omega/2\pi=614$. GHz, which differs from the observed value of 575 GHz by only 7%. In contrast, a frequency of 709 GHz is obtained with the theoretical formula of Ref. 7 and our values of m^* and ϵ_{eff}. [Our results disagree with the theoretical results of Ref. 7 because that work did not properly consider the averaging procedure leading to a factor of $\frac{3}{2}$ in Eq. (18). In addition, Ref. 7 apparently neglects an additional factor of $\frac{1}{2}$.]

The primary loss mechanisms considered in the present work are radiative decay and scattering. In terms of the mobility μ, the momentum relaxation time is

$$\tau_{\text{mr}}=\frac{m^*\mu}{e} . \tag{46}$$

Thus, the total relaxation rate Γ [see Eq. (12)] is given by

$$\Gamma=\tau_{\text{mr}}^{-1}+\gamma_{11}^r , \tag{47}$$

where γ_{11}^r is the radiative-decay rate given by Eq. (45). For the parameters of Ref. 7, the radiative-decay width $\gamma_{11}^r/2\pi$ is 0.16 GHz, which is negligible compared with the scattering width $(2\pi\tau_{\text{mr}})^{-1}=16.8$ GHz for the report-

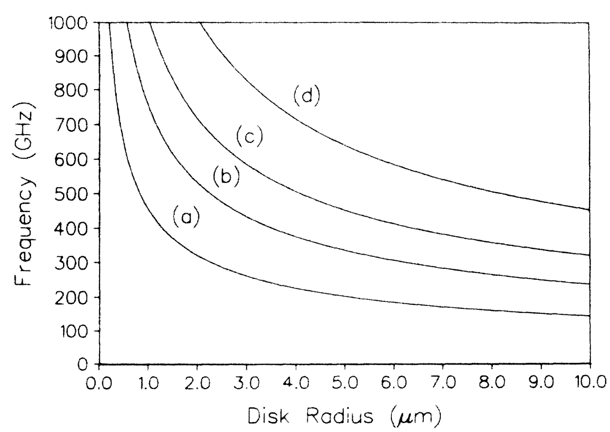

FIG. 3. Resonant frequency of the lowest symmetric mode (dipole mode) of a two-dimensional electron-gas disk as a function of disk radius for various sheet number densities N_s. (a) $n_s=2\times10^{11}$; (b) $n_s=5.5\times10^{11}$; (c) $n_s=1\times10^{12}$; (d) $n_s=2\times10^{12}$ (all in units of cm^{-2}).

ed mobility $\mu = 250\,000$ cm^2/V s. Therefore, the observed resonance linewidth of 50 GHz reported in Ref. 7 must arise from other causes, such as electron scattering from the edges of the disk.

An estimate of the edge-scattering contribution to the resonant linewidth can be obtained from a simple classical picture in which all electrons move at the Fermi velocity $v_F = \hbar(2\pi n_s)^{1/2}/m^*$. We first calculate the length s of a chord passing through a randomly chosen point (x,y) on the disk of radius a, where the angle between the chord and the x axis is θ (also chosen at random). By averaging s over all allowed values of x, y, and θ, we obtain the mean free path for edge collisions as $\langle s \rangle = 16a/3\pi$, and the mean time between edge collisions as $\tau_{ec} = \langle s \rangle / v_F$. From the parameters of Ref. 7, we obtain $(2\pi\tau_{ec})^{-1} = 20.2$ GHz; thus, the total resonance linewidth (momentum-relaxation plus radiative-decay plus edge-collision contributions) becomes 37.2 GHz, which is in much better agreement with the experimental result than the momentum-relaxation contribution alone. Further contributions to the linewidth could result from variations in radius from disk to disk and from the noncircular disk shapes observed in Ref. 7.

It is of interest to consider disks with a given surface number density and to find the disk radius a that maximizes the quality factor Q characterizing the resonance, defined by

$$Q = \frac{\omega_{11}}{\Gamma} \ . \tag{48}$$

If we ignore the radiative-decay contribution (negligible only for small disk radii) to the line width, we find that the maximum Q occurs when the momentum-relaxation and edge-scattering contributions to the line width are equal. This condition holds approximately in the experiment of Ref. 7. In contrast, for much larger disk radii, the edge-scattering contribution is negligible, and the radiative-decay rate can be made comparable to the inverse momentum-relaxation time. For example, for 45-μm-radius disks at $n_s = 10^{12}/\text{cm}^2$, these contributions are both equal to 16.8 GHz, and the resonant frequency is 150 GHz. In this case, radiation by the plasmons is a relatively efficient process.

The transmission for a square array of disks separated by a distance d is given by

$$T = \exp(-fU^{\max}) \ , \tag{49}$$

where $U^{\max}$ is defined in Eq. (41) and $f = \pi a^2/d^2$ is the area filling factor. For the experiment of Ref. 7, $U^{\max} = 2.79$ (we use the observed resonance linewidth) and $f = 0.442$; thus, the minimum transmission is 0.291, and the resonance should be observable (which it is[7]). Quantitative comparison in this case is not appropriate since the disks do not act as independent scatterers when they are spaced very closely together; consequently, Eq. (49) overestimates the absorption (underestimates the transmission).

Thus, we have shown that the theoretical formalism of Ritchie *et al.*[9] can be used to describe plasma oscillations in two-dimensional electron-gas disks. Calculations of the resonant frequency, the absorption cross section, and the radiative lifetime for the lowest-order symmetric disk mode compare favorably with the experimental results of Allen *et al.*,[7] obtained in a GaAs-$\text{Al}_x\text{Ga}_{1-x}$ heterojunction. For sheet charge densities of the order of 10^{12} electrons/cm^2 and disks of the order of 45 μm in radius, easily achievable with present-day GaAs technology, it appears that a reasonable fraction of the energy of plasma oscillations can be transferred to the radiation field at a frequency of about 150 GHz. It remains to be seen whether an efficient nonoptical mechanism (such as tunneling) can be devised to excite these plasma oscillations.

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