Hopping conductivity in one dimension with asymmetric transfer rates

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The frequency-dependent conductivity is calculated for a one-dimensional disordered system in the presence of an external biased electric field. Four classes of transfer-rate distributions are discussed. The singular distribution for which the inverse first moment does not exist gives the most interesting nonanalytic behavior. For the class-(a) distribution such that all inverse moments exist, numerical results are in complete agreement with the analytic expansions for weak disorder, as discussed in our previous work. For the class-(c) distribution such that no inverse moments exist, one finds nonuniversal crossover behavior as well as nonanalytic frequency dependence of the ac conductivity. We find a crossover region $\omega_1 \le \omega \le \omega_2$ such that for $\omega < \omega_1$, the conductivity $\sigma \sim \omega^{\alpha}$ $(0<\alpha<1)$, and for $\omega > \omega_2$, $\sigma \sim \omega^{\alpha/(2-\alpha)}$ with behavior the same as for the unbiased case. For the class-(b′) distribution such that only one inverse moment exists, nonanalytic leading corrections in the conductivity are also obtained. Finally, for the bond-percolation model, we find no crossover behavior in the frequency-dependent conductivity even in the strong biased case.

INTRODUCTION

Hopping transport in one dimension[1] has attracted great interest recently as it appears to explain conductivity experiments on quasi-one-dimensional conductors over a wide temperature range.[2] However, the theory focuses on symmetric transfer rates so that it is only good for small electric fields.

Transport with random and biased transfer rates seems to be even more interesting. This case was first studied by Derrida and Orbach[3] in an early paper in which the frequency-dependent conductivity of the one-dimensional disordered chain was calculated with an external biased electric field. The problem was solved in the weak-disorder limit and results very different from the symmetric case were found in the low-frequency region, crossing over to a behavior which was the same as the symmetric case.[1] Whether the weak-disorder results could be valid even when the disorder is not weak was also discussed.[3] Recent work on this problem[4,5] seems to confirm this point, and the calculations have been extended to general nonsingular distributions.

It would be more interesting to study the case when the transfer rates obey a singular distribution. In this paper, we wish to study distributions where the inverse first moment does not exist. In this case, there is no dc conductivity, and nonanalytic behavior in the ac conductivity will occur. Physically, infinitesimally small transfer rates correspond to infinitely high barriers and this is relevant to the conductivity experiments on the superionic conductor, hollandite.[2] On the other hand, with a scaling hypothesis, Bernasconi *et al.*[6] were able to obtain an exact asymptotic solution for the symmetric case. We shall see that such a scaling assumption is still correct in the biased case although the frequency dependence of a scaling length is modified at low frequencies.

The plan of this paper is as follows. In Sec. I we briefly discuss the model and the effective-medium theory. Four classes of distribution of transfer rates are introduced, in which the singular distribution gives the most interesting results. In Sec. II we present the results of the real and imaginary parts of the ac conductivity for all classes of distribution. The crossover behavior from drift to diffusion is discussed. Finally, we discuss the leading corrections to the small-frequency expansions which could be important for the singular distribution. In Sec. III we also discuss the high-frequency expansions for the class-(a) distribution. Comparison to the known results is discussed where appropriate.

I. THE MODEL AND THE EFFECTIVE-MEDIUM APPROXIMATION

A detailed description of the model has been presented in our recent paper[5] (hereafter referred to as paper I). We shall only discuss the results briefly for notation. We examine the master equation with asymmetric transfer rates

$$\frac{dP_n}{dt} = -\left[\sum_{\delta} a_{n+\delta,n} W_{n+\delta,n}\right] P_n + \sum_{\delta} a_{n,n+\delta} W_{n,n+\delta} P_{n+\delta} , \tag{1.1}$$

where $\delta = \pm 1$ in one dimension.

In this equation, $P_n(t)$ is the probability on site n at time t. $W_{m,n} = W_{n,m}$ are the nearest-neighbor transfer rates. They are all independent random variables and obey a probability distribution $P(W)$. We shall focus on the following classes of distribution:

Class (a). $P(W)$ is such that all $\langle W^n \rangle$ and $\langle W^{-n} \rangle$ exist. This is a nonsingular distribution. We shall examine the weak-disorder limit analytically; however, the weak-disorder results are valid for all class-(a) distributions in which the disorder is not necessarily weak, as we shall numerically confirm.

Class (c). $P(W)$ is such that no inverse moments exist and $\langle W^{-n} \rangle^{-1} = 0$ for all $n \ge 1$. As an example, we shall consider $P(W)=(1-\alpha)W^{-\alpha}$, $0\le W\le 1$, and $0<\alpha<1$.

Class (b′). $P(W)$ is such that only the first r inverse moments exist, $\langle W^{-n}\rangle$ being finite for all $n\le r$, and $\langle W^{-n}\rangle^{-1}=0$ for $n>r$. A particular example ($r=1$) is $P(W)=(1+\beta)W^{\beta}$, $0\le W\le 1$, and $0<\beta<1$. Note that a crossover from class (c) to class (b′) occurs when $\alpha=\beta=0$. This case is more general than the class (b) discussed in Ref. 1.

Class (d). $P(W)$ is such that there exists a finite probability for W to be zero. An example is the bond-percolation model.[7]

In Eq. (1.1), $a_{m,n}$ are the bias and are given by

$$a_{m,n}=e^{-(\mathbf{n}-\mathbf{m})\cdot e\mathbf{E}_0/k_BT}, \tag{1.2}$$

where $\mathbf{E}_0$ is an external electric field, so that the transfer rates are biased in the direction of the electric field.[5] Taking the Laplace transform, with the initial condition $P_n(t=0)=\delta_{n,0}$, one obtains

$$s\tilde{P}_n=\delta_{n,0}-\left[\sum_\delta a_{n+\delta,n}W_{n+\delta,n}\right]\tilde{P}_n+\sum_\delta a_{n,n+\delta}W_{n,n+\delta,n}\tilde{P}_{n+\delta}, \tag{1.3}$$

where $\delta=\pm 1$ in one dimension. $\tilde{P}_n(s)$ denotes the Laplace transform.

In paper I we discussed the crossover behavior in the biased case. If we consider the frequency response of a diffusing particle to an ac field at frequency ω, we obtain a crossover frequency

$$\omega_{\rm co}\sim(a^{-1}-a)^2\langle W\rangle, \tag{1.4}$$

where $\langle W\rangle$ is an averaged transfer rate and $a=\exp(-eE_0/k_BT)$ is the bias. For low frequencies, or long periods, the drift behavior always dominates and vice versa.

In order to give a quantitative description of our model, we shall solve the master equation [Eq. (1.3)] using the effective-medium approximation[5] (EMA). In this approximation, we replace the random transfer rates in Eq. (1.3) by a homogeneous value, which is to be determined self-consistently. As pointed out in Ref. 1, all classes of distribution can be well described by EMA. In particular, class-(a) distribution yields the normal diffusion behavior at long times. In the asymmetric case, however, it can be shown that[8] EMA still works well provided that either one of the averages $\langle W^-/W^+\rangle$ and $\langle W^+/W^-\rangle$ exists and is less than unity. In the present case, it is obvious that this condition is always satisfied. The details of EMA for the biased case have been established in paper I. Here we only briefly state the resulting equations. In the one-dimensional case, the self-consistency equations are

$$\left\langle\frac{W-\overline{W}}{1+Q(W-\overline{W})}\right\rangle=0, \tag{1.5}$$

where $\overline{W}$ is the EMA transfer rate and Q is the EMA impedance

$$Q=\frac{1}{\overline{W}}(1-s\tilde{P}_0), \tag{1.6}$$

and $\tilde{P}_0(s)$ is the zero-site probability:

$$\begin{aligned}\tilde{P}_0(s)&=\frac{1}{N}\sum_q\frac{1}{s+\overline{W}[a(1-e^{iq})+a^{-1}(1-e^{-iq})]}\\&=\frac{1}{(s^2+2\overline{W}sa_1+\overline{W}^2a_2^2)^{1/2}},\end{aligned} \tag{1.7}$$

where $a_1=a+a^{-1}$ and $a_2=a^{-1}-a$.

We shall solve these self-consistency equations to obtain the EMA transfer rate as a small-frequency expansion. To do the expansion, one should compare the magnitude of the two terms $2\overline{W}sa_1$ and $\overline{W}^2a_2^2$ in Eq. (1.7) for the zero-site probability. We shall see in Sec. II that for class-(a) and -(b′) distributions, the static value of $\overline{W}$ exists and is equal to a constant. This immediately gives us a crossover value of s,$s_{\rm co}\sim(a^{-1}-a)^2\overline{W}$, which is consistent with Eq. (1.4). For the singular class-(c) distribution, however, we shall see that $\overline{W}\sim O(s^{\alpha/(2-\alpha)})$ at low frequencies and $\overline{W}\sim O(s^{\alpha})$ at even lower frequencies. Thus, crossover will occur in a finite region of frequencies as we shall study in Sec. II. Finally, for the bond-percolation model, $\overline{W}\sim O(s)$ and we find no crossover behavior. Now in EMA, the frequency-dependent conductivity is given by the generalized Einstein relation[9] derived from the fluctuation-dissipation theorem

$$\sigma(\omega)=\frac{ne^2}{k_BT}\langle D(i\omega)\rangle=\frac{ne^2}{k_BT}\overline{W}(s=i\omega). \tag{1.8}$$

In the static limit $s\to 0$, $Q\to 1/\overline{W}$, and one can rewrite Eq. (1.5) as

$$\frac{1}{\overline{W}}=\left\langle\frac{1}{W}\right\rangle. \tag{1.9}$$

Thus, in one dimension, the dc transport depends on the existence of the inverse first moment of the distribution of transfer rates $P(W)$. As for class-(c) and -(d) distributions, we do not have any dc conductivity and we shall see that nonanalytic ac conductivity results.

Bernasconi *et al.*[6] proposed a scaling hypothesis for the probability $\langle\tilde{P}_n(s)\rangle$ in the small-s limit

$$\langle\tilde{P}_n(s)\rangle=\langle\tilde{P}_0(s)\rangle F(n/\xi(s)),\quad s\to 0 \tag{1.10}$$

where $\xi(s)$ is the frequency-dependent characteristic length and F is a scaling function. From the normalization $F(0)=1$ and

$$\sum_n\langle\tilde{P}_n(s)\rangle=s^{-1}, \tag{1.11}$$

they arrived at

$$[\xi(s)]^{-1}\sim 2s\langle\tilde{P}_0(s)\rangle,\quad s\to 0. \tag{1.12}$$

With this assumption, Alexander *et al.*[1] were also able to obtain the transport properties. Using the replica method, Stephen and Kariotis[10] verified the validity of this scaling hypothesis. Let us consider in the EMA

$$\begin{aligned}\langle\tilde{P}_n(s)\rangle&=\frac{1}{N}\sum_q\frac{e^{iqn}}{s+\overline{W}[a(1-e^{iq})+a^{-1}(1-e^{-iq})]},\\&=\langle\tilde{P}_0(s)\rangle z^{|n|},\end{aligned} \tag{1.13}$$

where

$$z=[(s+\overline{W}a_1)-(s^2+2\overline{W}sa_1+\overline{W}^2a_2^2)^{1/2}]/2a\overline{W} \ . \tag{1.14}$$

In the symmetric case, $a=a^{-1}=1$, where $s\to 0$,

$$z=1-(s/\overline{W})^{1/2} \ . \tag{1.15}$$

Thus, one finds $F(X)=\exp(-|X|)$ and

$$\xi(s)=(s/\overline{W})^{-1/2} \ , \tag{1.16}$$

which is consistent with the scaling hypothesis.[6] However, in the presence of a bias, where $s\to 0$,

$$z=1-s/a_2\overline{W} \ . \tag{1.17}$$

Thus, one finds

$$\xi(s)\sim(s/\overline{W})^{-1} \ , \tag{1.18}$$

and the scaling assumption is still correct but the frequency dependence of the characteristic length has changed.

II. FREQUENCY-DEPENDENT CONDUCTIVITY

Here we solve the EMA equations [Eqs. (1.5)—(1.7)] for all classes of distribution. The frequency-dependent conductivity is then calculated in both the drift and diffusion regions.

A. Class (a)

We discuss the weak-disorder limit analytically; however, the results are equally applied to the more general class-(a) distribution. In this limit we take

$$\frac{1}{W}=\frac{1}{W_0}+\epsilon \quad \text{with } \langle\epsilon\rangle=0 \text{ and } \langle\epsilon^2\rangle \ll \frac{1}{W_0^2} \ . \tag{2.1}$$

This allows us to expand $\overline{W}$ in a series expansion of s and $\langle\epsilon^2\rangle$. Let us write

$$\frac{1}{\overline{W}}=\frac{1}{W_0}+g \ , \tag{2.2}$$

where g depends on the bias and the disorder parameter ϵ as well as s, and

$$Q=\frac{1}{\overline{W}}\left[1-\frac{f}{\overline{W}}\right] . \tag{2.3}$$

From Eq. (1.5) we get

$$\left\langle \frac{g-\epsilon}{1-f(g-\epsilon)} \right\rangle=0 \tag{2.4}$$

or

$$g+f[g^2+\langle\epsilon^2\rangle]+f^2[g^3+3g\langle\epsilon^2\rangle-\langle\epsilon^3\rangle]+\cdots=0 \ , \tag{2.5}$$

by expanding the denominator in Eq. (2.4). To lowest order, one finds

$$g=-f\langle\epsilon^2\rangle \ . \tag{2.6}$$

From Sec. I one finds a crossover frequency

$$\omega_{\text{co}}=\frac{a_2^2W_0}{2a_1}=\frac{(a^{-1}-a)^2W_0}{2(a+a^{-1})} \ . \tag{2.7}$$

For $\omega<\omega_{\text{co}}$ from Eq. (1.7), $f=s/a_2$, one finds

$$g=-\frac{s}{a_2}\langle\epsilon^2\rangle \ . \tag{2.8}$$

For $\omega>\omega_{\text{co}}$,

$$f=\left[\frac{s\overline{W}}{2a_1}\right]^{1/2}\sim\left[\frac{sW_0}{2a_1}\right]^{1/2} .$$

Thus, one finds

$$g=-W_0\langle\epsilon^2\rangle\left[\frac{s}{2W_0a_1}\right]^{1/2} . \tag{2.9}$$

We obtain the conductivity from Eq. (1.8). For $\omega<\omega_{\text{co}}$,

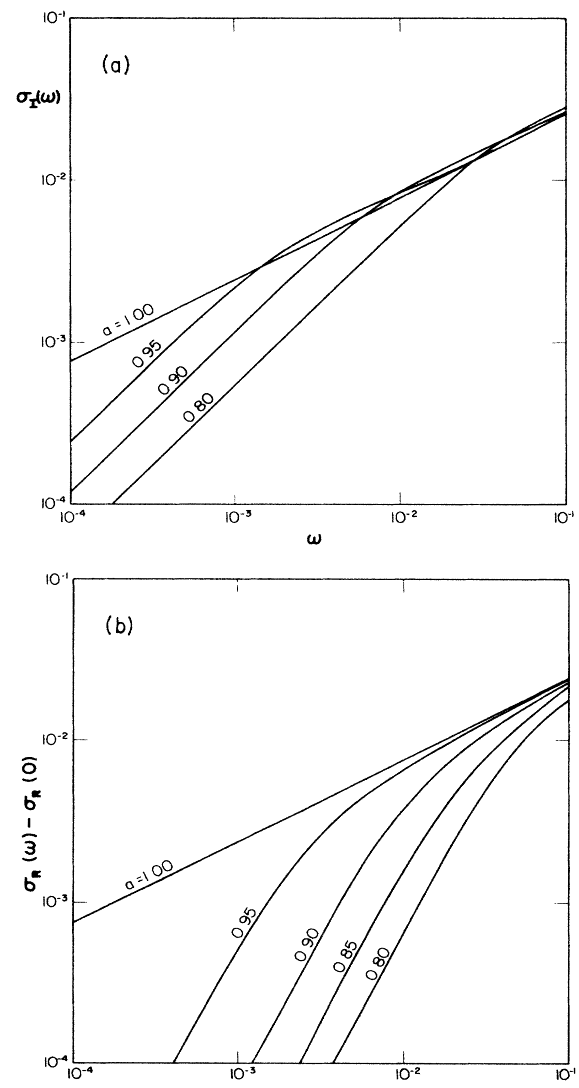

FIG. 1. (a) Imaginary part of the conductivity $\sigma_I(\omega)$ plotted as a function of the frequency for the class-(a) distribution $P(W)=\frac{1}{2}\delta(W-\frac{1}{2})+\frac{1}{2}\delta(W-\frac{3}{2})$ and for different values of bias strength a. Here, ω is reduced by the maximum transfer rate and $\sigma(\omega)$ is also normalized to the dc conductivity. (b) As in (a), the real part of the conductivity $\sigma_R(\omega)$ is plotted as a function of the frequency.

$$\sigma_R(\omega)=W_0\left[1+\frac{\langle\epsilon^2\rangle\omega^2(a+a^{-1})}{(a^{-1}-a)^3}\right], \quad (2.10a)$$

$$\sigma_I(\omega)=\frac{\langle\epsilon^2\rangle W_0^2\omega}{(a^{-1}-a)}. \quad (2.10b)$$

For $\omega>\omega_{\rm co}$,

$$\sigma_R(\omega)=W_0+W_0^3\langle\epsilon^2\rangle\left[\frac{\omega}{4W_0(a+a^{-1})}\right]^{1/2}, \quad (2.11a)$$

$$\sigma_I(\omega)=W_0^3\langle\epsilon^2\rangle\left[\frac{\omega}{4W_0(a+a^{-1})}\right]^{1/2}. \quad (2.11b)$$

These results have been obtained in paper I. We reproduce them here for the sake of completeness. A general series expansion of $\overline{W}$ for any nonsingular distribution can, in principle, be obtained order by order in terms of the inverse moments of $P(W)$.[4] However, as these expansions are fairly complicated, we do not present them here. We also solve the self-consistency equations numerically, using the binary distribution

$$P(W)=\tfrac{1}{2}\delta(W-\tfrac{1}{2})+\tfrac{1}{2}\delta(W-\tfrac{3}{2}).$$

Note that this distribution has all well defined inverse moments as well as the moments. One can easily show that $\langle W^{-1}\rangle^{-1}$ exists and is equal to $\frac{3}{4}$. The results are presented in Fig. 1(a) for the imaginary part of the conductivity σ_I, and in Fig. 1(b) for the ac part of the real part of the conductivity σ_R. One can see clearly that there exists a crossover frequency $\omega_{\rm co}$ which depends on the strength of bias in accord with Eq. (1.4). Figure 1 shows that $\sigma_I(\omega)$ behaves linearly as ω in the drift region, crossing over to $\omega^{1/2}$ in the diffusion region, while $\sigma_R(\omega)$ behaves as ω^2 in the drift region, crossing over to $\omega^{1/2}$ in the diffusion region. The results are consistent with the analytic expansions, Eqs. (2.10) and (2.11), and we thus confirm that the weak-disorder expansions can be equally applied to any general class-(a) distributions.

B. Class (c)

A particular example we shall consider is

$$P(W)=(1-\alpha)W^{-\alpha},\quad 0\le W\le 1,\quad 0<\alpha<1. \quad (2.12)$$

One can easily show that the inverse first moment diverges; therefore, one finds $s\to 0$, $\overline{W}\to 0$. We thus take

$$\overline{W}=0+g, \quad (2.13)$$

where g is a function of α and s, and

$$Q=\frac{1}{\overline{W}}(1-f). \quad (2.14)$$

From the self-consistency equation, one arrives at the following exact relation:

$$\left[\frac{g}{1-f}\right]\left\langle 1-\frac{g}{(1-f)W+fg}\right\rangle=0. \quad (2.15)$$

Thus,

$$g=0 \quad\text{or}\quad 1=\left\langle\frac{g}{(1-f)W+fg}\right\rangle. \quad (2.16)$$

Since we are interested in a nontrivial solution, we arrive at

$$(1-f)^{1-\alpha}g^{\alpha-1}f^{\alpha}=C_\alpha(1-\alpha), \quad (2.17)$$

where C_α is a real number;

$$C_\alpha=\int_0^\infty\frac{x^{-\alpha}dx}{1+x}=\frac{\pi}{\sin(\pi\alpha)}. \quad (2.18)$$

To lowest order, one can ignore $1-f\simeq 1$; thus,

$$g^{\alpha-1}f^{\alpha}=C_\alpha(1-\alpha). \quad (2.19)$$

One can show that there exists a crossover region ($s_{\rm co}^{\rm I}<s<s_{\rm co}^{\rm II}$)

$$s_{\rm co}^{\rm I}\sim\left[\frac{a_2^2}{a_1}\right]^{1/(1-\alpha)}, \quad (2.20)$$

$$s_{\rm co}^{\rm II}\sim\left[\frac{a_2^2}{a_1}\right]^{(2-\alpha)/2(1-\alpha)}. \quad (2.21)$$

In any case, $s_{\rm co}^{\rm I}<s_{\rm co}^{\rm II}$ because $0<\alpha<1$, such that for $s<s_{\rm co}^{\rm I}$, one finds [from Eqs. (1.7) and (2.19)]

$$g\sim s^{\alpha}, \quad (2.22)$$

and similarly for $s>s_{\rm co}^{\rm II}$, one finds

$$g\sim s^{\alpha/(2-\alpha)}. \quad (2.23)$$

This crossover behavior is in accord with the qualitative results discussed in Sec. I, and our result for the second region is still consistent with the symmetric case.

One can calculate the conductivity from Eq. (1.8). Note that

$$i^x=\cos\left[\frac{\pi}{2}x\right]+i\sin\left[\frac{\pi}{2}x\right]$$

is in general a complex number. For $\omega<\omega_1$ (corresponding to $s_{\rm co}^{\rm I}$),

$$\sigma_R(\omega)\sim\cos\left[\frac{\pi}{2}\alpha\right]\omega^{\alpha}, \quad (2.24a)$$

$$\sigma_I(\omega)\sim\sin\left[\frac{\pi}{2}\alpha\right]\omega^{\alpha}. \quad (2.24b)$$

For $\omega>\omega_2$ (corresponding to $s_{\rm co}^{\rm II}$),

$$\sigma_R(\omega)\sim\cos\left[\frac{\pi\alpha}{2(2-\alpha)}\right]\omega^{\alpha/(2-\alpha)}, \quad (2.25a)$$

$$\sigma_I(\omega)\sim\sin\left[\frac{\pi\alpha}{2(2-\alpha)}\right]\omega^{\alpha/(2-\alpha)}. \quad (2.25b)$$

Thus, in the presence of a bias, we obtain a lower-frequency region in which the conductivity behaves differently from the symmetric case, as in Ref. 1. Moreover, the crossover behavior obtained in this singular distribution is nonuniversal and is quite different from those of the class-(a) distribution.

On the other hand, the leading corrections to the results are also very important.[11,12] Let us consider the case $\omega > \omega_2$. Suppose $\alpha > \frac{1}{2}$, the correction from $(1-f)^{1-\alpha}$ is very important (ignoring corrections from $f^\alpha, g^{\alpha-1}$); one finds

$$g = C_\alpha^{(0)} s^{\alpha/(2-\alpha)}(1 - C_\alpha^{(1)} s^{(1-\alpha)/(2-\alpha)}), \quad \alpha \geq \tfrac{1}{2} \tag{2.26}$$

and furthermore, when $\alpha < \frac{1}{2}$, so that correction from f^α is more important, one finds

$$g = C_\alpha^{(0)} s^{\alpha/(2-\alpha)}(1 - C_\alpha^{(2)} s^{\alpha/(2-\alpha)}), \quad \alpha \leq \tfrac{1}{2} \ . \tag{2.27}$$

Richards and Renken[11] considered only the corrections for $\alpha \ll 1$. Our result for $\alpha \leq \frac{1}{2}$ is in agreement with theirs. In the lower-frequency region $\omega < \omega_1$ one finds the corresponding corrections,

$$g = C'_\alpha s^\alpha (1 - C_\alpha^{(1)'} s^{1-\alpha}), \quad \alpha \geq \tfrac{1}{2} \tag{2.28}$$

$$g = C'_\alpha s^\alpha (1 - C_\alpha^{(2)'} s^{\alpha}), \quad \alpha \leq \tfrac{1}{2} \ . \tag{2.29}$$

In Eqs. (2.26)–(2.29) the coefficients are functions of C_α and α.

C. Class (b')

A particular example ($r = 1$) is

$$P(W) = (1+\beta) W^\beta, \quad 0 \leq W \leq 1, \quad 0 < \beta < 1 \ . \tag{2.30}$$

This distribution has a nonzero inverse first moment, so that $\overline{W}(0) = \langle W^{-1} \rangle^{-1}$ is nonzero. But $\sigma - \sigma_{dc}$ will also show nonuniversal behavior because $\langle W^{-2} \rangle$ does not exist.

In order to study this case, we use the exact relationship from the class-(c) distribution [Eq. (2.16)],

$$\left\langle 1 - \frac{g}{(1-f)W + fg} \right\rangle = 0 \ .$$

If one defines

$$h = \frac{fg}{1-f} \ , \tag{2.31}$$

one finds

$$\left\langle \frac{1}{W+h} \right\rangle = \frac{1-f}{g} \ . \tag{2.32}$$

Let

$$\frac{1}{\overline{W}} = \frac{1}{W_0} - \xi \ , \tag{2.33}$$

$$W_0 = \frac{\beta}{1+\beta} \ . \tag{2.34}$$

One obtains

$$\xi = C_{1-\beta}(1+\beta) \left[\frac{f\overline{W}}{1-f} \right]^\beta - \frac{f}{\overline{W}} \ , \tag{2.35}$$

where

$$C_{1-\beta} = \int_0^\infty \frac{x^{\beta-1} dx}{1+x} = \frac{\pi}{\sin[\pi(1-\beta)]} \tag{2.36}$$

is a real number.

There exists a crossover frequency

$$\omega_{co} = \frac{a_2^2}{2a_1} W_0 = \frac{a_2^2}{2a_1} \left[\frac{\beta}{1+\beta} \right] , \tag{2.37}$$

such that for $\omega < \omega_{co}$,

$$\xi = C_{1-\beta}(1+\beta) \left[\frac{s}{a_2} \right]^\beta - \frac{1}{W_0^2 a_2} s \ , \tag{2.38}$$

and for $\omega > \omega_{co}$,

$$\xi = C_{1-\beta}(1+\beta) \left[\frac{W_0}{2a_1} \right]^{\beta/2} s^{\beta/2} - \frac{1}{W_0} \left[\frac{1}{2W_0 a_1} \right] s^{1/2} \ . \tag{2.39}$$

Since $0 < \beta < 1$, this correction is nonuniversal. To get $\overline{W}$, one inverts Eq. (2.33) and obtains

$$\overline{W} = W_0(1 + W_0 \xi) = W_0 + W_0^2 \xi \ . \tag{2.40}$$

We obtain the conductivity for $\omega < \omega_{co}$:

$$\sigma_R(\omega) - \sigma_R(0) \sim \left[\frac{1}{a_2} \right]^\beta \cos \left[\frac{\pi\beta}{2} \right] \omega^\beta \ , \tag{2.41a}$$

$$\sigma_I(\omega) \sim \left[\frac{1}{a_2} \right]^\beta \sin \left[\frac{\pi\beta}{2} \right] \omega^\beta \ . \tag{2.41b}$$

For $\omega > \omega_{co}$,

$$\sigma_R(\omega) - \sigma_R(0) \sim \left[\frac{W_0}{2a_1} \right]^{\beta/2} \cos \left[\frac{\pi\beta}{4} \right] \omega^{\beta/2} \ , \tag{2.42a}$$

$$\sigma_I(\omega) \sim \left[\frac{W_0}{2a_1} \right]^{\beta/2} \sin \left[\frac{\pi\beta}{4} \right] \omega^{\beta/2} \ . \tag{2.42b}$$

Equation (2.42) is in complete agreement with Richards and Renken.[11]

D. Class (d)

An example of this is the bond-percolation model,[7]

$$P(W) = (1-p)\delta(W) + p\delta(W-1), \quad 0 \leq p \leq 1 \ . \tag{2.43}$$

One finds from the self-consistency equation,

$$\left\langle \frac{W - \overline{W}}{1 + Q(W - \overline{W})} \right\rangle = \frac{p(1-\overline{W})}{1 + Q(1-\overline{W})} + \frac{(1-p)(-\overline{W})}{1 + Q(-\overline{W})} = 0 \ . \tag{2.44}$$

Since $\langle 1/W \rangle$ does not exist, we expect

$$\overline{W} \sim 0(s) \tag{2.45}$$

and we find no crossover behavior, as discussed in Sec. I.

After considerable simplification by taking $Q = (1-f)/\overline{W}$, one finds

$$(1 - \overline{W}) f = (1-p) \ . \tag{2.46}$$

Taking

$$f = 1 \Big/ \left[1 + \frac{2\overline{W} a_1}{s} + \frac{\overline{W}^2}{s^2} a_2^2 \right]^{1/2} , \tag{2.47}$$

one arrives at

$$-p(2-p)+2\overline{W}\left[\frac{a_1(1-p)^2}{s}+1\right]+\overline{W}^2\left[\frac{a_2^2(1-p)^2}{s^2}-1\right]=0\ . \quad (2.48)$$

In the case of no bias $a_2=0, a_1=2$, one finds

$$\overline{W}=\frac{p(2-p)}{4(1-p)^2}s\left[1-\frac{s}{2(1-p)^2}\right]. \quad (2.49)$$

This is consistent with Ref. 7. And in the scaling region $(1-p)^2 \ll s$,

$$\overline{W}=\frac{a_1 s}{a_2^2}\left\{\left[1+\left[\frac{a_2}{a_1}\right]^2\frac{1}{(1-p)^2}\right]^{1/2}-1\right\}$$
$$\sim\frac{s}{2a_1(1-p)^2}\ . \quad (2.50)$$

Numerical calculation for the conductivity for $p=\frac{1}{2}$ and $a=\frac{3}{4}$ is presented in Fig. 2. One can see clearly that in all cases, $\overline{W}\sim O(s)$, and the bias is irrelevant in this bond-percolation model. This result can be understood as follows. Random removal of bonds in the one-dimensional chain leads to isolated segments of finite clusters whose typical size is determined by the percolation correlation length. Far from the percolation threshold, the percolation correlation length is very small. The diffusing particle will never travel far enough to experience the drift, and thus diffusion always dominates as in the unbiased case.

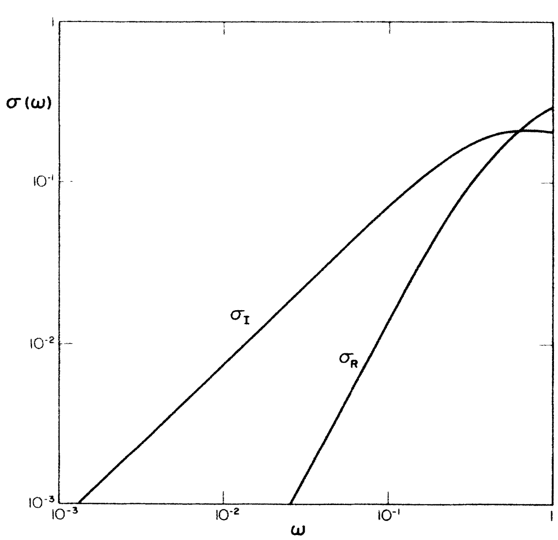

FIG. 2. Real and imaginary parts of the conductivity plotted as a function of the frequency for the percolation model $P(W)=p\delta(W-1)+(1-p)\delta(W)$ and $p=\frac{1}{2}$. Here the strength of bias $a=\frac{3}{4}$, and is fairly strong.

III. HIGH-FREQUENCY EXPANSIONS

Here we also discuss the high-frequency expansion for the class-(a) distributions. In this limit, the inverse moments are irrelevant and we take

$$W=W_0+\delta, \text{ with } \langle\delta\rangle=0 \text{ and } \langle\delta^2\rangle \ll W_0^2\ . \quad (3.1)$$

Let us write

$$\overline{W}=W_0+g\ , \quad (3.2)$$

where g depends on the bias and the disorder parameter δ, as well as s. We assume $\overline{W}\ll s$, so that

$$Q=\frac{a_1}{s}-\frac{\overline{W}}{2s^2}(3a_1^2-a_2^2)\ . \quad (3.3)$$

From the self-consistency equation, one finds

$$g+Q(g^2+\langle\delta^2\rangle)+Q^2(g^3+3g\langle\delta^2\rangle-\langle\delta^3\rangle)+\cdots=0\ . \quad (3.4)$$

To lowest order, one obtains

$$g=-Q\langle\delta^2\rangle\ , \quad (3.5)$$

$$\overline{W}=W_0-\frac{\langle\delta^2\rangle a_1}{s}+\frac{W_0\langle\delta^2\rangle(3a_1^2-a_2^2)}{2s^2}\ . \quad (3.6)$$

We obtain the conductivity

$$\sigma_R(\omega)=W_0-\frac{W_0\langle\delta^2\rangle(3a_1^2-a_2^2)}{2\omega^2}\ , \quad (3.7a)$$

$$\sigma_I(\omega)=\frac{a_1\langle\delta^2\rangle}{\omega}\ . \quad (3.7b)$$

These results are in complete agreement with Ref. 3.

IV. DISCUSSION

In summary we have extended the calculation of the frequency-dependent conductivity to singular distributions. Class-(c) distributions show nonanalytic behaviors for all frequencies. In the presence of a bias, nonuniversal crossover behaviors are also observed. For the percolation model, we do not find any crossover behavior even in the strong bias case.

Here we need to comment on our effective-medium results. As pointed out in Ref. 1, the EMA gives correct asymptotic behavior in the diffusion coefficient for all classes of distribution, including the singular class-(c) distribution. Furthermore, the higher corrections are also correctly predicted by EMA, except possibly that the coefficients may be different from those for the exact solution. Here in the biased case, one should expect EMA to have the same character, namely, that the exponents for the asymptotic behavior and higher corrections are exactly given by the EMA, except possibly that the coefficients may be different from those of the exact solution.

We have proven the validity of the scaling hypothesis[6] within the EMA for the biased case as well as the symmetric case. Such a scaling assumption is still correct in the biased case although the frequency dependence of a scaling length is modified. In this connection, the validity of this scaling assumption was verified recently for the symmetric case by Nieuwenhuizen and Ernst[13] to leading corrections of their asymptotic solutions.

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