Master-equation approach to shot noise in Josephson junctions

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(Received 5 December 1985)

We model the normal resistance of a hysteretic Josephson junction by writing a master equation to describe the individual quasiparticle tunneling. We solve the master equation by a WKB method near the zero-voltage state and near the nonzero-voltage state. We find that near the zero-voltage state the solution is given by the Boltzmann distribution with second-order corrections while near the nonzero-voltage state we obtain a nonsymmetric, non-Boltzmann distribution of voltage fluctuations, similar to the results obtained in a previous discussion based on the Fokker-Planck equation.

I. INTRODUCTION

Noise effects in tunnel and Josephson junctions are of primary importance and interest both from the theoretical aspect and, even more so, from the practical viewpoint. Consequently, extensive theoretical and experimental investigations of these effects have been conducted.[1] The elements under consideration are most commonly operated in a temperature range where the dominant noise effects are due to Johnson (thermal) noise of the normal resistance. The commonly accepted description of the noisy dynamics of these junctions has been the classical Langevin equation and, consequently, the corresponding Fokker-Planck equation.[2-4] The extensive theoretical results for this range of temperatures are in good agreement with experimental measurements in a variety of devices based on Josephson junctions. These investigations were primarily concerned with fluctuations about and transitions from the zero-voltage state of the junction. The nonzero-voltage state was studied in the Smoluchowski limit for the overdamped junction.[4] Recently, Johnson noise fluctuations about and transitions from the nonzero-voltage state into the zero-voltage state in an underdamped Josephson junction were studied.[5-7] These results have not been verified experimentally at this time.

At very low temperatures, as $k_B T = \hbar\omega_J$, where $\hbar\omega_J$ is the Josephson plasma frequency, the thermal-noise description of the fluctuations is no longer accurate. Therefore various generalizations of Langevin's equation to include quantum fluctuations have been proposed for this range of temperatures.[8-10] In addition, transition rates from the zero-voltage state were calculated via macroscopic quantum tunneling (MQT) theories.[11,12] Extensive experimental effort has been devoted to the verification of these theories.[13-15] At low, but finite temperatures, at which the electrostatic energy increment due to the tunneling of a single quasiparticle is comparable with $k_B T$, another quantum effect is likely to become important. This is the discreteness of electric charge that manifests itself as shot noise.[16] This effect is certainly present in the nonzero-voltage state of the Josephson junction. This discreteness naturally calls for a master-equation description. However, in Ref. 6 we tried to simplify the problem by using an effective Fokker-Planck equation to describe the shot-noise effect and obtained transition rates from the dissipative nonzero-voltage state into the zero-voltage state which are considerably higher than those predicted by classical thermal-noise models. Similar analysis of the zero-voltage state[17] led to results which show surprising agreement with the experimental measurements of Ref. 13.

In this paper we show that the effective Fokker-Planck equation we have assumed in Refs. 6 and 17 is the one obtained from a master equation by truncating the Kramers-Moyal series[18,19] after the second moment. However, as shown below, such truncation is not justified, and all moments have to be accounted for. We show that the truncation is justified for the description of small fluctuations, but leads to increasingly large errors when applied to large fluctuations about steady states.

We derive the master equation by using the classical theory of quasiparticle tunneling and construct its steady-state solution by employing the recently developed theory of Ref. 20.

In Sec. II we present the derivation of the master equation. In Sec. III we construct the solution of the master equation near the zero-voltage state and obtain a Boltzmann distribution of fluctuations with small higher-order corrections due to shot noise. Thus, in contrast with previous expectations,[17] this model cannot explain the measurements of Ref. 13.

In Sec. IV we construct the solution near the nonzero-voltage state and calculate the height of the effective barrier for transitions from that state into the zero-voltage state. Finally, in Sec. V we present a discussion of the results.

II. THE MASTER EQUATION

In this section we describe the individual tunneling of quasiparticles across the junction. This tunneling gives rise to both the normal resistance and the fluctuations associated with it.

In the absence of quasiparticle tunneling the dynamics of the order parameter θ (the superconducting phase difference across the junction) is governed by the following equation,

$$C\frac{dV}{dt}+I_J\sin\theta=I_{\mathrm{dc}}\ , \tag{2.1}$$

where C is the capacitance, $V(t)$ is the instantaneous voltage, I_J is the critical Josephson current, and I_{dc} is the driving external current through the junction. In addition, we have the Josephson relation

$$\dot{\theta}(t)=\frac{2e}{\hbar}V(t)\ . \tag{2.2}$$

The usual way dissipation is introduced into this model is the phenomenological addition of Ohmic resistance in parallel with the junction.[21,22] This, together with Eqs. (2.1) and (2.2), leads to the following dimensionless equation,[5]

$$\ddot{\theta}+G\dot{\theta}+\sin\theta=I\ , \tag{2.3}$$

where

$$\begin{aligned} G&=(\omega_J RC)^{-1}\ ,\\ I&=I_{\mathrm{dc}}/I_J\ ,\\ \omega_J^2&=\frac{2eI_J}{\hbar C}\ , \end{aligned} \tag{2.4}$$

time is measured in units of ω_J^{-1}, and energy is measured in units of

$$E_J=\frac{\hbar I_J}{2e}\ . \tag{2.5}$$

The dynamics of Eq. (2.3) has been thoroughly investigated in Refs. 5, 23, and 24. We consider here the hysteretic junction in which two stable states coexist: a stable equilibrium state with

$$\dot{\theta}=0 \ \text{ and } \ \theta=\arcsin I\ ,$$

and a stable nonequilibrium steady state with $\dot{\theta}$ a 2π-periodic function of θ.

As mentioned in the Introduction, in the range of temperatures where the voltage fluctuations are primarily due to Johnson noise, the noise dynamics of the junction is described by the Langevin equation

$$\ddot{\theta}+G\dot{\theta}+\sin\theta=I+L(t)\ , \tag{2.6}$$

where $L(t)$ is Gaussian white noise with

$$\langle L(t)L(t+s)\rangle=2GT\delta(s)\ , \tag{2.7}$$

and T is temperature measured in units of E_J/k_B.

The probability density function $\rho(\theta,\dot{\theta},t)$ of these fluctuations obeys the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t}=-\dot{\theta}\frac{\partial\rho}{\partial\theta}+U'(\theta)\frac{\partial\rho}{\partial\dot{\theta}}+G\frac{\partial}{\partial\dot{\theta}}\left[(\dot{\theta}\rho+T)\frac{\partial\rho}{\partial\dot{\theta}}\right]\ , \tag{2.8}$$

where

$$U(\theta)=-I\theta-\cos\theta\ . \tag{2.9}$$

This leads to the Boltzmann quasistationary distribution of energies about the stable equilibrium state. Near the nonequilibrium steady state the fluctuations must be described by a different distribution, since the Boltzmann distribution

$$\rho_B\equiv\exp(-E/T)\ , \tag{2.10}$$

while it is a stationary solution of Eq. (2.8), is nonperiodic and even unbounded as $\theta\to\infty$. Moreover, ρ_B implies a vanishing probability current in phase space. Instead, we seek a stationary solution of Eq. (2.8) that is bounded and periodic in θ with the same period as the force term $I-\sin\theta$, and that produces a nonzero current in the appropriate direction. This approach was pursued in Ref. 5, where a stationary distribution of the form

$$\rho\propto\exp(-W/T) \tag{2.11}$$

was found and W was calculated using a WKB approximation. This led to an expression for W of the form

$$W=\frac{(A-A_0)^2}{2} \tag{2.12}$$

near the nonzero-voltage steady state. Here, A is the action of a constant probability contour and A_0 is the action of the nonequilibrium steady-state solution of Eq. (2.3), i.e.,

$$A=\frac{1}{2\pi}\int_0^{2\pi}\dot{\theta}\,d\theta\ , \tag{2.13}$$

where the integral in Eq. (2.13) is a line integral on a W contour (see Appendix and Ref. 5). Note that W is not equal to the energy E and, consequently, $\rho\neq\rho_B$ even very close to the steady-state trajectory.

As mentioned in the Introduction, at low temperatures the discreteness of the quasiparticle tunneling should be taken into account and a master-equation approach is called for rather than the Langevin-equation description.

We begin with Eqs. (2.1) and (2.2) for the description of the dynamics in the absence of quasiparticle tunneling.

Next, we model the quasiparticle tunneling rates as follows. We assume elastic tunneling of the quasiparticles so the tunneling rates at energy E are given by

$$\begin{aligned} l &= cf_R(E)[1-f_L(E)]\,, \\ r &= cf_L(E)[1-f_R(E)] \end{aligned} \tag{2.14}$$

for some constant c, where l is the tunneling rate from right to left and r is the tunneling rate from left to right, and $f_L(E)$ and $f_R(E)$ are the occupation probabilities of single quasiparticle states on the two sides of the junction. The functions $f_L(E)$ and $f_R(E)$ are equilibrium Fermi distributions with Fermi levels which are displaced by the amount eV,

$$\begin{aligned} f_R(E) &= \frac{1}{\exp\left[\dfrac{E+eV-E_F}{k_BT}\right]+1}\,, \\ f_L(E) &= \frac{1}{\exp\left[\dfrac{E-E_F}{k_BT}\right]+1}\,, \end{aligned} \tag{2.15}$$

and hence

$$\begin{aligned} \frac{l}{r} &= \exp\left[\frac{-eV}{k_BT}\right]\,, \\ \frac{r+l}{r-l} &= \coth\left[\frac{eV}{2k_BT}\right]\,. \end{aligned} \tag{2.16}$$

[Equation (2.15) should include the shift in the Fermi level due to the transfer of a single charge across the junction, since the Fermi levels after a transition are shifted relative to their positions before the transition. In the limit of small e/C considered here, such a correction was found to be negligible. However, if e/C is not negligible relative to V, and $e^2/2C$ is not smaller than k_BT, such a correction can be important. See Sec. VI for a discussion of this case.]

Next, we identify $\int N(E)dE\, e(r-l)$ as the mean net electric current I across the junction, so that

$$\int N(E)dE \frac{e(r+l)}{\coth\left[\dfrac{eV}{2k_BT}\right]} = \frac{V}{R}\,, \tag{2.17}$$

where $N(E)$ is the density of states per unit energy and R is the normal Ohmic resistance of the junction. Thus,

$$\int (r+l)N(E)dE = \frac{V}{eR}\coth\left[\frac{eV}{2k_BT}\right]\,. \tag{2.18}$$

We consider the case when r and l can be replaced by their mean values [with respect to $N(E)$], and consequently we obtain

$$\begin{aligned} l &= \frac{V}{eR\left[1-\exp\left[\dfrac{-eV}{k_BT}\right]\right]}\,, \\ r &= \frac{V}{eR\left[\exp\left[\dfrac{eV}{k_BT}\right]-1\right]}\,. \end{aligned} \tag{2.19}$$

Thus the dynamic equations (2.1) and (2.2) in presence of quasiparticle tunneling are replaced by the stochastic Eqs. (2.20) and (2.21),

$$V(t+dt) = V(t) + \frac{I_J}{C}\sin\theta(t)dt + \frac{I_{\text{dc}}}{C}dt + o(dt)\,, \tag{2.20}$$

with probability $[1-(r+l)dt]+o(dt)$,

$$V(t+dt) = V(t) + \frac{I_J}{C}\sin\theta(t)dt + \frac{I_{\text{dc}}}{C}dt \pm \frac{e}{C} + o(dt)\,, \tag{2.21}$$

with probability $\binom{r}{l}dt+o(dt)$. The transition probability density $\rho(\theta,V,t)$, defined by

$$\rho(\theta,V,t) = \Pr(\theta(t)=\theta,\ V(t)=V \mid \theta(0)=\theta_0,\ V(0)=V_0)\,, \tag{2.22}$$

satisfies the forward Kolmogorov equation or master equation[18]

$$\begin{aligned} \frac{\partial\rho}{\partial t} = &-\frac{2eV}{\hbar}\frac{\partial\rho}{\partial\theta} - \left[\frac{I_{\text{dc}}}{C}-\frac{I_J}{C}\sin\theta\right]\frac{\partial\rho}{\partial V} + r\left[V-\frac{e}{C},\theta\right]\rho\left[V-\frac{e}{C},\theta,t\right] \\ &+ l\left[V+\frac{e}{C},\theta\right]\rho\left[V+\frac{e}{C},\theta,t\right] - [r(V,\theta)+l(V,\theta)]\rho(V,\theta,t)\,, \end{aligned} \tag{2.23}$$

with the initial conditions

$$\rho(\theta,V,t)\to\delta(\theta-\theta_0,\ V-V_0) \text{ as } t\to 0\,. \tag{2.24}$$

To be consistent with the notations of Ref. 5, we introduce the dimensionless variables

$$T\to\frac{2ek_BT}{\hbar I_J},\quad t\to t\omega_J,\quad E\to\frac{E}{E_J} \tag{2.25}$$

and

$$\rho(\theta,V,t)\to\rho(\theta,\dot\theta,t)\,,$$

where E_J is defined in Eq. (2.5). We obtain

$$eV = \frac{\hbar\omega_J\dot\theta}{2},\quad \frac{V}{RI_J} = G\dot\theta\,. \tag{2.26}$$

In these dimensionless units the elementary change in charge is given by

$$q = \frac{e\omega_J}{I_J} = \frac{\hbar\omega_J}{2E_J}\,. \tag{2.27}$$

Thus q is the ratio of the zero-point energy of the junction to the Josephson coupling energy. It is also the inverse of the number of quasiparticles transferred across the junction in one period $2\pi/\omega_J$. Note that q is also the parameter which determines the range where MQT becomes important; that is, for q not too small, MQT can be appreciable.[11,12]

Our analysis concerns the case where q is small, so we can solve the master equation asymptotically. In the chosen units we have

$$\frac{r}{l}=\exp\left[\frac{q\dot\theta}{T}\right],\quad \frac{r+l}{r-l}=\coth\left[\frac{q\dot\theta}{2T}\right],$$
$$r=\frac{G\dot\theta}{q\left[1-\exp\left[\frac{-q\dot\theta}{T}\right]\right]},\quad l=\frac{G\dot\theta}{q\left[\exp\left[\frac{q\dot\theta}{T}\right]-1\right]}. \tag{2.28}$$

Now the master equation (2.23) takes the form

$$\frac{\partial\rho}{\partial t}=-\dot\theta\frac{\partial\rho}{\partial\theta}-(I-\sin\theta)\frac{\partial\rho}{\partial\dot\theta}+r(\theta,\dot\theta-q)\rho(\theta,\dot\theta-q,t)+l(\theta,\dot\theta+q)\rho(\theta,\dot\theta+q,t)-[r(\theta,\dot\theta)+l(\theta,\dot\theta)]\rho(\theta,\dot\theta,t). \tag{2.29}$$

In summary, the master equation (2.29) holds under the following assumptions:

(1) The dynamics of θ obeys the continuous classical Josephson relations (2.1) and (2.2) between jumps.

(2) The quasiparticle tunneling is elastic and is controlled by the instantaneous Fermi occupation probability of single-particle states on the two sides of the junction.

(3) The tunneling matrix elements and the density of states are essentially energy independent.

(4) The individual quasiparticle tunneling is uncorrelated.

III. THE KRAMERS-MOYAL EXPANSION AND MOMENT TRUNCATION

For small q the Kramers-Moyal expansion is obtained by expanding the master equation (2.29) in a formal Taylor series in powers of q. It is given by

$$\frac{\partial\rho}{\partial t}=L\rho+\sum_{n=1}^{\infty}\frac{(-q)^n}{n!}\frac{\partial^n}{\partial\dot\theta^n}[M_n(\theta,\dot\theta)\rho(\theta,\dot\theta,t)], \tag{3.1}$$

where L is the Liouville operator, given by

$$L\rho=-\dot\theta\frac{\partial\rho}{\partial\theta}-(I-\sin\theta)\frac{\partial\rho}{\partial\dot\theta}, \tag{3.2}$$

and $M_n(\theta,\dot\theta)$ are the conditional jump moments. These moments are given by

$$\begin{aligned} q^nM_n(\theta,\dot\theta)&\equiv q^n\Pr(\Delta\dot\theta=q\mid\theta,\dot\theta)+(-q)^n\Pr(\Delta\dot\theta=-q\mid\theta,\dot\theta)\\ &=q^n[r(\theta,\dot\theta)+(-1)^nl(\theta,\dot\theta)], \end{aligned} \tag{3.3}$$

that is,

$$M_n(\theta,\dot\theta)=r(\theta,\dot\theta)+(-1)^nl(\theta,\dot\theta). \tag{3.4}$$

Usually, the Kramers-Moyal series in (3.1) is truncated at $n=2$ to obtain the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t}=L\rho-q\frac{\partial(M_1\rho)}{\partial\dot\theta}+\frac{q^2}{2}\frac{\partial^2(M_2\rho)}{\partial\dot\theta^2}. \tag{3.5}$$

This procedure has been criticized in the literature.[18–20,25] To understand the difficulty in this truncation procedure, we show next the intrinsic inconsistency of such a procedure. The truncation of higher-order terms in Eq. (3.1) is based on the premise that higher powers of q are negligible relative to the lower ones. This is, however, not always the case here, since, for example, the stationary solution of Eq. (3.5) has the WKB form

$$\rho=\exp\left[\frac{-W(\theta,\dot\theta)}{q}\right], \tag{3.6}$$

where $W(\theta,\dot\theta)$ is a solution of the eikonal (Hamilton-Jacobi) equation,[20]

$$\dot\theta\frac{\partial W}{\partial\theta}+(I-\sin\theta)\frac{\partial W}{\partial\dot\theta}+M_1\frac{\partial W}{\partial\dot\theta}+\frac{M_2}{2}\left[\frac{\partial W}{\partial\dot\theta}\right]^2=0. \tag{3.7}$$

Hence,

$$\frac{\partial^n\rho}{\partial\dot\theta^n}=O(1/q^n),$$

so that a typical term in Eq. (3.1) has the form

$$\frac{q^n}{n!}\frac{\partial^n(M_n\rho)}{\partial\dot\theta^n}=O(1). \tag{3.8}$$

Thus all terms in the Kramers-Moyal expansion are of the same order in q and therefore all moments have to be considered in general.

As shown in Refs. 19 and 20, the WKB structure (3.6) of the solution to the master equation (2.29) leads to the eikonal equation

$$\dot\theta\frac{\partial W}{\partial\theta}+(I-\sin\theta)\frac{\partial W}{\partial\dot\theta}+M\left[\frac{\partial W}{\partial\dot\theta},\theta,\dot\theta\right]=1, \tag{3.9}$$

where $M(z,\theta,\dot\theta)$ is the conditional-moment generating function of the jump process

$$M(z,\theta,\dot\theta)=r(\theta,\dot\theta)(e^z-1)+l(\theta,\dot\theta)(e^{-z}-1)+1. \tag{3.10}$$

It is easy to see that for small deviations from equilibrium, $W=0$ and $\operatorname{grad}W=0$, Eq. (3.7) is obtained by truncating Taylor's expansion of Eq. (3.9) at second order. Thus, the leading quadratic terms in Taylor's expansion of both solutions [to (3.7) and (3.9)] are the same. It follows that Eq. (3.5) is adequate for the description of small fluctuations about equilibrium up to size $q^{1/2}$.

This discussion implies that the Fokker-Planck equation (3.5), used in Refs. 6 and 17 to describe shot noise in a Josephson junction, provides a good approximation for small fluctuations. However, the transition rates calculated from (3.5) are different from those calculated from (2.29), as shown in Secs. IV and V.

IV. THE ZERO-VOLTAGE STATE

In this section we construct an asymptotic solution to the master equation (2.29) in the WKB form (3.6) near the stable equilibrium state $\dot\theta=0$, $\theta=\arcsin I$. To this end we construct a solution to the eikonal equation (3.9) as an asymptotic series in powers of G. We assume that G is sufficiently small so the junction is hysteretic. Thus, setting $W_\theta=\partial W/\partial\theta$, we obtain

$$Y(\theta,\dot\theta,W_\theta,W_{\dot\theta})=\dot\theta W_\theta+(I-\sin\theta)W_{\dot\theta}+G\dot\theta\left[\frac{\exp\left[W_{\dot\theta}-\frac{q\dot\theta}{T}\right]}{1-\exp\left[\frac{-q\dot\theta}{T}\right]}+\frac{\exp(-W_{\dot\theta})}{1-\exp\left[-\frac{q\dot\theta}{T}\right]}+\coth\left[\frac{q\dot\theta}{T}\right]\right]=0 \tag{4.1}$$

and

$$W=W_0+GW_1+\cdots . \tag{4.2}$$

We obtain, to leading order,

$$\dot\theta\frac{\partial W_0}{\partial\theta}+(I-\sin\theta)\frac{\partial W_0}{\partial\dot\theta}=0 , \tag{4.3}$$

so that W_0 is a function only of the energy

$$E=\tfrac{1}{2}\dot\theta^2+U(\theta) , \tag{4.4}$$

where $U(\theta)$ is given by (2.9). At the next order we obtain

$$\dot\theta\frac{\partial W_1}{\partial\theta}+(I-\sin\theta)\frac{\partial W_1}{\partial\dot\theta}=-\dot\theta\frac{\exp\left[\frac{\partial W_0}{\partial\dot\theta}-\frac{q\dot\theta}{T}\right]+\exp\left[-\frac{\partial W_0}{\partial\dot\theta}\right]-\exp\left[-\frac{q\dot\theta}{T}\right]-1}{1-\exp\left[-\frac{q\dot\theta}{T}\right]} . \tag{4.5}$$

The solvability condition for Eq. (4.5) is that the integral of the right-hand side over a constant-energy contour vanishes. Hence, using

$$\frac{\partial W_0(E)}{\partial\dot\theta}=\dot\theta\frac{\partial W_0(E)}{\partial E}\equiv\dot\theta W_0'(E) , \tag{4.6}$$

we obtain

$$\int_0^{2\pi}\frac{\cosh\{[2W_0'(E)-1](q\dot\theta/2T)\}}{\sinh(q\dot\theta/2T)}d\theta=\int_0^{2\pi}\coth\left[\frac{q\dot\theta}{2T}\right]d\theta \tag{4.7}$$

for all $E<E_J$. It follows that

$$2\frac{\partial W_0(E)}{\partial E}-1=1$$

and

$$W_0(E)=E . \tag{4.8}$$

Therefore, to leading order in q and G the quasistationary distribution of fluctuations about equilibrium is of Boltzmann type. Consequently, the transition rate over the potential barrier is determined by the Boltzmann factor $\exp(-\delta U/T)$, where δU is the height of the potential barrier.

We see from Eq. (4.2) that the correction of order G for the Boltzmann distribution can be calculated by the same method. This correction represents the contribution of the quasiparticle tunneling Hamiltonian to the energy. A similar correction has been observed in Refs. 10, 16, 26, and 27. Further corrections of order q and higher orders can be calculated as well. They represent the effect of shot noise near equilibrium. Even for the small junctions made today we have $q=10^{-2}$, so these corrections are negligible. Furthermore, if the refinements mentioned following Eq. (2.16) are introduced, the corrected form of W is given by

$$\frac{W}{q}=\frac{E}{k_BT}\left[1-\frac{1}{90}\left[\frac{e^2}{k_BTC}\right]^2\right] . \tag{4.9}$$

This corresponds to a correction to the usual Boltzmann distribution, which is only of order q^2.

V. THE NONZERO-VOLTAGE STATE

First, we consider the averaged dynamics of the "stochastic" junction obtained by averaging Eqs. (2.20) and (2.21) over all jumps. This also corresponds to retaining only the first term in the Kramers-Moyal series in Eq.

(3.1). We obtain a damped Liouville equation,

$$\frac{\partial \rho}{\partial t}=L\rho+q\frac{\partial[(r-l)\rho]}{\partial\dot\theta}=L\rho+G\frac{\partial(\dot\theta\rho)}{\partial\dot\theta}\ , \quad (5.1)$$

by Eq. (2.28). Equation (5.1) is Liouville's equation for the damped dynamics (2.3). Hence, on the average, the motion is bistable, as described in Ref. 5. In particular, there is a nonequilibrium steady-state solution whose phase-space trajectory is given by

$$\dot\theta=\frac{I}{G}+\frac{G}{I}\cos\theta+\cdots\ .$$

We write the eikonal equation (4.1) in the form

$$\dot\theta W_\theta+(I-\sin\theta-G\dot\theta)W_{\dot\theta}+G\dot\theta\left[W_{\dot\theta}+\frac{\exp\left[W_{\dot\theta}-\frac{q\dot\theta}{T}\right]+\exp(-W_{\dot\theta})-\exp\left[-\frac{q\dot\theta}{T}\right]-1}{1-\exp\left[-\frac{q\dot\theta}{T}\right]}\right]=0\ , \quad (5.2)$$

or, alternatively, in the form

$$LW+G\dot\theta\frac{\cosh\left[\frac{q\dot\theta}{2T}-W_{\dot\theta}\right]-\cosh\left[\frac{q\dot\theta}{2T}\right]}{\sinh\left[\frac{q\dot\theta}{2T}\right]}=0\ . \quad (5.3)$$

Equations (5.2) and (5.3) comprise the generalization of the eikonal equation of Refs. 5 and 6.

We require W to be a periodic function of θ on the steady-state trajectory S. We show in the Appendix (see also Refs. 5 and 28) that this implies that $W=$ const on S (e.g., $W=0$) and $\nabla W=0$ as well. We proceed now as in Refs. 5 and 6. We show in the Appendix that, on constant-W contours and for small G,

$$\frac{\partial W}{\partial\dot\theta}=\frac{q}{T}\dot\theta K(W)+O(G)\ ,$$

where $K(W)$ is defined in Eq. (A9) of the Appendix [also see Eq. (5.8) below] and that the constant-W contours are the steady-state trajectories of the equation

$$\ddot\theta+g(\dot\theta,K)+\sin\theta=I\ , \quad (5.4)$$

where $g(\dot\theta,K)$ is a nonlinear dissipative force given by

$$g(\dot\theta),K)=-\frac{GT}{2qK}\frac{\cosh\left[\frac{q\dot\theta(1-2K)}{2T}\right]-\cosh\left[\frac{q\dot\theta}{2T}\right]}{\sinh\left[\frac{q\dot\theta}{2T}\right]}\ . \quad (5.5)$$

In the limit $K\to 0$ (i.e., on S) we have

$$g(\dot\theta,K)\to G\dot\theta\ .$$

Away from S, K determines a renormalized value for the nonlinear dissipative term g in Eq. (5.4). These contours are given by

$$\dot\theta=\langle\dot\theta\rangle+\frac{I}{\langle\dot\theta\rangle}\cos\theta+O(G^2)\ , \quad (5.6)$$

with

$$\langle\dot\theta\rangle=-\left[\frac{T}{2qK}\right]\ln\left[1-\frac{2qIK}{GT}\right] \quad (5.7)$$

($\langle\ \rangle$ denotes averaging over constant-W contours, see the Appendix). In Fig. 1 we show the constant-W contours for high ($T\gg q$) and low ($T\ll q$) temperatures. We note that for $T\gg q$ these contours are identical to those of Ref. 5, which are obtained under pure Johnson noise assumptions.

From Eqs. (5.6) and (5.7) and Fig. 1 we see, as in Ref. 5, that through each point $(\theta,\dot\theta)$ in D, the basin of attraction of S, there is a unique constant-W contour, and hence a unique K. We can express W as a function of K by

$$W(K)-W(0)=(q/2T)\int_0^K K\frac{\partial\langle\dot\theta^2\rangle}{\partial K}dK+O(G)\ . \quad (5.8)$$

Since K is a function of θ and $\dot\theta$ through Eqs. (5.6) and (5.7), the probability density of finding the system at a point $(\theta,\dot\theta)$ in D is given by

$$\rho=\exp[-W(K(\theta,\dot\theta))/q]\ , \quad (5.9)$$

with $W(K)$ given by Eq. (5.8).

In Fig. 2 we show the graphs of $W(K)$ for $T\gg q$ and $T\ll q$. We note that for $T\gg q$ the graph agrees with that of Ref. 5. [Our W/q corresponds to W/T of Ref. 5, while our K corresponds to $G(1-K)$ of Eq. (3.18) there.] The transition rate from S to the stable equilibrium state, i.e., from the nonzero-voltage state to the zero-voltage state of the Josephson junction, is again determined by an exponential factor $\exp(-W_c/q)$, where $W_c=W(K_c)$ and where K_c is the value of the parameter K in Eqs. (5.4) and (5.5) for which the steady-state trajectory touches the θ axis. We refer to this as the critical trajectory. To calculate the critical value K_c, we assume that in the limit of small current and small dissipation, the critical trajectory satisfies

$$2\pi I=\int_0^{2\pi}g(\dot\theta(\theta),K)d\theta\ , \quad (5.10)$$

where $\dot\theta(\theta)=|\cos(\theta-d)/2|$ and $d=$ const. Thus, using Eq. (5.5) we have that K_c is the solution of the following implicit equation:

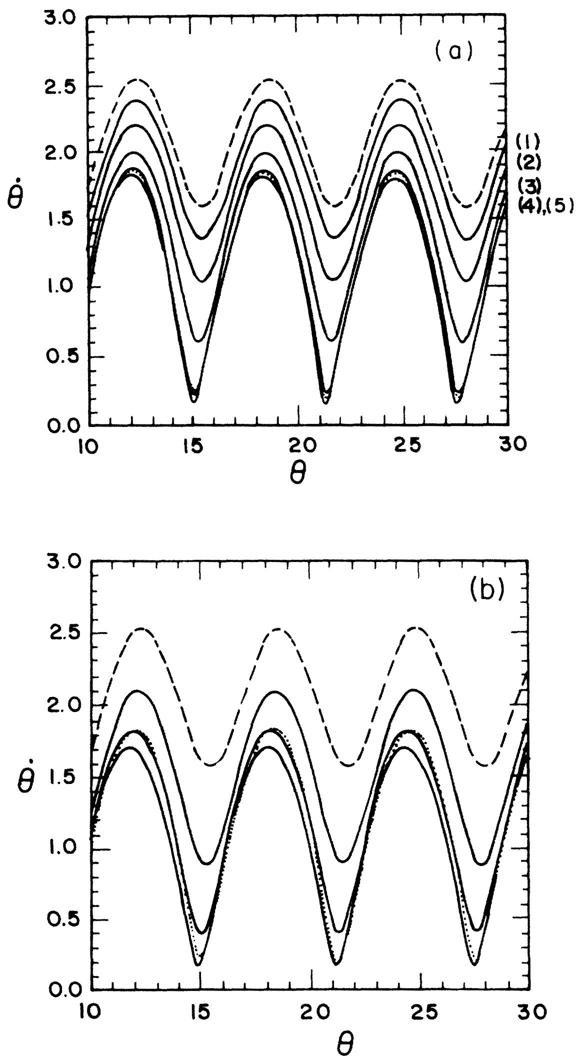

FIG. 1. The constant-W contours for $I_J = 10\ \mu$A, $I_{dc} = 8\ \mu$A, $C = 1$ pF, and $R = 15\ \Omega$. (a) For $T = 10$ K. The upper dashed line is the steady-state solution. Lines (1)–(4) are the constant-W contours for $K = -0.25$, -0.5, -0.75, -0.85, and -0.9, respectively. The dotted line is steady-state trajectory of the equation of motion with different dissipation G. (b) For $T = 0.1$ K. The solid lines are the constant-W contours for $K = -0.05$, -0.1, and -0.125, respectively. The dashed line is the steady-state trajectory and the dotted line is the steady-state trajectory with different dissipation.

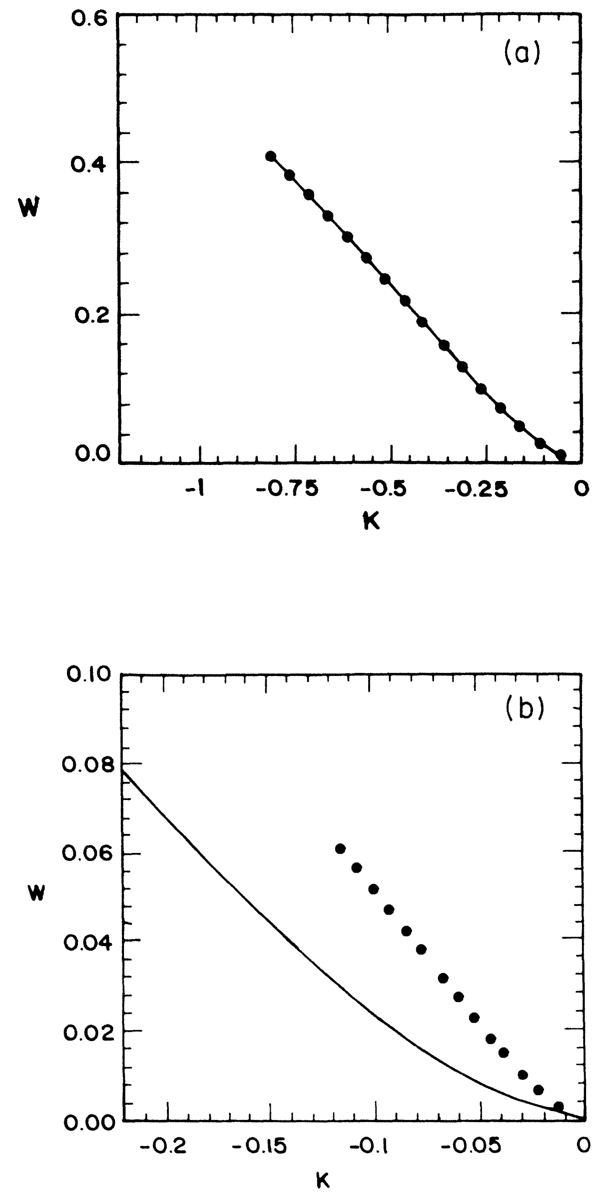

FIG. 2. The effective energy W as function of K for $I_J = 10\ \mu$A, $I_{dc} = 8\ \mu$A, $C = 1$ pF, and $R = 15\ \Omega$. The solid line is the effective energy for the Johnson limit, while the circles are the effective energy calculated by the master equation (see text). (a) for $T = 10$ K and (b) for $T = 0.1$ K.

$$\frac{qIK_c}{G} = -\int_0^{2\pi} \frac{\cosh\left[\frac{q(1-2K_c)}{T}\cos\frac{\theta}{2}\right] - \cosh\left[\frac{q}{T}\cos\frac{\theta}{2}\right]}{\sinh\left[\frac{q}{T}\cos\frac{\theta}{2}\right]}\, d\theta\ .$$

The graphs of K_c versus I are given in Fig. 3 for $T \gg q$ and $T \ll q$. In Fig. 4 we compare the graphs of the effective barrier height (ΔE) of the present theory and that in the thermal (Johnson noise) theory of Ref. 5 and the approximated one for shot noise of Ref. 6. The graphs were obtained from numerical solutions of the critical trajectories. To emphasize the effect of shot noise, relative to that of Johnson noise, we calculate the ratio of the transi-

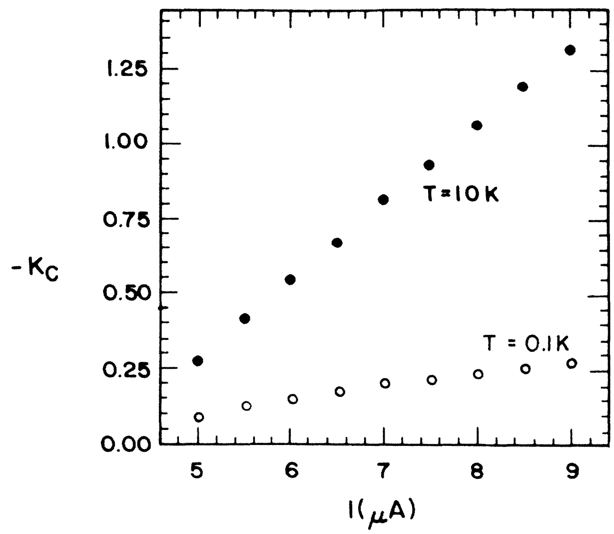

FIG. 3. The parameter K_C which characterizes the critical trajectory and determines the transition rate, plotted as a function of the applied current for $I_J = 10\ \mu$A, $C = 1$ pF, and $R = 15\ \Omega$.

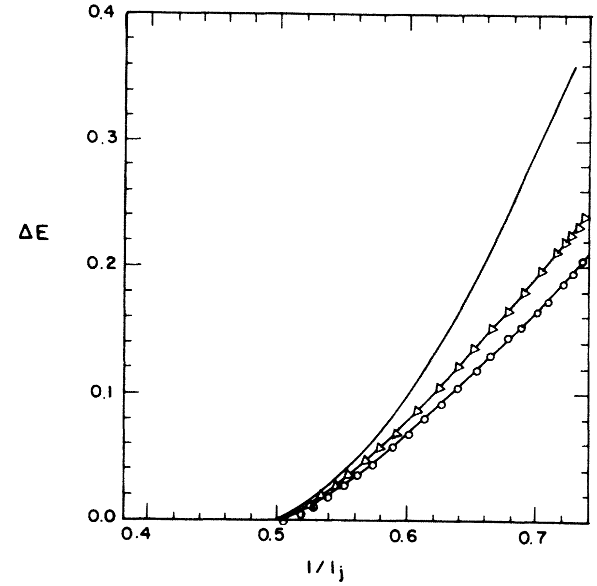

FIG. 4. The effective barrier height ΔE as function of the applied current. The solid line is the thermal limit (Ref. 5), the triangles are the Fokker-Planck description of shot noise (Ref. 6), and the circles are the master-equation description.

tion rates from Fig. 4. For example, even at $I = 0.6$ the shot-noise rate is about 8 orders of magnitude greater than that of the Johnson noise rate.

VI. DISCUSSION

First, we discuss the results of Ref. 6 in view of the present model. Near the nonzero-voltage resistive steady state the truncated Kramers-Moyal equation (3.5) is identical to the Fokker-Planck (FP) equation used in Ref. 6. The transition rate predicted in Ref. 6 is not the same as here; however, the dependence of the rate on parameters is similar to the one obtained in this paper and both theories give rates which are orders of magnitude higher than those predicted by Johnson noise alone. We also observe that, because of the strong stability of the nonzero resistive state, transitions can be observed only at driving currents which are close to $I_{\min}$, the minimal current for which that state exists. In the nonequilibrium state, the probability of voltage fluctuations is always non-Boltzmann, even when the Johnson noise dominates. To observe the effect of shot noise, the junction should be small enough and the temperature low enough so that $T < q$, or in usual physical units, $k_B T < \frac{1}{2}\hbar\omega_J$. For example, for a junction with the parameters given in Fig. 1, namely $I_J = 10 \mu$A and $C = 1$ pF, the shot noise will govern the distribution of fluctuations as well as the transition rate when $T \lesssim 1$ K: When $I = 6\ \mu$A and $R = 15\ \Omega$, we obtain for the steady state a lifetime of $\tau = 1.7$ msec independent of temperature. By contrast, if the shot noise is ignored, the calculated lifetime at $T = 0.8$ K is about 7 sec, and increases to 3×10^{21} sec when T drops to 0.3 K.

Near the zero-voltage stable equilibrium state the FP equation (3.5), which was used in Ref. 17, is a good approximation to the master equation. However, it is not a valid approximation for describing large fluctuations around that state.[19] The master equation (2.29) predicts a Boltzmann distribution of energies for both small and large fluctuations. We have to bear in mind the fact that the model (2.29) neglects macroscopic quantum effects of the junction which should be accounted for. At the present time a more general master equation that incorporates these effects is under study.

We studied the Josephson junction in a range of temperatures for which $q \ll 1$. With the advent of miniaturization of the junction, the limit $q = 1$ is being approached. In this limit a new approach to modeling the junction is called for. To see the effect of $q > 1$ we consider a low-capacitance tunnel junction. In an unbiased junction the Fermi levels on both sides are at the same height. When a dc current I_{dc} flows, the junction is charged at the rate I_{dc}. This causes the Fermi level on one side (say, on the right) to be shifted relative to the Fermi level on the other side at the rate I_{dc}/C. At first it may seem that for any finite voltage across the junction there is a finite probability of charge transfer. However, we have to take into consideration the change in voltage caused by such a transfer: When a single charge e moves across the junction, the gap between the Fermi levels on the two sides decreases by $e^2/2C$. Thus, as long as the junction voltage before the transfer is less than $e/2C$ (and thus the gap before the electron transfer is less than $e^2/2C$), and as long as $k_B T < e^2/2C$, the number of empty states into which the moving charge can tunnel directly is negligible. The tunneling probability is also correspondingly small. It follows that whenever the junction voltage reaches the value $e/2C$ the tunneling probability increases drastically, and as a result the voltage soon drops by e/C to about $-e/2C$. Thus the voltage exhibits sawtooth oscillations between $\pm e/2C$ at a frequency I_{dc}/e.[29,30] The oscillations described here, though apparently similar to the ones described in Ref. 31, actually arise from a different mechanism. In particular, the two treatments lead to different results for the voltage fluctuations.[30]

ACKNOWLEDGMENTS

We are grateful to V. Ambegaokar, G. Schön, and Y. Imry for useful discussions, and to R. Landauer for bringing to our attention the importance of using a master equation for the description of discrete stochastic processes such as shot noise. This research was supported in part by the Department of Energy, under Grant No. DE-AC02-78ER0-4650, by the National Science Foundation under Grant No. DMS-84-06110, by a grant from the U.S.-Israel Binational Science Foundation, and by a grant from the Israel Academy of Sciences.

APPENDIX

In this Appendix we construct the asymptotic expansion of $W(\theta,\dot\theta)$ around the resistive state. First, we show that on S we have $W=\text{const}=0$ and grad $W=0$. To this end we write Eq. (5.2) on S in the form

$$\frac{dW}{dt}=-F(z,\theta,\dot\theta)\;, \tag{A1}$$

where d/dt denotes differentiation along S, $z=W_{\dot\theta}$, and

$$F(z,\theta,\dot\theta)=G\dot\theta\left[z+\frac{\exp\left[z-\dfrac{q\dot\theta}{T}\right]+\exp(-z)-\exp\left[-\dfrac{q\dot\theta}{T}\right]-1}{1-\exp\left[-\dfrac{q\dot\theta}{T}\right]}\right]. \tag{A2}$$

We have

$$F(0,\theta,\dot\theta)=F_z(0,\theta,\dot\theta)=0$$

and (A3)

$$F_{zz}(z,\theta,\dot\theta)=G\dot\theta\frac{\exp\left[z-\dfrac{q\dot\theta}{T}\right]+\exp(-z)}{1-\exp\left[-\dfrac{q\dot\theta}{T}\right]}>0\;.$$

Hence $F(z,\theta,\dot\theta)>0$ for all $z\neq 0$. Now it follows from (A1) that for W to be a periodic function on S we must have $\partial W/\partial\dot\theta=0$, and then $dW/dt=0$, so that $W=\text{const}$ on S and consequently grad $W=0$ on S.

Next, we need to consider the following sets of equations in the phase space $(\theta,\dot\theta)$. (a) The averaged equations of motion of the variables $\theta,\dot\theta$ [see Eq. (2.3)],

$$\frac{d\theta}{dt}=\dot\theta,\quad \frac{d\dot\theta}{dt}=I-\sin\theta-G\dot\theta\;. \tag{A4}$$

(b) The parametric equations for the constant-W contours,

$$\frac{d\theta}{dt}=\dot\theta\;,$$
$$\frac{d\dot\theta}{dt}=I-\sin\theta+\frac{G\dot\theta}{W_{\dot\theta}}\,\frac{\cosh\left[\dfrac{q\dot\theta}{2T}-W_{\dot\theta}\right]-\cosh\left[\dfrac{q\dot\theta}{2T}\right]}{\sinh\left[\dfrac{q\dot\theta}{2T}\right]}\;. \tag{A5}$$

(c) The parametric equations for the characteristic lines of the eikonal equation (4.1),

$$\frac{d\theta}{dt}=\frac{\partial Y}{\partial W_\theta}=\dot\theta\;,$$

$$\frac{d\dot\theta}{dt}=\frac{\partial Y}{\partial W_{\dot\theta}}=I-\sin\theta-G\dot\theta\frac{\sinh\left[\dfrac{q\dot\theta}{2T}-W_{\dot\theta}\right]}{\sinh\left[\dfrac{q\dot\theta}{2T}\right]}\;,$$
$$\frac{dW_\theta}{dt}=-\frac{\partial Y}{\partial\theta},\quad \frac{dW_{\dot\theta}}{dt}=-\frac{\partial Y}{\partial\dot\theta}\;, \tag{A6}$$
$$\frac{dW}{dt}=W_\theta\frac{\partial Y}{\partial W_\theta}+W_{\dot\theta}\frac{\partial W}{\partial W_{\dot\theta}}\;.$$

Next, we find $\partial W/\partial\dot\theta$ on the constant-W contours. To this end we consider the function

$$H(\theta,\dot\theta)\equiv\frac{\dot\theta^2}{2}-\cos\theta-I\theta-\int_{\theta_0}^{\theta}\frac{G\dot\theta}{W_{\dot\theta}}\,\frac{\cosh\left[\dfrac{q\dot\theta}{2T}-W_{\dot\theta}\right]-\cosh\left[\dfrac{q\dot\theta}{2T}\right]}{\sinh\left[\dfrac{q\dot\theta}{2T}\right]}d\theta\;, \tag{A7}$$

where the integral in (A7) is a line integral along the constant-W contour that passes through the point $(\theta,\dot\theta)$ in phase space. The function H was chosen so that $H=\text{const}$ on constant-W contours, i.e.,

$$\left.\frac{dH}{dt}\right|_{W=\text{const}}=0\;; \tag{A8}$$

hence, $H=H(W)$. We can now calculate the rate of change of $H(W)$ on a characteristic curve,

$$\left.\frac{dH}{dt}\right|_{\text{char}}=H_\theta\frac{d\theta}{dt}+H_{\dot\theta}\frac{d\dot\theta}{dt}\;,$$

where $d\theta/dt$ and $d\dot\theta/dt$ are given by Eq. (A6).

After some tedious calculations, we find

$$\left.\frac{dH}{dW}\right|_{\text{char}}=\frac{dH}{dt}\Big/\frac{dW}{dt}=\dot\theta/W_{\dot\theta}+O(G^2)$$
$$=\left[\frac{T}{q}K(W)\right]^{-1}+O(G^2)\ , \qquad (A9)$$

where $K(W)$ is constant on constant-W contours. The quantity $K(W)$, which is defined by (A9), is equal to zero on the steady-state trajectory S where $W=0$. Using (A9) in (A5), we can rewrite (A5) for small G as

$$\frac{d\theta}{dt}=\dot\theta\ ,$$
$$\frac{d\dot\theta}{dt}=-g(\dot\theta,K(W))+I-\sin\theta\ , \qquad (A10)$$

where $g(\dot\theta,K(W))$ is defined in Eq. (5.5). Hence,

$$\dot\theta\frac{d\dot\theta}{d\theta}=-g(\dot\theta,K)+I-\sin\theta\ . \qquad (A11)$$

We find the constant-W contours $\dot\theta(\theta,K)$ by expanding the solution of (A11) in powers of G,

$$\dot\theta=\frac{\dot\theta_{-1}}{G}+\dot\theta_0+G\dot\theta_1+\cdots\ ,$$

and scaling $K=GK_0$. Proceeding as in Ref. 28, we obtain Eqs. (5.6) and (5.7). The calculation of $W(\theta,\dot\theta)$ follows Eq. (5.8).

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