Nonanalytic behavior of ultrasonic attenuation in disordered electronic systems

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The sound attenuation coefficient α is calculated in perturbation theory around the Boltzmann result, α^0, for two- and three-dimensional (3D) disordered electronic systems. For 3D systems we calculate impurity density corrections to α^0 up to second order. The second-order correction is found to be nonanalytic in the impurity density. We also calculate the leading nonanalytic low-frequency corrections to α^0 due to electron localization effects up to terms of second order in the impurity density. The theory suggests that for 2D systems there will be singular low-frequency corrections to α^0 even in the presence of a magnetic field. The perturbation theory also shows that the behavior of α near an electronic mobility edge cannot be obtained by exponentiating an ϵ expansion around $d=2$.

I. INTRODUCTION

In recent years there has been a considerable amount of work on nonanalytic low-frequency corrections to the Boltzmann result for the electrical conductivity in disordered two- and three-dimensional electronic systems.[1] For models of noninteracting electrons, these corrections are due to Anderson localization effects: because of disorder, electronic motion is diffusive, which leads to an enhanced low-frequency current relaxation, and to a strong frequency dependence of the current spectrum. The leading nonanalytic corrections to the Boltzmann conductivity are of order $(\hbar/E_F\tau)\ln(\omega\tau)$, and $(\hbar/E_F\tau)^2\sqrt{\omega\tau}$, for 2D and 3D systems, respectively. The small expansion parameter is $\hbar/E_F\tau$, with E_F the Fermi energy and τ the Boltzmann collision time, and $\hbar/E_F\tau$ is proportioned to the impurity density n_i. Much less attention has been given to nonanalytic terms encountered in the static conductivity as a function of n_i. It has been known since the work of Langer and Neal[2] that in 3D systems the leading nonanalytic terms are of order $(\hbar/E_F\tau)^2\ln(E_F\tau/\hbar)$. In 2D, the corresponding result is of order $(\hbar/E_F\tau)\ln(E_F\tau/\hbar)$. This observation is of fundamental importance since it shows that for transport coefficients there is no analog to the virial expansion for equilibrium quantities.

Since the principal reason for ultrasonic attenuation in electronic systems is the decay of phonons into electron-hole pairs, the sound attenuation coefficient α is sensitive to electronic correlations similar to the one which governs the electrical conductivity σ. It is therefore natural to expect for α anomalies of the same kind as the one found for σ, and it is the purpose of the present paper to study this problem.

This paper is organized as follows. In Sec. II we briefly outline the model, show how α can be related to the electronic stress-tensor correlation function, and establish the rules for the perturbation theory. In Sec. III A we calculate for 3D systems the first two impurity-density corrections to the low-frequency Boltzmann attenuation coefficients α_L^0 and α_T^0 for both longitudinal (L) and transverse (T) phonons. They turn out to be of order $(\hbar/E_F\tau)+(\hbar/E_F\tau)^2\ln(E_F\tau/\hbar)$. For completeness we also give the corresponding result for the electrical conductivity σ. The impurity-density corrections to σ have been considered before in Ref. 3, where all processes that lead to logarithmic contributions have been identified. However, there were some minor algebraic errors in the original calculations. Impurity-density corrections for 2D systems will not be considered since in 2D transport is anomalous at low frequencies. In fact, the sound attenuation in the low-frequency limit is not yet understood in 2D.

In Sec. III B we calculate the leading nonanalytic low-frequency corrections to α_L^0 and α_T^0. This problem has been addressed recently by Kotliar and Ramakrishnan[4] and by Houghton and Won.[4(b)] We disagree with some of the results of Ref. 4(a), as we will discuss below. In agreement with Ref. 4(b), we find leading corrections of order $(\hbar/E_F\tau)\ln(\omega\tau)$ and $(\hbar/E_F\tau)^2\sqrt{\omega\tau}$ for $d=2$ and $d=3$ for both α_L and α_T. For $d=2$ we also consider the next order of perturbation theory, in which we find leading singular terms of order $(\hbar/E_F\tau)^2\ln^2(\omega\tau)$. In the case of the conductivity these terms are known to cancel,[5] and this cancellation is a necessary condition for the scaling theory of localization[6] to be valid. For α we find the prefactor of these terms to be nonzero. In contrast to the contribution of order $(\hbar/E_F\tau)\ln(\omega\tau)$, the existence of the terms of order $(\hbar/E_F\tau)^2\ln^2(\omega\tau)$ is not restricted to systems invariant under time reversal. That is, even if time-reversal invariance is broken, e.g., in the presence of a magnetic field, there will be important singular low-frequency contributions to the sound-attenuation coefficient.

We conclude this paper by a discussion of our results. In particular we point out that the existence of the term proportioned to $\ln^2(\omega\tau)$ together with the numerical value

of its prefactor sets rather narrow limits on attempts to extend the perturbation results to describe the sound attenuation near an electronic mobility edge. We find that an ϵ expansion around $d=2$ and exponentiating, a procedure which yields for the conductivity a critical behavior consistent with results obtained otherwise, yields for the sound attenuation a result which is in contradiction to our perturbation theory results. We also comment on the relationship between the longitudinal and transverse sound-attenuation coefficients that was established in Sec. II, and discuss the experimental relevance of our results

II. BASIC FORMALISM

A. The model

To calculate ultrasonic attenuation in the presence of impurities, we use a model derived by Blount and by Tsuneto[7] (see also Kadanoff and Falko[8]) and Schmid.[9] The model starts with bare electrons and ions interacting via long-range Coulomb forces. If the ionic lattice undergoes thermal motion, the electrons will almost coherently follow the ions because of the tendency of the system to maintain local charge neutrality. This observation suggests transformation to a frame of reference which moves locally with the ions. This can be accomplished by means of a unitary transformation,[7] which indeed makes most of the electron-ion interaction vanish.[9] The remaining effective electron-phonon interaction arises from a coupling between the lattice strain and the electronic stress tensor. This feature is in contrast to the more common coupling to the electronic density in the Fröhlich model. An uncritical application of this model to impure systems would yield a spurious extra coupling to long wavelength phonons. Proper elimination of this problem within the Fröhlich model is possible only by taking into account inelastic scattering of electrons off the moving impurities, and it is the main advantage of the Blount-Tsuneto-Schmid model to avoid this complication.

The model Hamiltonian is given by[9] (in this section we choose units such that $\hbar=1$)

$$H=H_0+H_{e\text{-}e}+H_{e\text{-ph}}+H_{\text{ph}} \tag{2.1}$$

with the electronic Hamiltonian

$$H_0=\frac{1}{2m}s(q=0)+\sum_{\mathbf{q}}V(\mathbf{q})n^{\dagger}(\mathbf{q}) , \tag{2.2a}$$

$$H_{e\text{-}e}=\tfrac{1}{2}\sum_{\mathbf{q}}(e^2/q^2)n^{\dagger}(\mathbf{q})n(\mathbf{q}) . \tag{2.2b}$$

$H_{e\text{-ph}}$ is the electron-phonon interaction,

$$H_{e\text{-ph}}=i\sum_{\mathbf{q},b}\frac{[\omega_b(q)]^{1/2}}{mc_b\sqrt{2\rho}}\tau_b(\mathbf{q})[B_b(\mathbf{q})+B_b^{\dagger}(-\mathbf{q})] , \tag{2.3}$$

and H_{ph} is the phonon Hamiltonian in the Debye approximation,

$$H_{\text{ph}}=\sum_{\mathbf{q},b}\omega_b(\mathbf{q})B_b^{\dagger}(\mathbf{q})B_b(\mathbf{q}) . \tag{2.4}$$

In these equations, n,s, and τ denote electronic density, kinetic energy, and stress operators, respectively,

$$n(\mathbf{q})=\sum_{\mathbf{k},\sigma}c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,\sigma}c_{\mathbf{k}+\mathbf{q}/2,\sigma} , \tag{2.5a}$$

$$s(\mathbf{q})=\sum_{\mathbf{k},\sigma}k^2c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\sigma}c_{\mathbf{k}-\mathbf{q}/2,\sigma} , \tag{2.5b}$$

$$\tau_b(\mathbf{q})=\sum_{\mathbf{k},\sigma}(\mathbf{k}\cdot\mathbf{q}/q)[\mathbf{k}\cdot\mathbf{e}_b(\mathbf{q})]c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,\sigma}c_{\mathbf{k}+\mathbf{q}/2,\sigma} . \tag{2.5c}$$

Here $c^{\dagger}_{\mathbf{k},\sigma}$,$c_{\mathbf{k},\sigma}$ denote creation and annihilation operators for electrons with momentum $\mathbf{k}$ and spin index σ. $V(\mathbf{q})$ is the Fourier transform of the electron impurity scattering potential, m is the effective electron mass, and ρ is the ion mass density. $B_b^{\dagger}(\mathbf{q})$ and $B_b(\mathbf{q})$ are creation and annihilation operators for phonon with wave vector $\mathbf{q}$ and polarization index b ($b=$L and $b=$T for longitudinal and transverse phonons), and $\mathbf{e}_b$ is the phonon polarization vector. Finally, $\omega_b(\mathbf{q})=c_bq$ is the bare phonon dispersion relation with sound velocity c_b. In our calculation, we will assume only s-wave electron-impurity scattering. For simplicity, we also restrict ourselves to zero temperature.

B. Correlation function representation for the sound attenuation

It is important that the electronic Hamiltonian, Eq. (2.2), still contains the long-range Coulomb interaction. Its main effect is to screen the electron-phonon interaction, Eq. (2.3). Kadanoff and Falko[8] have treated this effect within a generalized random-phase approximation (RPA), and have shown that within the perfect screening approximation, the ultrasonic attenuation coefficient can be written as

$$\alpha_b(q,\omega)=\frac{\omega}{m^2\rho c_b^3}\mathrm{Im}\{\chi_{\tau_b\tau_b}(q,\omega) -[\chi_{\tau_bn}(q,\omega)]^2/\chi_{nn}(q,\omega)\} . \tag{2.6}$$

The χ are commutator-correlation functions

$$\chi_{AB}(q,\omega)=i\int_0^{\infty}dt\,e^{i(\omega+i0)t} \times\langle\langle[A^{+}(\mathbf{q},t),B(\mathbf{q},0)]\rangle\rangle_{\text{av}} , \tag{2.7}$$

formed with the stress operator τ_b, Eq. (2.5c), and the number density operator n, Eq. (2.5a), respectively. The brackets in Eq. (2.7) stand for the quantum-mechanical expectation value, while the symbol $\langle\ \rangle_{\text{av}}$ denotes the impurity average. These correlation functions are to be calculated for a system given by H_0 alone, *without* electron-phonon and electron-electron interactions.[10] Because of symmetry, one has $\chi_{\tau_Tn}(q,\omega)\equiv0$ (there is no screening of transverse modes), and α_T is just given by the transverse stress-correlation function $\chi_T\equiv\chi_{\tau_T\tau_T}$,

$$\alpha_T(q,\omega)=\frac{\omega}{m^2\rho c_T^3}\mathrm{Im}\chi_T(q,\omega) . \tag{2.6'}$$

A similar simplification for α_L can be obtained for small frequencies. To see this, we use that for small q and ω, χ_{nn} has a diffusive pole,

$$\chi_{nn}(q,\omega)=\frac{dn}{d\mu}\frac{Dq^2}{-i\omega+Dq^2}\ , \tag{2.8}$$

where $dn/d\mu$ is the compressibility, and D is the diffusion coefficient. Consequently, there will be a region of low frequencies $\omega=c_b q$, for which the condition $Dq^2\ll\omega$ holds. In this region the wave number is small compared with the frequency in the natural units of the system, and we need to consider only the long-wavelength limit. Kadanoff and Falko[8] have shown that for low frequencies Eq. (2.6) for α_L simplifies to

$$\alpha_L(q,\omega)=\frac{\omega}{m^2\rho c_L^3}\mathrm{Im}\left[\chi_L(q,\omega)-2\frac{g_{Ln}(q)}{g_{nn}(q)}\chi_{Ln}(q,\omega)+\left[\frac{g_{Ln}(q)}{g_{nn}(q)}\right]^2\chi_{nn}(q,\omega)\right],$$

where $g_{Ln}(q)=\int_{-\infty}^{\infty}(d\omega/\pi)\mathrm{Im}\chi_{Ln}(q,\omega)/\omega$, etc. If we introduce the new operators $\tau(q)\equiv\tau_L(q)-s(q)/d$, and $\delta(q)\equiv s(q)/d-[g_{Ln}(q)/g_{nn}(q)]n(q)$, we can write this as

$$\alpha_L(q,\omega)=\frac{\omega}{m^2\rho c_L^3}\mathrm{Im}[\chi_{\tau\tau}(q,\omega)+2\chi_{\tau\delta}(q,\omega)+\chi_{\delta\delta}(q,\omega)]\ . \tag{2.6''}$$

In the clean limit, $n_i\to 0$, one has $g_{Ln}(q)/g(q)=k_F^2/d$, and $\chi_{\tau\delta}(q,\omega)\sim q^2$, $\chi_{\delta\delta}(q,\omega)\sim q^4$, hence these terms are negligible in the low-frequency region. On the other hand, for $q=0$ we find $\chi_{\tau\delta}(q=0,\omega\to 0)\sim n_i$, $\chi_{\delta\delta}(q=0,\omega\to 0)\sim n_i$, while $\chi_{\tau\tau}(q=0,\omega\to 0)\sim n_i^{-1}$. From this we see that the last two terms in the square brackets in Eq. (2.6'') are of relative order n_i^2, and can again be neglected for our purposes. We conclude that the effect of screening in the longitudinal case is simply to replace the longitudinal stress vertex $(\mathbf{k}\cdot\mathbf{q}/q)^2$ by $(\mathbf{k}\cdot\mathbf{q}/q)^2-k^2/d$, which eliminates the diffusive pole inherent in $\chi_L(q,\omega)$. This is how the elimination of a spurious extra coupling to long wavelength phonons, which are mentioned earlier, works technically. As a result, it is sufficient to calculate the long wavelength transverse sound attenuation,

$$\alpha_T(\omega)=\frac{\omega}{m^2\rho c_T^3}\mathrm{Im}\chi_T(q=0,\omega)\ . \tag{2.9}$$

By symmetry α_L is given by

$$\alpha_L(\omega)=\alpha_T(\omega)[(c_T/c_L)^3]2(d-1)/d,\quad d=2,3\ . \tag{2.10}$$

C. Diagrammatic representation

The remaining task is to calculate the transverse stress correlation function $\chi_T(q,\omega)=\chi_{\tau_T\tau_T}(q,\omega)$. We do this by means of standard diagrammatic techniques.[11] Accordingly, we define retarded and advanced one-particle functions as

$$\left.\begin{matrix}\tilde{G}^R_{\mathbf{k},\mathbf{p}}(\omega)\\ \tilde{G}^A_{\mathbf{k},\mathbf{p}}(\omega)\end{matrix}\right\}=\left\langle\mathbf{k}\left|\frac{1}{\omega-H\pm i0}\right|\mathbf{p}\right\rangle, \tag{2.11a}$$

and their impurity-averaged counterparts,

$$\left.\begin{matrix}G^R_{\mathbf{k},\mathbf{p}}(\omega)\\ G^A_{\mathbf{k},\mathbf{p}}(\omega)\end{matrix}\right\}\equiv\delta_{\mathbf{k},\mathbf{p}}\times\left\{\begin{matrix}G^R_{\mathbf{k}}(\omega)\\ G^A_{\mathbf{k}}(\omega)\end{matrix}\right\}=\left\{\begin{matrix}\langle\tilde{G}^R_{\mathbf{k},\mathbf{p}}(\omega)\rangle_{\mathrm{av}}\\ \langle\tilde{G}^A_{\mathbf{k},\mathbf{p}}(\omega)\rangle_{\mathrm{av}}\end{matrix}\right\}=\delta_{\mathbf{k},\mathbf{p}}[\omega-k^2/2m+\Sigma^{\pm}_{\mathbf{k}}(\omega)]^{-1}\ , \tag{2.11b}$$

where $\Sigma^{\pm}_{\mathbf{k}}(\omega)$ is the self-energy. To lowest order in the impurity density, we have

$$\Sigma^{\pm}_{\mathbf{k}}(\omega)\equiv\pm i/2\tau=\pm\tfrac{1}{2}iE_F/(E_F\tau)\ , \tag{2.11c}$$

due to our restriction to s-wave scattering. In terms of these functions, χ_T has the representation

$$\mathrm{Im}\chi_T(\mathbf{q},\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}dE\int_{-\infty}^{\infty}dE'\,\delta(\omega+E-E')[f(E)-f(E')]\sum_{\mathbf{k},\mathbf{p}}v_T(\mathbf{k})v_T(\mathbf{p})\times\langle\Delta\tilde{G}_{\mathbf{k}-\mathbf{q}/2,\mathbf{p}-\mathbf{q}/2}(E')\Delta\tilde{G}_{\mathbf{p}+\mathbf{q}/2,\mathbf{k}+\mathbf{q}/2}(E)\rangle_{\mathrm{av}}\ , \tag{2.12a}$$

where $f(E)$ is the Fermi function,

$$\Delta\tilde{G}_{\mathbf{k},\mathbf{p}}(\omega)=[\tilde{G}^R_{\mathbf{k},\mathbf{p}}(\omega)-\tilde{G}^A_{\mathbf{k},\mathbf{p}}(\omega)]/(2i)\ , \tag{2.12b}$$

and $v_T(\mathbf{k})=k_xk_y$ (with $\mathbf{q}$ in x direction) is the transverse stress vector. For our purposes the products G^RG^R and G^AG^A occurring in Eq. (2.12a) do not contribute. For small ω we finally obtain

$$\mathrm{Im}\chi_T(q=0,\omega)=\frac{\omega}{2\pi}\mathrm{Re}\left[\sum_{\mathbf{k},\mathbf{p}}v_T(\mathbf{k})\langle\tilde{G}^R_{\mathbf{k},\mathbf{p}}(E_F+\omega)\tilde{G}^A_{\mathbf{p},\mathbf{k}}(E_F)\rangle_{\mathrm{av}}v_T(\mathbf{p})\right]. \tag{2.13}$$

We note that the conductivity is given by a very similar formula, viz.,

$$\mathrm{Re}\sigma(\omega)=\frac{1}{\pi m^2}\mathrm{Re}\left[\sum_{\mathbf{k},\mathbf{p}}v_c(\mathbf{k})\langle\tilde{G}^R_{\mathbf{k},\mathbf{p}}(E_F+\omega)\tilde{G}^A_{\mathbf{p},\mathbf{k}}(E_F)\rangle_{\mathrm{av}}v_c(\mathbf{p})\right], \tag{2.14}$$

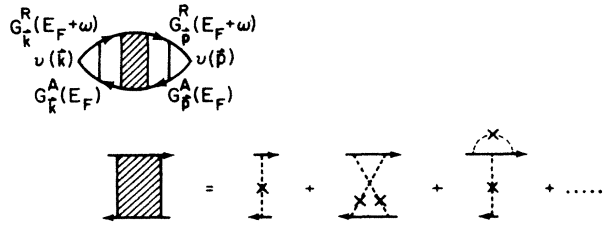

FIG. 1. Diagrammatic representation of $\sum v\langle G^R G^A\rangle v$, and the first few diagrams of the perturbation series.

with the current vertex $v_c(\mathbf{k})=k_x$.

Diagrammatically, we denote G^R and G^A, Eq. (2.11b), by straight lines, and the electron-impurity interaction by dashed lines. The impurity density factor $u=1/2\pi N_F\tau$ is denoted by a cross, where N_F is the density of states per spin at the Fermi energy and $N_F=k_F m/2\pi^2$ and $N_F=m/2\pi$ for $d=3$ and $d=2$, respectively. $k_F=\sqrt{2mE_F}=\sqrt{\varepsilon}$ is the Fermi wave number. In Fig. 1, we show the general graphic representation of $\sum_{\mathbf{k},\mathbf{p}} v(\mathbf{k})\langle \tilde{G}^R_{\mathbf{k},\mathbf{p}}\tilde{G}^A_{\mathbf{k},\mathbf{p}}\rangle v(\mathbf{p})$, where the triangle stands for either v_T or v_c.

III. PERTURBATION THEORY FOR SOUND ATTENUATION COEFFICIENT AND CONDUCTIVITY

A. Density expansion of α_T and σ for $d=3$

In this section the two leading terms of the expansion of α_T in powers of the impurity density, or of $1/E_F\tau$, in three dimensions is calculated. For completeness, and because there were minor algebraic errors in the original calculations,[3] we also quote the first two density corrections to the Boltzmann electrical conductivity σ_0 in three dimensions.

To proceed, we first list all of the relevant diagrams in Fig. 2. Diagram 2(a) determines the lowest order or Boltzmann contribution to α_T, which we denote by α_T^0. From Fig. 2(a) and Eqs. (2.13) and (2.9) we obtain

$$\alpha_T^0=\frac{2}{15\pi^2}\left[\frac{v_F}{c_T}\right]^3\frac{m^3}{\rho\hbar^2}\left[\frac{E_F\tau}{\hbar}\right]\omega^2,\quad d=3\ . \tag{3.1}$$

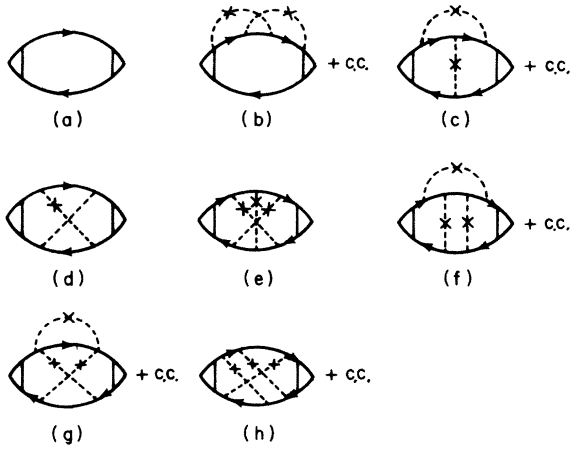

FIG. 2. These diagrams and the complex conjugates (c.c.) of (b), (c), (f), (g), and (h) contribute to the first three terms of the density expansion of α in 3D.

This is the low-frequency limit of the result first obtained by Pippard.[12] Corrections arising from this diagram are of relative order $(\hbar/E_F\tau)^2$, and hence beyond our scope. The remaining diagrams in Fig. 2 determine the corrections to α_T^0 which are of relative order $\hbar/E_F\tau$, and $(\hbar/E_F\tau)^2\ln(E_F\tau/\hbar)$. Diagrams 2(b) and 2(d) contribute to both the leading correction and to the logarithmic terms. In diagram 2(c) the contributions to the leading correction cancel, and it contributes together with the remaining diagrams to the logarithmic term. All other possible diagrams are at least of relative order $(\hbar/E_F\tau)^2$.

The diagrams in Fig. 2 can be expressed in terms of two integrals which appear in the analytic expressions corresponding to Fig. 2. These two integrals are

$$J^{++}(q)=\int d\mathbf{k}\frac{1}{(k^2-\varepsilon-i\gamma)[(\mathbf{k}-\mathbf{q})^2-\varepsilon-i\gamma]}\ , \tag{3.2a}$$

and

$$J^{+-}(q)=\int d\mathbf{k}\frac{1}{(k^2-\varepsilon-i\gamma)[(\mathbf{k}-\mathbf{q})^2-\varepsilon+i\gamma]}\ , \tag{3.2b}$$

where $\varepsilon=2mE_F$, and $\gamma=\hbar m/\tau$. Note that J^{+-} is real. In terms of these integrals, the contribution of diagram 2(b) to α_T (denoted by $\alpha_T^{(b)}$) can be written as

$$\alpha_T^{(b)}=\alpha_T^0\frac{m\hbar}{4\pi^5\varepsilon^{3/2}\tau}\mathrm{Re}\left[\int_0^\infty dq\,q^2J^{++}(q)\left[J^{++}(q)-J^{+-}(q)-\gamma\frac{d}{d\gamma}J^{++}(q)\right]\right]. \tag{3.3}$$

Equation (3.3) is valid to lowest order in the frequency. Within the same accuracy, we obtain the contribution of diagram 2(c) as

$$\alpha_T^{(c)}=\alpha_T^0\frac{m\hbar}{16\pi^5\varepsilon^{7/2}\tau}\mathrm{Re}\left[\int_0^\infty dq\,q^2J^{++}(q)[J^{+-}(q)-\mathrm{Re}J^{++}(q)](8\varepsilon^2-12\varepsilon q^2+q^4)\right], \tag{3.4}$$

that of diagram 2(d) as

$$\alpha_T^{(d)}=\alpha_T^0\frac{m\hbar}{32\pi^5\varepsilon^{7/2}\tau}\int_0^\infty dq\,q^2J^{+-}(q)[J^{+-}(q)-\mathrm{Re}J^{++}(q)](8\varepsilon^2-12\varepsilon q^2+q^4)\ , \tag{3.5}$$

and that of diagram 2(e) as

$$\alpha_T^{(e)}=\alpha_T^0\frac{m^2\hbar^2}{64\pi^7\varepsilon^4\tau^2}\int_0^\infty dq\,q^2[J^{+-}(q)]^2[J^{+-}(q)-\mathrm{Re}J^{++}(q)](8\varepsilon^2-12\varepsilon q^2+q^4)\ . \tag{3.6}$$

For the combination of diagrams 2(f), 2(g), and 2(h) we obtain

$$\alpha_T^{(f)+(g)+(h)} = -\alpha_T^0 \frac{m^2\hbar^2}{32\pi^7\epsilon^4\tau^2} \int_0^\infty dq\, q^2 J^{+-}(q)[\mathrm{Re}J^{++}(q) - J^{+-}(q)]^2 , \tag{3.7}$$

where we have neglected terms that lead to contributions of relative order n_i^2 or higher.

To extract the two leading density corrections from these expressions, the integrals J^{++} and J^{+-} have to be evaluated. J^{++}, Eq. (3.2a), is familiar from quantum electrodynamics and can be evaluated by using Feynman's method of folding denominators together. A representation for small γ, which is sufficient for calculating the first two density corrections to α_T^0, is

$$J^{++}(q) = \frac{\pi^2}{q}\left[\arctan[(q-2\sqrt{\epsilon})\sqrt{\epsilon}/\gamma] + \arctan[(q+2\sqrt{\epsilon})\sqrt{\epsilon}/\gamma] - \frac{i}{2}\ln\left[\frac{(\gamma/\sqrt{\epsilon})^2+(q-2\sqrt{\epsilon})^2}{(\gamma/\sqrt{\epsilon})^2+(q+2\sqrt{\epsilon})^2}\right]\right] . \tag{3.8}$$

J^{+-} can be evaluated by using an auxiliary parameter as discussed before.[3] A small γ representation is

$$J^{+-}(q) = \frac{2\pi^2}{q}\arctan(q\sqrt{\epsilon}/\gamma) . \tag{3.9}$$

In Ref. 3, the last two terms on the right-hand side of Eq. (4.10) should not be present.

With the help of Eqs. (3.3)—(3.9), calculation of the two leading density corrections to α_T^0 is straightforward. We obtain

$$\alpha_T = \alpha_T^0\left[1 - \frac{\pi}{5}\left(\frac{\hbar}{E_F\tau}\right) - \frac{3\pi^2}{128}\left(\frac{\hbar}{E_F\tau}\right)^2 \ln\left(\frac{\hbar}{E_F\tau}\right) + O((\hbar/E_F\tau)^2)\right] . \tag{3.10}$$

For the static conductivity, an essentially identical calculation yields

$$\sigma = \sigma^0\left[1 - \frac{\pi}{3}\left(\frac{\hbar}{E_F\tau}\right) + \frac{\pi^2-4}{32}\left(\frac{\hbar}{E_F\tau}\right)^2 \ln\left(\frac{\hbar}{E_F\tau}\right) + O((\hbar/E_F\tau)^2)\right] . \tag{3.11}$$

In Ref. 3, the corresponding results are incorrect. We will further discuss these results in Sec. IV.

B. Low-frequency behavior of α_T in 2D and 3D

In this section we calculate the leading low-frequency corrections to α_T^0 in both two and three dimensions. For simplicity we ignore the density corrections discussed in Sec. III A. For two-dimensional systems we also calculate the most singular terms of second order in $\hbar/E_F\tau$. The result will be used in Sec. IV B to rule out the behavior for α at an Anderson transition proposed earlier.[4(a)]

To leading order in the impurity density the most important low-frequency corrections are due to the maximally crossed diagrams shown in Fig. 3. Due to quantum-interference effects these diagrams contribute nonanalytic low-frequency corrections to α_T^0 which are of order $\hbar/E_F\tau$ in $d=2$, and of order $(\hbar/E_F\tau)^2$ in $d=3$. The calculation is exactly analogous to that for the conductivity,[5] and the result is

$$\alpha_T = \alpha_T^0\begin{cases} 1 + \dfrac{\hbar}{2\pi E_F\tau}\ln\left(\dfrac{1}{\omega\tau}\right), & d=2 \quad (3.12a) \\ 1 - \dfrac{3\sqrt{6}}{16}\left(\dfrac{\hbar}{E_F\tau}\right)^2\sqrt{\omega\tau}, & d=3 . \quad (3.12b) \end{cases}$$

Here $\alpha_T^0(d=3)$ is given by Eq. (3.1), and in 2D we have from diagram 2(a)

$$\alpha_T^0 = \frac{1}{4\pi^2}\left(\frac{v_F}{c_T}\right)^3 \frac{m^3}{\rho\hbar^2}\left(\frac{E_F\tau}{\hbar}\right)\omega^2, \quad d=2 . \tag{3.13}$$

Equations (3.12) have been obtained before in Ref. 4(b). In Eq. (3.12a) [Eq. (3.12b)] we have neglected a positive (negative) low-frequency contribution [cf. Eq. (3.10)]. Comparison with the corresponding result for the conductivity (Ref. 5) shows that the corrections are identical apart from a different sign: Interference effects decrease σ, but increase α. This is physically obvious. As electron diffusion slows down with increasing disorder, the noninteracting electron system becomes stiffer with respect to shear deformations. Technically the sign difference comes from different symmetry of the current and stress vertices under parity operations. The fact that the prefactors of the leading nonanalytical terms are of the same absolute value for both σ and α is rather obvious from the structure of the theory. We disagree with the result of Kotliar and Ramakrishnan,[4(a)] who obtain an additional factor of $\frac{1}{2}$ in 2D, and a correction of order $(\hbar/E_F\tau)\sqrt{\omega\tau}$ in 3D.

As in the case of the conductivity, the leading correction in α_T in 2D diverges for low frequencies. To investigate this point further, we have calculated the contributions of order $(\hbar/E_F\tau)^2\ln^2\omega\tau$. For σ, these contributions are known to cancel.[5] We find this not to be the case for

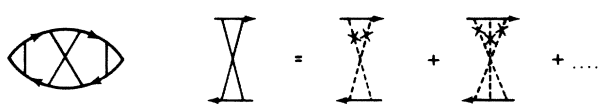

FIG. 3. Maximally crossed diagrams responsible for the leading low-frequency anomaly.

α. In Fig. 4, we show all diagrams contributing to this term. Note that due to the form of the stress vertex the number of diagrams is much smaller than in the conductivity case. The extra diagrams occurring for σ cancel internally, though. Calculation shows that the diagrams 4(b) as well as those in 4(e) cancel each other. The sum of contributions 4(a) and 4(c) is equal to 4(d). Combining the result with Eq. (3.12a) we obtain

$$\alpha_T=\alpha_T^0\left[1+\frac{\hbar}{2\pi E_F\tau}\ln\left(\frac{1}{\omega\tau}\right)+2\left(\frac{\hbar}{2\pi E_F\tau}\right)^2\ln^2\left(\frac{1}{\omega\tau}\right)+O((\hbar/E_F\tau)^2\ln(1/\omega\tau),\ (\hbar/E_F\tau)^3\ln^3(1/\omega\tau))\right]. \tag{3.14}$$

In the case of the conductivity, contributions (4a) and (4c) are equal and opposite to 4(d), hence the coefficient of the $\ln^2(\omega\tau)$ term is zero instead of two. We will discuss the implications of Eq. (3.14) in Sec. IV.

In the presence of a magnetic field B, the singularities stemming from the maximally crossed diagrams are cut off, while those arising from the ordinary ladder diagrams persist, since these singularities are a consequence of particle-number conservation. Consequently, of all the diagrams considered only those shown in Fig. 4(c) are still divergent and we have

$$\alpha_T=\alpha_T^0\left[1+\frac{1}{2}\left(\frac{\hbar}{2\pi E_F\tau}\right)^2\ln^2\frac{1}{\omega\tau}+O((\hbar/E_F\tau)\ln(1/\omega_c\tau),\ (\hbar/E_F\tau)^2\ln(1/\omega_c\tau)\ln(1/\omega\tau))\right], \tag{3.15}$$

where in the omitted terms the cyclotron frequency $\omega_c=eB/mc$ acts as an infrared cutoff. We conclude that singular low-frequency behavior in α_T persists even in the presence of a magnetic field.

IV. DISCUSSION

We conclude this paper with a discussion of our results.

A. Connection between α_L and α_T

In Sec. II B we have shown that for small frequencies $\omega\ll c^2/D$ the longitudinal and traverse sound-attenuation coefficients are connected in the same way as their lowest-order (Boltzmann) approximations. This followed from the fact that the screening correction to the longitudinal stress-correlation function χ_L in Eq. (2.6), apart from negligible terms, just replaces the longitudinal stress vertex k_x^2 by $k_x^2-k^2/d$, thus subtracting the diffusion-pole contribution from χ_L. The physical interpretation of this subtraction is that since the impurities move with the lattice, and the electrons follows almost adiabatically, there is no extra coupling to phonons due to the broken translational invariance. For the perturbation theory this replacement of the vertex results in a reduction in the number of nonzero diagrams, contributing to α_L. As a consequence, the diagrams that determine α_T also determine α_L. Technically, the subtraction is equivalent to rearranging the perturbation series in such a way that convergence is greatly improved. These facts seem not to have been taken into account by Kotliar and Ramakrishnan,[4] who tried to calculate α_L by using the original vertex. Their results disagree with ours, including a different power of $1/\tau$ and a different sign of the low-frequency correction in 3D. We believe that these discrepancies arise partly from the fact that their list of diagrams for α_L is far from being complete if the original vertex is used. For the same reason, they find α_L to be affected by interaction effects, while we find it is not.[10]

B. ε expansion, exponentiation, and critical behavior

In the case of the conductivity it is possible to draw some conclusions from perturbation theory about the critical behavior of σ at an Anderson transition for $d=2+\varepsilon$ ($\varepsilon>0$) dimensions.[1] The argument starts from the fact that the renormalization group beta function for $d=2+\varepsilon$ has a zero which to lowest order in ε is given by $\hbar/\pi E_F\tau=\varepsilon$. This zero corresponds to an unstable fixed point of the renormalization procedure and implies an Anderson transition. Restricting ourselves to leading logarithmic terms in an $\varepsilon\log(\omega\tau)$ expansion, we can write the perturbation results for σ and α_T for $d=2+\varepsilon$ dimensions as

$$\sigma=\sigma_0\{1+(\varepsilon/2)[\ln(\omega\tau)]_\varepsilon\}, \tag{4.1}$$

$$\alpha_T=\alpha_T^0\{1-(\varepsilon/2)[\ln(\omega\tau)]_\varepsilon+2(\varepsilon/2)^2[\ln(\omega\tau)]_\varepsilon^2+\cdots\}. \tag{4.2}$$

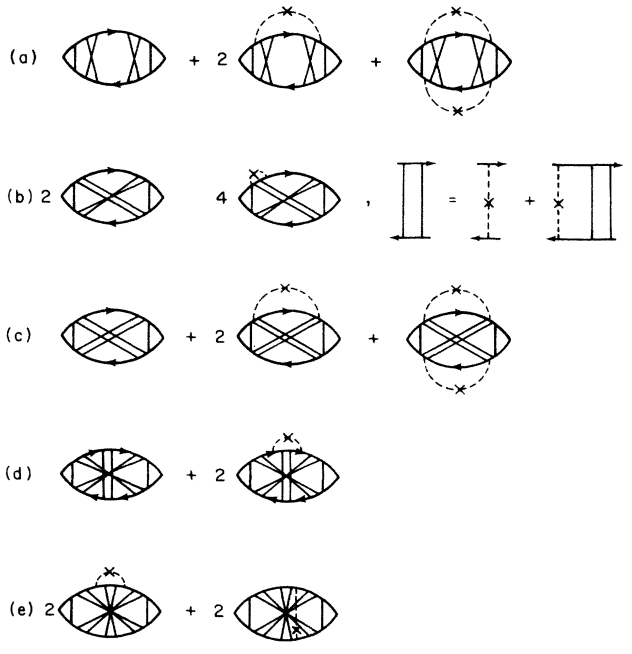

FIG. 4. Diagrams and their multiplication factors contributing to terms of $O((\hbar/E_F\tau)^2\ln^2(\omega\tau))$ for α_T in 2D.

Here we have assumed that for σ there are no terms of order $(\hbar/E_F\tau)^n\ln^n(\omega\tau)$ with $n>3$ (existence of these terms would invalidate the scaling theory of localization). $[\ln(\omega\tau)]_\varepsilon$ is an abbreviation for the d-dimensional integral

$$[\ln(\omega\tau)]_\varepsilon \equiv -\frac{1}{\pi}\mathrm{Re}\left[\int d^dx \frac{1}{-i\omega\tau+x^2}\right]$$

which reduces to $\ln(\omega\tau)+$(nonsingular terms) for $\varepsilon=0$. Performing an ε expansion on these integrals and, again, retaining only the leading logarithmic terms, yields

$$\sigma=\sigma_0[1+(\varepsilon/2)\ln(\omega\tau)+\tfrac{1}{2}(\varepsilon/2)^2\ln^2(\omega\tau)+\cdots], \quad (4.3)$$

$$\alpha_T=\alpha_T^0[1-(\varepsilon/2)\ln(\omega\tau)+\tfrac{3}{2}(\varepsilon/2)^2\ln^2(\omega\tau)+\cdots]. \quad (4.4)$$

Equation (4.3) is just the first few terms of a exponential series. Furthermore, if only leading terms are kept, one indeed obtains an exponential series to all orders, and the final result for the conductivity at the mobility edge in $2+\varepsilon$ dimensions is

$$\sigma=\sigma_0(\omega\tau)^{\varepsilon/2}.$$

To leading order in ε, this result agrees with the one obtained by more sophisticated methods by Wegner.[13] On the other hand, Eq. (4.4) shows that the expansion for α_T is by no means just an exponential series due to the existence and prefactor of the $(\hbar/E_F\tau)^n\ln^n\omega\tau$ term with $n=2$ in 2D. Furthermore, there is no reason to assume that the terms with $n=3$ will be absent either, so one cannot guess how the series (4.4) will continue. Kotliar and Ramakrishnan[4] have calculated only the first-order terms in Eq. (4.4) and assumed exponentiation to be valid and obtained $\alpha_T=\alpha_T^0(\omega\tau)^{-\varepsilon/2}$. This result is in contraction to second-order perturbation theory. Unfortunately, the behavior of α at a mobility edge seems to require much more sophisticated work. One particular effect to be expected is that because of the breakdown of screening occurring at the mobility edge, the simple relation between α_L and α_T, Eq. (2.10), will probably no longer be valid. In this region one should expect α_L to be sensitive to the slowing down of density fluctuations and to rise sharply.

Finally, we remark that we have also tried to apply the self-consistent technique,[14] used successfully for the conductivity problem, to calculate α near the mobility edge. Although this method correctly reproduces our perturbation results for α in 2D, Eq. (3.14), it does not lead to physically meaningful results for α near the mobility edge.

C. Lowest order density correction

The leading-order density correction to the Boltzmann conductivity is linear in the impurity density, Eq. (3.11), as it is in classical dynamical systems.[15] This means that in approximate theories, when the correction to σ^0 is obtained as a wave number integral over a diffusive propagator, the integral must not be cut off at the inverse mean-field path since one then erroneously obtains a leading correction proportioned to n_i^2 (cf. the discussion given in Ref. 16). Cutting off density propagators approximated by diffusive poles seems to be a popular procedure in the literature, though, in this context, as well as elsewhere (usually with the argument that there is no diffusive behavior outside the hydrodynamic region). One should keep in mind that this is valid *only* for extracting singular low-frequency contributions, which do not depend on this upper cutoff anyway.

D. Experimental relevance of the results for α

In Sec. II A we have already discussed that our results are restricted to small frequencies $\omega \ll c^2/D$. Using $D=(\hbar k_F/m)^2\tau/d$, and introducing Mott's resistivity $\rho_M=(\hbar/e^2)3\pi^2/k_F$, we find that the low-frequency region is given by $\omega \ll \omega^*=3mc^2\rho/\hbar\rho_M$. For solids with $c\simeq 5\times 10^5$ cm/sec, $m=$free electron mass, and $\rho\leq\rho_M$, this yields $\omega^*\lesssim 600$ GHz. On the other hand, the scale for the low-frequency anomalies in three dimensions, Eq. (3.12b), is $1/\tau=2(E_F/\hbar)(\rho/\rho_M)$. For metals, $E_F/\hbar$ is typically of the order of 10^{15} Hz, and we are far outside the region of validity of our small ω calculation. However, in semiconductors with Fermi energies of a few meV, the low-frequency scales are comparable, and the effects should be observable. Note that the disorder determines only the absolute frequency scale, the crucial parameter $\omega^*\tau$ is independent of (ρ/ρ_M).

The longitudinal sound-attenuation coefficient has been investigated in Sb-doped Ge by Sakurai and Suzuki.[17] These authors measured α at a fixed frequency $\omega=300$ MHz as a function of temperature on either side of the Anderson transition occurring in this system. Unfortunately, for samples on the metal side of the Anderson transition, the data do not extend on low-enough temperatures to see anomalies if there are any. Let us briefly discuss what one should expect at lower temperatures. If we adopt the view[1] that finite temperatures essentially cut off the low-frequency anomalies at $\omega=1/\tau_{in}(T)$ with an elastic life time $\tau_{in}=\tau(T_0/T)^p$, we obtain

$$\alpha/\alpha^0=1-\frac{3\sqrt{6}}{4}(\rho/\rho_M)^2(T/T_0)^{p/2},\quad d=3. \quad (4.5)$$

If inelastic electron-electron collisions dominate, one would expect $p=2$. It would be interesting to have precise measurements of α at low temperatures in the metallic region to compare with Eq. (4.6). For insulating samples, Sakurai and Suzuki found α_L to rise sharply if the temperature was lowered. This might be due to the breakdown of screening mentioned in Sec. IV B.

In the case of the conductivity, experiments on 2D materials have proven particularly successful for an investigation of the low-frequency anomalies. The existence of the $\ln(\omega\tau)$ term and its sensitivity to magnetic fields and spin-orbit coupling have been convincingly demonstrated in many beautiful experiments.[18] To our knowledge, no attempts have been made so far to measure the sound attenuation in these materials with equal precision. Using again the inelastic lifetime argument given above, one expects

$$\alpha/\alpha^0=1+(p/\pi)(\rho/\rho_M)\ln(T_0/T),\quad d=2. \quad (4.6)$$

This behavior is identical to the one that has frequently

been observed for the resistance. It would be interesting to check this prediction experimentally. Since the singularity in second order is much stronger than it is for the conductivity, it may even be possible to observe the $[(\rho/\rho_M)\ln(T_0/T)]^2$ behavior following from Eq. (3.14). This behavior should persist even in the presence of a magnetic field, cf. Eq. (3.15).

The nonanalytic density dependence, Eqs. (3.10) and (3.11), is probably even harder to observe experimentally than the low-frequency anomalies. Up to now it has not been observed for either the conductivity or for the sound attenuation. We also note that similar nonanalytic density corrections have been predicted and discussed for many years in the theory of transport in classical gases.[19] Indeed, the work of Langer and Neal was motivated by results from classical kinetic theory. However, even in classical systems these logarithmic terms have never been convincingly observed experimentally. There are several problems associated with their observation. The most important one is that we do not have a reliable theoretical estimate of the n_i^2 term in Eqs. (3.10) and (3.11) which is closely correlated with the $n_i^2 \log n_i$ contribution. For the simple model considered here, however, it may be possible to calculate the n_i^2 terms. An apparent experimental difficulty in measuring these contributions in solid-state systems is that unlike in a gas where the density can be easily varied, the impurity density in a solid is more or less fixed for a particular sample of material.

Finally, we mention that it may be possible to measure the nonanalytic density corrections discussed here in electron-mobility experiments in low-temperature helium gas.[20] On the time scale of electronic motion the helium atoms act as stationary scatterers and the standard or Edwards model of electron transport in disordered solids is applicable.

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