Twisting in superfluid ^{3}He-A and consequences for hydrodynamics at $T = 0$

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We study the effects of twisting on the low-energy excitations of superfluid ^{3}He-A. They are small compared to the effects of bending, but they have a different symmetry which makes them important. The structure of the excitation wave functions has strong analogies with the eigenstates of a charged particle in a magnetic field. The group velocity is parallel to curl $\hat{l}$. The density of states at zero energy has a contribution proportional to twisting which is odd with respect to $\hat{k}$. Because of this term a density fluctuation produces a fluctuation of the current carried by the excitations. This current fluctuation turns out to be exactly equal to the fluctuation of the C_0 term in the current. This shows that the C_0 term fluctuation is carried by the normal liquid and leads us to believe that the C_0 term as a whole is linked to the normal liquid. Since calculations making use of the Bardeen-Cooper-Schrieffer wave function consider only the condensate, our results offer a possible explanation for the "angular-momentum paradox." We physically interpret the $\mathbf{v}_n$ term in the Josephson equation at $T = 0$ and show that it is readily obtained from the microscopic description of the normal liquid. In the same way we deduce from the momentum conservation law an equation of motion for $\mathbf{v}_n$ and we show that it can be entirely understood from the microscopic framework.

I. INTRODUCTION

In several papers[1-4] we have studied the low-energy excitation spectrum of ^{3}He-A in the presence of textures. We have found that the density of states for zero energy is nonzero, which gives rise to a nonzero normal density at $T=0$, together with a corresponding change in the superfluid density. These effects are dominantly due to the bending $(\hat{l}\cdot\nabla)\hat{l}$ of the texture.

In this paper, we shall be concerned by the more subtle effects of twisting $\hat{l}\cdot(\nabla\times\hat{l})$. Although the modifications of the excitation spectrum and wave functions due to twisting are small, they are of essential importance for the statics and the hydrodynamics of ^{3}He-A at $T=0$. The reason is that twisting gives rise to a qualitatively new feature: the density of states is no longer even with respect to $\mathbf{k}$. Precisely, we obtain for the density of states at zero energy

$$N_\epsilon(0) = \frac{N_0}{4k_F}\left\{\left[\left[\frac{2E_F}{\delta}(\hat{l}\cdot\nabla)\hat{l}\right]^2 + (\hat{l}\cdot\nabla\times\hat{l})^2\right]^{1/2} - \epsilon\hat{l}\cdot\nabla\times\hat{l}\right\}, \tag{1}$$

where $\epsilon = \hat{k}\cdot\hat{l} = \pm 1$, depending on whether we consider excitations with wave vectors $\hat{k}$ parallel or antiparallel to $\hat{l}$; N_0 is the density of states in the normal state and δ the maximum of the A-phase gap $|\Delta_k| = \delta\sin(\hat{k}\cdot\hat{l})$. If we introduce the notations

$$\mathbf{B} = \mathrm{curl}\,\hat{l}, \quad B_{||} = \mathbf{B}\cdot\hat{l}, \quad B_\perp = \mathbf{B} - B_{||}\hat{l}, \tag{2}$$

and take into account $(\hat{l}\cdot\nabla)\hat{l} = -\hat{l}\times\mathrm{curl}\,\hat{l}$, we can rewrite

$$N_\epsilon(0) = \frac{N_0}{4k_F}\left[\left[B_{||}^2 + 4\frac{E_F^2}{\delta^2}B_\perp^2\right]^{1/2} - \epsilon B_{||}\right]. \tag{3}$$

In a standard situation, $B_{||} \sim B_\perp$, and since $\delta \ll E_F$,

$$N_\epsilon(0) \simeq \frac{N_0}{4k_F}\left[\frac{2E_F}{\delta}|B_\perp| - \epsilon B_{||}\right]$$

$$= \frac{N_0}{4}\left[\frac{v_F}{\delta}|B_\perp| - \frac{\epsilon}{k_F}B_{||}\right] \tag{4}$$

is an excellent approximation, except in the peculiar case where $|B_\perp| \lesssim |B_{||}|\,\delta/E_F$. In all other cases we see that twisting $B_{||}$ leads to a small correction compared to the effect of bending $B_\perp$, in agreement with our earlier approximation.[1-4] The total density of states

$$N(0) = N_+(0) + N_-(0) = \frac{N_0}{2k_F}\left[B_{||}^2 + \frac{4E_F^2}{\delta^2}B_\perp^2\right]^{1/2}. \tag{5}$$

agrees with our earlier result[1,3]

$$N(0) = N_0\frac{v_F}{2\delta}|B_\perp| = N_0\frac{v_F}{2\delta}|(\hat{l}\cdot\nabla)\hat{l}| \tag{6}$$

in the standard case $|B_\perp| \gg |B_{||}|\,\delta/E_F$.

We note that our distinction between excitations with $\hat{k}$ parallel or antiparallel to $\hat{l}$ is somewhat artificial. Indeed, for the energy band which produces the density of states at zero energy, one can show that there is no physical difference between creating an excitation $\hat{k}$ and destroying an excitation $-\hat{k}$. Therefore one can very well work out everything by considering only excitations with $\hat{k}\simeq\hat{l}$. However, we feel that our picture is physically more convenient and more appealing. We shall defer the proof of this result to Sec. V. In the following sections we shall instead look for the consequences.

II. THE CURRENT

We shall first investigate the expression for the current. There is a general agreement[5-7] on the following result for the superfluid current under static conditions at $T=0$,

$$\mathbf{g}=\rho\mathbf{v}_s+\frac{\hbar}{4m}\nabla\times(\rho\hat{l})-\frac{\hbar}{2m}C_0\hat{l}(\hat{l}\cdot\nabla\times\hat{l})\ ,\tag{7}$$

if we forget momentarily the contribution due to the small normal density.[1-3,8] Here, $C_0\simeq\rho$ to a very good approximation. The correction is of order $(T_c/E_F)^2$, which is too small to be considered in this paper.

In hydrodynamics one must use expression (7) since one assumes local equilibrium. One naturally has to wonder about the relaxation time necessary to reach this equilibrium since hydrodynamics is valid only for very slow motion compared to this time. In $^3\mathrm{He}$-B at $T=0$ there is no problem since the excitations have completely disappeared. Physically, only the superfluid is present and its relaxation time is very fast, of order of $\hbar/\Delta$. In $^3\mathrm{He}$-A, as we will see, the situation is quite different because there is no gap for the excitations and the density of states $N(0)$ is nonzero, which opens the possibility of normal liquid effects.

In order to look for these effects we want to calculate the current under dynamical conditions. Phase fluctuations are linked to density fluctuations through the Josephson equation. To lowest order, which is enough for our present purpose, we have

$$\delta\rho\equiv\rho-\rho_0=-\frac{\hbar}{2}N_0\partial_t\phi\ ,\tag{8}$$

where $\delta\rho$ is the particle density fluctuation around some equilibrium value ρ_0. Phase fluctuations of the order parameter induce corresponding fluctuations in the current. We will exhibit normal liquid effects by calculating the response under two different boundary conditions. First, we can calculate in a collisionless situation: We start with the system in equilibrium with an external bath and give a time dependence to the phase of the order parameter. The superfluid reacts, but there is no relaxation process which allows the excitations to come into equilibrium with the superfluid. In practice, the current is calculated by directly solving Gorkov's equations with an explicit time dependence in the order parameter. On the other hand, we can easily perform a second calculation where, in the same dynamical situation, the excitations are allowed to relax toward the instantaneous equilibrium given by the superfluid, which corresponds to the hydrodynamic regime. This is done by performing a gauge transformation which eliminates the time dependence of the order parameter. One is left with a static order parameter in the presence of a gauge field, which in our case corresponds merely to a shift in the chemical potential, $\delta\mu=\delta\rho/N_0=-\hbar\partial_t\phi/2$. If we calculate the current by solving Gorkov's equations under these conditions, the excitations are physically in equilibrium with the gauge field, that is, the superfluid. The result of this last calculation must obviously be Eq. (7) with $\rho=\rho_0+\delta\rho$, and this is indeed what is found. This means that the current fluctuation $\delta\mathbf{g}_h$ under hydrodynamic conditions is given by

$$\delta\mathbf{g}_h=\delta\rho\,\mathbf{v}_s+\frac{\hbar}{4m}\mathrm{curl}(\hat{l}\,\delta\rho)-\frac{\hbar}{2m}\delta\rho\,\hat{l}(\hat{l}\cdot\mathrm{curl}\hat{l})\ .\tag{9}$$

The difference $\delta\mathbf{g}_h-\delta\mathbf{g}_c$ between $\delta\mathbf{g}_h$ and the current fluctuation $\delta\mathbf{g}_c$ under collisionless conditions will give us the part of the current fluctuation carried by the excitations. Since the result is fairly obvious physically, the details of the calculation are given in Appendix A. The difference between the two terms comes from the fact that in the hydrodynamic situation the excitations have relaxed to the equilibrium corresponding to the chemical-potential shift $\delta\mu$, while in the collisionless situation they have not. Therefore, in the hydrodynamic situation, we have an additional number of excitations, $N_+(0)\delta\mu$, each carrying a momentum $\hbar\mathbf{k}=\hbar k_F\hat{l}$, and an additional number of excitations, $N_-(0)\delta\mu$, each carrying a momentum $\hbar\mathbf{k}=-\hbar k_F\hat{l}$. This gives rise to an additional current,

$$\delta\mathbf{g}_h-\delta\mathbf{g}_c=\frac{\hbar k_F}{m}\hat{l}N_+(0)\delta\mu-\frac{\hbar k_F}{m}\hat{l}N_-(0)\delta\mu$$

$$=-\frac{\hbar}{2m}\delta\rho\,\hat{l}\cdot(\hat{l}\cdot\mathrm{curl}\hat{l})\ ,\tag{10}$$

if we take Eq. (1) into account. Upon comparison with Eq. (9), we obtain

$$\delta\mathbf{g}_c=\delta\rho\,\mathbf{v}_s+\frac{\hbar}{4m}\mathrm{curl}(\hat{l}\,\delta\rho)\ .\tag{11}$$

The physical picture revealed by Eqs. (7), (9), and (11) is very simple, even if it is somewhat surprising. The current in Eq. (7) was considered until now as being all supercurrent. Actually, we see that the fluctuations of the first two terms of Eq. (7) are carried by the superfluid since it already exists in the collisionless situation. However, the fluctuation of the C_0 term is carried by the excitations since it appears only when we let these excitations relax to equilibrium with the superfluid.

It is natural to generalize this result and consider that excitations are responsible for the whole C_0 term in Eq. (7). This can be done in the following way. We start with a superfluid with a very small density, which therefore carries almost no current. We work with a fixed texture; that is, $\mathbf{v}_s$ and $\hat{l}$ do not change. We raise the density step by step. In each step we slightly increase the chemical potential, while keeping, at first, the excitation distribution frozen. This gives an increase in density, $\delta\rho$, and a corresponding increase, Eq. (11), for the current carried by the superfluid. Then we let the excitations relax to the new equilibrium, which gives rise to an additional current, Eq. (10). If we stop at some density ρ, we find that the sum of all the superfluid-current increases is given by the first two terms of Eq. (7), while the sum of all the currents due to excitations will just give the C_0 term.

Our result provides a physical explanation of all the difficulty[1,2,8] encountered in trying to build the hydrodynamics of $^3\mathrm{He}$-A at $T=0$. This hydrodynamics would actually be a dynamics of the superfluid alone. Let us recall that all the problems come from the C_0 term. Without it the current would be the same[6,8] as in a Bose condensate of diatomic molecules, each carrying an intrinsic angular momentum $\hbar\hat{l}$. More generally, the hydro-

dynamic equations would be the same as for this Bose condensate. However, if we have in the current a term which is physically linked to the excitations and not to the superfluid, it is easy to understand that one cannot obtain a satisfactory hydrodynamics without allowing the excitations to be described by some degrees of freedom.

This physical interpretation of the C_0 term may also facilitate understanding of the discrepancy between various calculations of the angular momentum of ^{3}He-A samples. Indeed, direct calculations[9] using the symmetry of the Bardeen-Cooper-Schrieffer (BCS) wave function agree with the calculations using Eq. (7) for the current only if $C_0 = 0$: they apparently take properly into account the first two terms of Eq. (7) only, which are related to the superfluid, as we have seen. This can be understood if we consider that the C_0 term does not originate from the condensate but from the excitations; in other words, it is a term related to the normal liquid. In this view it is not astonishing that calculations working directly with the BCS wave function miss the C_0 term, since they clearly consider only the condensate and ignore any normal liquid degree of freedom. We may also say that the condensate alone corresponds to a highly excited state with the normal liquid far from equilibrium with the superfluid. In contrast, Eq. (7) corresponds to a different state where superfluid and normal liquid are in equilibrium. Naturally, the real physical situation is described by this last state. It would obviously be advantageous to have a direct calculation explicitly showing that the contribution of the normal liquid to the current is the C_0 term. However, this would involve the consideration of very-high-energy excitations which are beyond our study. We just note that the expression of Mermin and Muzikar[6] for C_0 is suggestive of such a result since it contains the one-particle density matrix for $\mathbf{k}$ parallel to $\hat{l}$ and integrated over k.

We note, finally, a result[10] of Volovik and Balatskii that is rather similar to ours. They studied the response of the current to a time-dependent external potential and found that C_0 does not change, except at zero frequency. However, they interpreted this result as an intrinsic dynamical invariance of C_0 within the hydrodynamical regime. This is in contrast to our physical picture, where the C_0 term is linked to the normal liquid. In their paper this normal liquid plays no role in hydrodynamics. We remark also that their result is obtained within a gradient expansion that is invalid and leads to singular results.

III. THE JOSEPHSON EQUATION

We will now make use of the same kind of arguments as were used in the preceding section to show that the problems arising in the Josephson equation come from the presence of the excitations, and that it is possible to understand them in detail. The problem is the following.[2,8] Formal hydrodynamics[11] gives the Josephson equation

$$-\frac{\hbar}{2}\frac{\partial \phi}{\partial t} = \mu + m\mathbf{v}_n \cdot \mathbf{v}_s + \frac{\hbar}{4}\hat{l} \cdot \nabla \times \mathbf{v}_n \ , \tag{12}$$

where the chemical potential μ is the functional derivative with respect to the density ρ of the energy density ϵ at

constant current $\mathbf{g}$. By making use of a Legendre transform, we express it with derivatives at constant $\mathbf{v}_n$:

$$\mu = \frac{\partial}{\partial \rho}(\epsilon - \mathbf{g}\cdot\mathbf{v}_n) - \partial_i\left[\frac{\partial}{\partial(\partial_i\rho)}(\epsilon - \mathbf{g}\cdot\mathbf{v}_n)\right] \ . \tag{13}$$

Since we have a small normal density $\rho_n \sim |(\hat{l}\cdot\nabla)\hat{l}|$, we must consider the term $\frac{1}{2}\mathbf{v}_n\rho_n\mathbf{v}_n$ in ϵ. However, this term gives a third-order contribution in μ, which must be neglected since hydrodynamics retains only second-order terms. At this level we have

$$\epsilon - \mathbf{g}\cdot\mathbf{v}_n = \mu_0(\rho) + m(\mathbf{v}_s - \mathbf{v}_n)\cdot\mathbf{g} - \frac{1}{2}m\rho\mathbf{v}_s^2 + \cdots \ , \tag{14}$$

where the ellipsis corresponds to all the terms which do not contain $\mathbf{v}_s$ or $\mathbf{v}_n$, and $\mu_0(\rho) \simeq \mu_0(\rho_0) + \delta\rho/N_0$ is the chemical potential of the homogeneous system at rest. Equations (12), (13), and (7) lead to [we omit the unimportant constant $\mu_0(\rho_0)$]

$$-\frac{\hbar}{2}\frac{\partial \phi}{\partial t} = \frac{\delta\rho}{N_0} + \frac{m}{2}\mathbf{v}_s^2$$

$$-\frac{\hbar}{2}[(\mathbf{v}_s - \mathbf{v}_n)\cdot\hat{l}][\hat{l}\cdot\text{curl}\,\hat{l}] + \cdots \ . \tag{15}$$

If we want to forget about the normal liquid, it is impossible to understand the second term since it contains $\mathbf{v}_n$ at $T = 0$. Dropping the v_n term leads to a Josephson equation which is no longer Galilean invariant, as was found by Volovik and Mineev.[8]

Now we show that this term is directly produced by the excitations. Indeed, we know that in the presence of a superfluid velocity $\mathbf{v}_s$ all the excitation energies are shifted by an amount $\hbar\mathbf{k}\cdot\mathbf{v}_s$. If we let the system relax to equilibrium, a number $\hbar k_F\mathbf{v}_s\cdot\hat{l}N_+(0)$ of excitations with $\hat{\mathbf{k}} = \hat{l}$ disappear and it appears a number of $\hbar k_F\mathbf{v}_s\cdot\hat{l}N_-(0)$ of excitations with $\hat{\mathbf{k}} = -\hat{l}$. The result is a contribution:

$$\delta\rho = -\hbar k_F\mathbf{v}_s\cdot\hat{l}[N_+(0) - N_-(0)]$$

$$= N_0\frac{\hbar}{2}\mathbf{v}_s\cdot\hat{l}(\hat{l}\cdot\text{curl}\,\hat{l}) \tag{16}$$

to the density fluctuation which agrees exactly with Eq. (15).

In the presence of a normal velocity $\mathbf{v}_n$, the distribution of the excitations of energy E_k is given by $f(E_k - \mathbf{k}\cdot\mathbf{v}_n)$, where $f(E_k)$ is the Fermi distribution, in our case a step function. Since we have excitations with only $\hat{\mathbf{k}} = \pm\hat{l}$, the normal velocity amounts to a chemical potential shift $\pm k_F\hat{l}\cdot\mathbf{v}_n$ for the excitations with $\hat{\mathbf{k}} = \pm\hat{l}$. This modifies Eq. (16) into

$$\frac{\delta\rho}{N_0} = \frac{\hbar}{2}(\mathbf{v}_s - \mathbf{v}_n)\cdot\hat{l}(\hat{l}\cdot\text{curl}\,\hat{l}) \ . \tag{17}$$

In agreement with Eq. (15). This result is also obvious from Eq. (16) by Galilean invariance. Since hydrodynamics is independent of any microscopic picture, it is a rather remarkable feature of hydrodynamics that the troublesome terms containing $\mathbf{v}_n$ at $T = 0$ are only functions of $\hat{l}\cdot\mathbf{v}_n$. This feature finds a simple explanation in terms of our $\hat{\mathbf{k}} = \pm\hat{l}$ excitations. Conversely, since the term

$(\mathbf{v}_s - \mathbf{v}_n) \cdot \hat{l}(\hat{l} \cdot \mathrm{curl}\,\hat{l})$ in Eq. (15) originates from the C_0 term in the current, this is an additional reason to believe that this term has nothing to do with the condensate but rather originates from the normal liquid.

We note finally that if we consider a collisionless situation where the excitations do not have time to relax, the above contribution to the density fluctuation will not arise. Therefore in this regime the Josephson equation should not contain the term $(\mathbf{v}_s - \mathbf{v}_n) \cdot \hat{l}(\hat{l} \cdot \mathrm{curl}\,\hat{l})$. Such an equation would describe physically the motion of the superfluid alone.

IV. THE MOMENTUM CONSERVATION LAW

In ^{4}He as well as in ^{3}He-B, the momentum conservation law becomes an identity at $T=0$. Indeed, the supercurrent is expressed in terms of the order parameter and the conserved quantities, and one can calculate its time derivative from the conservation laws and equations of motion for the order parameter. Then one can check that momentum is conserved and identify the stress tensor. In contrast at $T \neq 0$, the momentum conservation law is an independent equation and it provides an equation of motion for $\mathbf{v}_n$.

In ^{3}He-A one can try to proceed in the same way, starting from the supercurrent Eq. (7). As pointed out by Volovik and Mineev,[8] it appears impossible to write the time derivative of this supercurrent as the divergence of a stress tensor: the supercurrent is not conserved. This prompted Volovik and Mineev to propose the existence of a normal fluid at $T=0$.

The excitations that we have found provide a quantitative description of the normal fluid and we will show that this allows an understanding, in detail, of the momentum conservation law. Actually, we are in situation analogous to ^{4}He and ^{3}He-B at $T \neq 0$: the momentum conservation law gives an equation for $\mathbf{v}_n$. It is obtained by starting from the full expression for the current,

$$\mathbf{g} = \mathbf{g}^s + \rho_n[\hat{l} \cdot (\mathbf{v}_n - \mathbf{v}_s)] \ , \tag{18}$$

where $\mathbf{g}^s$ is the superfluid current given by Eq. (7) and ρ_n is the small normal density at $T=0$:

$$\rho_n = N(0)k_F^2 = \frac{N_0}{2}k_F \left[B_{\parallel}^2 + \frac{4E_F^2}{\delta^2}B_{\perp}^2 \right]^{1/2}$$

$$\simeq \tfrac{3}{2}\rho v_F \frac{|(\hat{l} \cdot \nabla)\hat{l}|}{\delta} \ . \tag{19}$$

We know the time derivative of $\mathbf{g}$ from the momentum conservation law:

$$\frac{\partial g_i}{\partial t} + \partial_j \sigma_{ij} = 0 \ . \tag{20}$$

The time derivative of $\mathbf{g}_s$ is obtained from Eq. (7) and hydrodynamic equations. Finally, hydrodynamics gives that $\partial \hat{l}/\partial t$ is second order in gradients. This makes the terms coming from $\partial \hat{l}/\partial t$ and $\partial \rho_n/\partial t$ in the last term of Eq. (18) negligible, because they are fourth order in gradients. This leads to

$$\rho_n \hat{l}_i \left[\hat{l} \cdot \frac{\partial}{\partial t}(\mathbf{v}_n - \mathbf{v}_s) \right] = -\partial_j \sigma_{ij} - \frac{\partial g_i^s}{\partial t} \ . \tag{21}$$

The tedious explicit calculation of the right-hand side (rhs) of Eq. (21) is done in Appendix B and here we only give the result. We first note that the rhs of Eq. (21) is indeed proportional to $\hat{l}_i$ as required by the lhs. This is a remarkable result since again the calculation of this rhs involves only hydrodynamics. However, this result is not obtained for the most general hydrodynamics. We have to assume some specific values for the unknown coefficients entering the hydrodynamic equation[11] for $\partial \hat{l}/\partial t$, namely

$$\beta = \frac{\hbar}{2m}\frac{1}{\rho - C_0}, \quad \alpha_1 - \alpha_2 = -\frac{\hbar}{2m}, \quad \alpha_1 + \alpha_2 = \beta C_0 \ . \tag{22}$$

The second relation is imposed by hydrodynamics.[11] The third one can be obtained from the requirement that for $T=0$ the terms proportional to $\partial_i v_j^n$ disappear from the $\partial \hat{l}/\partial t$ equation. This is a natural requirement since these terms are present even if there is no texture. In this case there is no normal liquid and the $\mathbf{v}_n$ terms should all disappear. The first relation comes from microscopic theory[12,8] (naturally, we have to consider here that $\rho - C_0 \neq 0$).

Since

$$\partial(\hat{l} \cdot \mathbf{v}_{n,s})/\partial t \simeq \hat{l} \cdot \partial \mathbf{v}_{n,s}/\partial t \ ,$$

we can introduce the scalars $v_{n,s} = \hat{l} \cdot \mathbf{v}_{n,s}$. Then Eq. (21) can be transformed into (cf. Appendix B)

$$\rho_n \frac{\partial}{\partial t}(v_n - v_s) - \frac{1}{2m}\frac{\partial C_0}{\partial t}\hat{l} \cdot \nabla \times \hat{l}$$

$$- \frac{3}{2m}C_0 \left[\frac{\partial \hat{l}}{\partial t} + \nabla v_n \right] \cdot \nabla \times \hat{l} = 0 \ , \tag{23}$$

a rather simple result for the equation of motion for v_n. The $\partial \hat{l}/\partial t$ term has already been obtained under another form by Volovik and Mineev.[8]

Now we interpret Eq. (23) as the equation for the time evolution of the momentum carried by the normal liquid. Since $\rho_n \partial v_n/\partial t \simeq \partial(\rho_n v_n)/\partial t$, the $\partial v_n/\partial t$ term is the time derivative of the normal current $\rho_n v_n$ (all the currents are parallel to $\hat{l}$). We have also seen that the fluctuations of the C_0 term in the current are carried by the excitations. Therefore the $\partial v_n/\partial t$ term and the $\partial C_0/\partial t$ term together represent the rate of change of the momentum carried by the normal liquid. This rate would be zero if the superfluid did not act on the excitations. The other terms in Eq. (23) represent the momentum transferred from the superfluid to the excitations, as we will see now microscopically.

If we insist we are at $T=0$ and in the hydrodynamic regime, the excitation distribution is very simple to describe since we have only excitations for $\hat{\mathbf{k}} = \pm \hat{l}$. Indeed, by definition of the normal velocity, the excitation distribution is given by $\Theta(\hbar\mathbf{k} \cdot \mathbf{v}_n - E_k)$, where Θ is the unit step function. This gives a distribution $\Theta(\hbar k_F v_n - E_k)$ if $\mathbf{k} = k_F\hat{l}$ and $\Theta(-\hbar k_F v_n - E_k)$ if

$\mathbf{k} = -k_F \hat{l}$. We note that v_n is the only physically meaningful quantity, as opposed to $\mathbf{v}_n$, since $\hat{\mathbf{k}} = \pm \hat{l}$. Now since collisions conserve momentum they are unable to give a net transfer of an excitation from $\hat{\mathbf{k}} = \hat{l}$ to $\hat{\mathbf{k}} = -\hat{l}$ vice versa. Therefore the two excitation distributions are frozen and no excitation will be created or destroyed. If because of the motion of the superfluid the energy of an excitation changes, this will not change the occupation of this excited state. The condition that the distributions are frozen can be written explicitly as

$$\frac{d}{dt}(E_k + \mathbf{k}\cdot\mathbf{v}_s - \mathbf{k}\cdot\mathbf{v}_n) = 0 \ , \tag{24}$$

where we have written explicitly the contribution $\mathbf{k}\cdot\mathbf{v}_s$ of the superfluid velocity to the excitation energy. The excitation energy $E_k + \mathbf{k}\cdot\mathbf{v}_s$ can only vary because of the motion of the superfluid (otherwise the excitation moves at constant energy). On the other hand, $\mathbf{v}_n$ can vary because of its time dependence or because the excitation moves with its group velocity $\mathbf{v}_g$. This leads to

$$\frac{\partial E_k}{\partial t} - \frac{\partial}{\partial t}\mathbf{k}\cdot(\mathbf{v}_n - \mathbf{v}_s) - \mathbf{v}_g\cdot\nabla(\mathbf{k}\cdot\mathbf{v}_n) = 0 \ . \tag{25}$$

We see that the freezing of the excitation distribution leads to a time dependence for $\mathbf{v}_n$.

We obtain $\partial E_k/\partial t$ from the expression[3] for E_k,

$$E_k = -\delta\hat{\mathbf{k}}\cdot\hat{\mathbf{B}}_\perp \ . \tag{26}$$

Because of the motion of $\hat{l}$, $\hat{\mathbf{B}}_\perp$ changes. Since $\hat{\mathbf{k}} \simeq \pm \hat{l}$, the main contribution to $\partial_t E_k$ comes from the components of $\partial_t \hat{\mathbf{B}}_\perp$ parallel to $\hat{l}$. With $\hat{l}\cdot\partial_t\hat{\mathbf{B}}_\perp = -\hat{\mathbf{B}}_\perp\cdot\partial_t\hat{l}$, this leads to

$$\frac{\partial E_k}{\partial t} = \delta(\hat{\mathbf{k}}\cdot\hat{l})\hat{\mathbf{B}}_\perp\cdot\frac{\partial\hat{l}}{\partial t} = \frac{\delta\hat{\mathbf{k}}\cdot\hat{l}}{|(\hat{l}\cdot\nabla)\hat{l}|}\frac{\partial\hat{l}}{\partial t}\cdot\nabla\times\hat{l} \tag{27}$$

since $\hat{l}\cdot\partial_t\hat{l} = 0$. This result is justified in detail in Appendix C, where the effect of twisting, neglected in Eq. (27), is also considered. The result is to replace $\hat{\mathbf{B}}_\perp$ by $B_\perp/\gamma B$.

We will show in the next section that the group velocity $\mathbf{v}_g$ of the excitations is given by

$$\mathbf{v}_g = -\frac{\delta}{k_F}\frac{\mathbf{B}}{[B_\perp^2 + (\delta^2/4E_F^2)B_\parallel^2]^{1/2}} \simeq -\frac{\delta}{k_F}\frac{\nabla\times\hat{l}}{|(\hat{l}\cdot\nabla)\hat{l}|} \ , \tag{28}$$

where the last approximation holds for the standard situation $|B_\perp| \gg |B_\parallel|/E_F$. This result and Eq. (27) leads from Eq. (25) to

$$\left[\frac{\partial\hat{l}}{\partial t} + \nabla v_n\right]\cdot\nabla\times\hat{l} - \frac{k_F}{\delta}|(\hat{l}\cdot\nabla)\hat{l}|\frac{\partial}{\partial t}(v_n - v_s) = 0 \ . \tag{29}$$

If we take $C_0 = \rho$ and take into account Eq. (19) for ρ_n, we see that Eq. (29) agrees exactly with Eq. (23) for the superfluid contribution to $\partial(\rho_n v_n)/\partial t$. We note that the $\partial C_0/\partial t$ term represents the momentum change of the excitations in the hydrodynamic regime under a change of chemical potential of the superfluid. Therefore we could also obtain this term from Eq. (24) by taking into account the effect on the excitations of a shift in chemical potential.

In closing this section let us point out that hydrodynamics is not expected to apply to the real system and our discussion is only of formal interest. Indeed, hydrodynamics holds only if the excitation mean free path l is small compared to the typical length L for variations of $\hat{l}$, so that we can define a local $\mathbf{v}_n(\mathbf{r})$. We have $l \sim v_g\tau \sim \delta\tau/k_F$ and we can estimate the relaxation time from an extrapolation of the "high"-temperature expression[13]

$$\hbar/\tau \sim (k_B T)^4/E_F(k_B T_c)^2$$

down to the crossover temperature T_L, where the thermal energy $k_B T_L$ is equal to the typical energy of the quantized spectrum $(\delta\hbar v_F/L)^{1/2}$ [this is the temperature where only the zeroth energy level is occupied and where the expression of the normal density saturates to its zero-temperature expression, Eq. (19)]. The relaxation time for $T < T_L$ will be longer than this crossover value τ_L. We have $\tau_L \sim mL^2/\hbar$, which leads to $l/L \sim L/\xi_0 \gg 1$, where $\xi_0 = \hbar v_F/\delta$ is the ^{3}He-A coherence length. Therefore the condition $l \ll L$ is never satisfied.

V. EXCITATION WAVE FUNCTIONS

In this section we want to find the wave functions of the quasiparticles trapped in a texture. We have already considered this problem[1-4] and found the corresponding Green's functions, excitation spectrum, and wave functions. But, in this treatment we have systematically neglected correction of order δ/E_F. However, it is known from gradient expansions that such terms give important contributions in hydrodynamics because their symmetry is different from the dominant ones. For example, it is important to retain the $\mathbf{q}$ in the argument of the gap Δ_{k+q}, even if this contribution is of order δ/E_F compared to similar terms coming from the kinetic energy. Here we want to handle our problem to a similar level of approximation.

For our purpose we do not need the Green's functions. All the information that we want is contained in the excitation wave functions, and we will solve directly for them as it has been done by Ho et al.[14] (the mathematics is basically the same, but the physics is more transparent). The spinor wave function (ϕ_1, ϕ_2) is solution of the Bogoliubov equations:[15]

$$\begin{bmatrix} p^2/2m - \mu & \frac{1}{2}(\Delta\cdot\mathbf{p} + \mathbf{p}\cdot\Delta) \\ \frac{1}{2}(\Delta^*\cdot\mathbf{p} + \mathbf{p}\cdot\Delta^*) & -(p^2/2m - \mu) \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \omega\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \ , \tag{30}$$

where $\mu = k_F^2/2m$ is the chemical potential, $\mathbf{p} = -i\nabla$, and in ^{3}He-A

$$\Delta \equiv \frac{\delta}{k_F}[\hat{\Delta}_1(\mathbf{r}) + i\hat{\Delta}_2(\mathbf{r})] \ , \tag{31}$$

with $\hat{\Delta}_1\cdot\hat{\Delta}_2 = 0$ and $\hat{l} = \hat{\Delta}_1\times\hat{\Delta}_2$.

We consider a low-energy excitation and look for its wave function around some arbitrary origin O in space.

The spinor wave function (ϕ_1, ϕ_2) for such an excitation must have around O a rapid spatial variation $e^{i\mathbf{k}\cdot\mathbf{r}}$ with $\mathbf{k} \simeq k_F \hat{l}(0)$. We take the x, y, and z axes along $\hat{\Delta}_1(0)$, $\hat{\Delta}_2(0)$, and $\hat{l}(0)$, respectively. We can set, without loss of generality,

$$(\phi_1, \phi_2) = e^{ik_F z}(\psi_1, \psi_2) , \tag{32}$$

where (ψ_1, ψ_2) is slowly varying on an atomic scale. This gives the Bogoliubov equations:

$$\begin{bmatrix} -\tfrac{1}{2}m^{-1}\nabla^2 - iv_F\partial_z & -i\mathbf{\Delta}\cdot\nabla + k_F\Delta_z - \tfrac{1}{2}i\,\mathrm{div}\nabla \\ -i\mathbf{\Delta}^*\cdot\nabla + k_F\Delta_z^* - \tfrac{1}{2}i\,\mathrm{div}\mathbf{\Delta}^* & \tfrac{1}{2}m^{-1}\nabla^2 + iv_F\partial_z \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \omega \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} . \tag{33}$$

The approximation made in our preceding work corresponds to retaining only the $iv_F\partial_z$ and the $k_F\Delta_z$ terms. The others are expected to give corrections of order δ/E_F at most. However, the term $\mathbf{\Delta}(\mathbf{r})\cdot\nabla$ has a different symmetry which corresponds precisely to the $\mathbf{q}$ dependence in $\Delta_{\mathbf{k}+\mathbf{q}}$ mentioned above. So we keep this term, but we can simplify it somewhat: Since we are interested in the wave function around the origin, we may approximate $\mathbf{\Delta}(\mathbf{r})$ by $\mathbf{\Delta}(0)$. Since this amounts to neglecting the spatial dependence of $\mathbf{\Delta}(\mathbf{r})$, we have consistently to drop the $\mathrm{div}\mathbf{\Delta}$ term. This is also necessary if we want to keep the Hamiltonian Hermitian. In the spirit of our preceding work, we expand—in the dominant term $k_F\Delta_z(\mathbf{r})$—the spatial variation of Δ to first order, which gives $\Delta_z(\mathbf{r}) \simeq \mathbf{r}\cdot\nabla\Delta_z$. Since $(d\mathbf{\Delta})\cdot\hat{l} = -\mathbf{\Delta}\cdot d\hat{l}$, we obtain $\Delta_z(\mathbf{r}) \simeq -\hat{l}(\mathbf{r})\cdot\mathbf{\Delta}(0)$ with $\hat{l}(\mathbf{r}) \simeq \hat{l}(0) + (\mathbf{r}\cdot\nabla)\hat{l}$. Finally, as before, we neglect the $\nabla^2/2m$, which gives a correction with no particular symmetry. This leads to

$$\begin{bmatrix} v_F p_z & \mathbf{\Delta}(0)\cdot[\mathbf{p} - k_F\hat{l}(\mathbf{r})] \\ \mathbf{\Delta}^*(0)\cdot[\mathbf{p} - k_F\hat{l}(\mathbf{r})] & -v_F p_z \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \omega \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} . \tag{34}$$

We note that this approximation is in agreement with what we would obtain from a semiclassical approximation[16] by assuming $\hat{\mathbf{p}}$ and $\hat{l}(\mathbf{r})$ almost parallel to the z axis in order to have a small energy. Then

$$\mathbf{p}\cdot\mathbf{\Delta}(r) \simeq k_F\Delta_z(\mathbf{r}) + \mathbf{p}\cdot\mathbf{\Delta}(0)$$

to lowest order in the small quantities $\Delta_z(\mathbf{r})$ and $\mathbf{p}\cdot\mathbf{\Delta}(0)$, in agreement with our off-diagonal term. In the same way, if we set $\mathbf{p} = k_F\hat{l}(0) + \mathbf{p}'$, we have, for the kinetic energy, $\mathbf{p}^2/2m - \mu \simeq v_F p_z'$, which corresponds to our approximation. We note also that some features of our preceding solution[3] will be missing since we used a rapid variation $e^{i\mathbf{k}\cdot\mathbf{r}}$ for the wave function instead of $e^{ik_F z}$, as we do here. In order to recover the same results, we should keep the $\mathbf{r}$ dependence in $\mathbf{\Delta}(\mathbf{r})\cdot\mathbf{\Delta}$. However, the missing terms are small and they do not have the symmetry we are looking for.

We now solve Eq. (34). This rather simple if we notice that the only operator appearing in the Hamiltonian is $\mathbf{p} - \mathbf{A}(\mathbf{r})$, where

$$\mathbf{A}(\mathbf{r})/k_F = \hat{l}(\mathbf{r}) - \hat{z}l_z(\mathbf{r}) .$$

This is similar to the problem of a charged particle in a magnetic field with vector potential $\mathbf{A}(\mathbf{r})$. We know that the only physically relevant field is $\nabla \times \mathbf{A}(\mathbf{r})$, and since we have linearized the $\mathbf{r}$ dependence of $\hat{l}(\mathbf{r})$, this field is actually constant over space. We find

$$\nabla \times \mathbf{A}(\mathbf{r}) = k_F \nabla \times \hat{l}(\mathbf{r}) = k_F \mathbf{B} \tag{35}$$

because $\nabla l_z(\mathbf{r}) = 0$ since $\hat{l}^2 = 1$ implies $\hat{l}_i \partial_j \hat{l}_i = \partial_j \hat{l}_z = 0$. Therefore a more convenient vector potential is

$$\mathbf{A}'(\mathbf{r}) = \tfrac{1}{2}k_F \mathbf{B} \times \mathbf{r} , \tag{36}$$

and we can eliminate all the physically irrelevant parts of $\hat{l}(\mathbf{r})$ by the gauge transformation $\psi_{1,2} = \exp[i\phi(\mathbf{r})]\psi_{1,2}'$, with $\nabla\phi = \mathbf{A} - \mathbf{A}'$. This amounts to replacing $\mathbf{A}$ and $\mathbf{A}'$ and ψ by ψ' in Eq. (34). It is clear that our eigenstates will be closely analogous to Landau levels.

We choose a new set of axes X, Y, Z. We can safely assume that $\mathbf{B}$ is in the x-z plane since a rotation of $\hat{\Delta}_1$ and $\hat{\Delta}_2$ in the plane perpendicular to $\hat{l}$ is equivalent to an irrelevant overall change of the phase of the gap. We take Y along y and Z along $\mathbf{B}$. The component p_Z of $\mathbf{p}$ along $\mathbf{B}$ is a good quantum number and the wave functions have a plane-wave factor $\exp(ip_Z Z)$. As for Landau levels, we have degeneracy because the operators $p_X + A_X'$ and $p_Y + A_Y'$ commute with the Hamiltonian but not with each other. This is clearly seen if we use a Landau gauge, for example, $A_Y'' = k_F BX$, $A_X'' = A_Z'' = 0$, instead of $\mathbf{A}'$. The two preceding operators become $p_X + k_F BY$ and p_Y, and we can take p_Y as quantum number. Physically, the wave function will be centered around $X_0 = p_Y/k_F B$ and the freedom in the choice of this point gives rise to the degeneracy. We set $X' = X - X_0$.

After the change of axes, Eq. (34) gives, with the Landau gauge $\mathbf{A}''$, the one-dimensional problem:

$$\left[v_F(p_Z\sin\theta - p_X\cos\theta)\sigma_3 + \frac{\delta}{k_F}[(p_Z\cos\theta + p_X\sin\theta)\sigma_1 + k_F BX'\sigma_2] \right]\psi'' = \omega\psi'' , \tag{37}$$

where $\sigma_{1,2,3}$ are the Pauli matrices and θ the angle between $\mathbf{B}$ and the x axis. As in our preceding work, this can be solved by a pseudospin rotation. By a rotation around axis 2 we can eliminate $p_X\sigma_3$ and keep only $p_X\sigma_1$, which makes p_X only appear in the off-diagonal terms. This is done by setting

$$\psi'' = \exp\left[i\frac{u}{2}\sigma_2\right]\chi' , \tag{38}$$

which makes $\sigma_1 + i\sigma_3 \to (\sigma_1 + i\sigma_3)e^{iu}$. We choose

$$\sin u = \frac{1}{\gamma}\cos\theta ,$$

$$\cos u = -\frac{\delta}{2E_F\gamma}\sin\theta , \tag{39}$$

$$\gamma = \left[\cos^2\theta + \frac{\delta^2}{4E_F^2}\sin^2\theta\right]^{1/2} ,$$

which gives

$$[\omega_0\sigma_3 - \gamma v_F(p_X - p_0)\sigma_1 + \delta BX'\sigma_2]\chi' = \omega\chi' , \tag{40}$$

where

$$\omega_0 = -\frac{\delta}{\gamma k_F}p_Z, \quad p_0 = \frac{p_Z}{\gamma^2}\sin\theta\cos\theta\left[1 - \frac{\delta^2}{4E_F^2}\right] . \tag{41}$$

The term p_0 gives a plane-wave dependence in X and is eliminated by setting $\chi' = \exp(ip_0X)\chi$. The resulting equation is solved by introducing the new variable $\rho = (\delta B/\gamma v_F)^{1/2}X'$ and the harmonic-oscillator operator:

$$a = \frac{1}{\sqrt{2}}(\rho + \partial_\rho), \quad a^+ = \frac{1}{\sqrt{2}}(\rho - \partial_\rho), \quad [a,a^\dagger] = 1 . \tag{42}$$

Then Eq. (40) reduces to

$$(\omega - \omega_0)\chi_1 + i(2\gamma B\,\delta v_F)^{1/2}a^\dagger\chi_2 = 0 ,$$

$$(\omega + \omega_0)\chi_2 - i(2\gamma B\,\delta v_F)^{1/2}a\chi_1 = 0 , \tag{43}$$

which leads to

$$2\gamma B\,\delta v_F\, a^\dagger a\chi_1 = (\omega^2 - \omega_0^2)\chi_1 . \tag{44}$$

The solutions are

$$\chi_1(\rho) = f_p(\rho) ,$$

$$\chi_2(\rho) = i\frac{(2p\gamma B\,\delta v_F)^{1/2}}{\omega + \omega_0}f_{p-1}(\rho) , \tag{45}$$

where $f_p(\rho)$ are the harmonic-oscillator eigenfunctions, and the eigenvalues are given by

$$\omega = \begin{cases} \omega_0, & p = 0 \\ \pm\omega_p = \pm(\omega_0^2 + 2p\gamma B\,\delta v_F)^{1/2}, & p \geq 1 . \end{cases} \tag{46}$$

As we mentioned, our wave functions are very similar to those of a charged particle in a magnetic field. They have a plane-wave dependence along $\mathbf{B}$ and are localized in the plane perpendicular to $\mathbf{B}$ with an arbitrary center leading to the degeneracy. Our solutions (45) are free along Y and localized along X with a typical localization

length $(\gamma v_F/\delta B)^{1/2} \sim (\xi_0 L)^{1/2}$ [our preceding results with wave functions localized along $\hat{\mathbf{k}} \simeq \hat{l}(0)$ correspond to cross sections of the present ones along $\hat{l}(0)$; this gives a localization length $(\gamma v_F/\delta B)^{1/2}/\cos\theta \simeq (\xi_0/B_\perp)^{1/2}$, in agreement with our earlier results]. Naturally, as for Landau levels, we could use another gauge and have another set of eigenstates free along X and localized along Y. The localization length

$$(\delta/2E_F k_F B\gamma)^{1/2} \sim (\xi_0 L)^{1/2}\delta/E_F$$

would be much smaller. This corresponds to the very strong anisotropy factor $\delta/2E_F$, which makes—in our problem—the effects of twisting $B_\parallel$ much smaller than those of bending $B_\perp$.

As expected, the spectrum (46) is almost identical to our earlier result.[3] Indeed, from the expression of the wave function, the component k_x of the wave vector $\mathbf{k}$ along $\mathbf{B}_\perp$ is given by

$$k_x = p_Z\cos\theta + p_0\sin\theta = p_Z\frac{\cos\theta}{\gamma^2} , \tag{47}$$

which gives

$$\omega_0 = -\delta\hat{k}_x\frac{\gamma}{\cos\theta} \simeq -\delta\hat{k}_x , \tag{48}$$

and $\gamma B \simeq B_\perp$. Therefore the effect of twisting is very small, except when bending is completely absent, in which case $\gamma = \delta/2E_F$ instead of $\gamma \simeq \cos\theta$. We notice that when only twisting is present, the energy bands still have a gap $\delta(2pB/k_F)^{1/2}$ which is much smaller than the gap produced by bending. This is in contrast with the result of Ho et al.,[14] who do not take effects of order δ/E_F into account. We also note that the similarity between the spectrum (46) and the one of a charged particle in a magnetic field is quite deep. Indeed, in the classical limit the Hamiltonian of our problem would be

$$\mathcal{H}^2 = \frac{\delta^2}{k_F^2}[(p_x - k_F l_x)^2 + (p_y - k_F l_y)^2] + v_F^2 p_z^2 , \tag{49}$$

which is basically the Hamiltonian of a charged particle with an anisotropic mass tensor in a magnetic field.

The group velocity is easily obtained from Eq. (41). The energy depends only on p_Z and, therefore, the group velocity is along $\mathbf{B}$. This is what is expected physically: In the classical limit the particle is trapped by the magnetic field in closed elliptical orbits and is only free to move along $\mathbf{B}$. The group velocity for the $p = 0$ energy band is

$$v_g = \frac{\partial\omega_0}{\partial p_Z} = -\frac{\delta}{\gamma k_F} , \tag{50}$$

which coincides with Eq. (28).

The density of states $N_+(0)$ at the origin at zero energy for states with $\hat{\mathbf{k}} = \hat{l}(0)$ is given by

$$N_+(0) = \sum |\psi_1(0)|^2\delta(\omega_0) , \tag{51}$$

where the sum is over all states of the $p = 0$ energy band. In this case χ_2 is zero and we have, from our solution,

$$\psi_1(0)=\chi_1\left[-p_y\left[\frac{\delta}{2E_Fk_FB\gamma}\right]^{1/2}\right]\cos\left[\frac{u}{2}\right].\qquad(52)$$

The summation is performed by $\sum\rightarrow\int dp_Y dp_Z$, but one needs to add a prefactor $(\delta B/\gamma v_F)^{1/2}/(2\pi)^2$ in order to have properly normalized eigenstates. This leads to

$$N_+(0)=\frac{N_0}{2}\frac{\gamma Bv_F}{\delta}\cos^2\left[\frac{u}{2}\right],\qquad(53)$$

with

$$\cos^2\left[\frac{u}{2}\right]=\frac{1}{2}\left[1-\frac{\delta}{2E_F\gamma}\sin\theta\right],\qquad(54)$$

and therefore

$$N_+(0)=\frac{N_0}{4k_F}\left[\frac{2E_F}{\delta}\gamma B-B\sin\theta\right],\qquad(55)$$

in agreement with Eq. (1). It is easy to perform the same calculations with $\hat{\mathbf{k}}\simeq-\hat{l}(0)$. It amounts to making $v_F\rightarrow-v_F$ in (most of) our equations and gives

$$N_-(0)=\frac{N_0}{4k_F}\left[\frac{2E_F}{\delta}\gamma B+B\sin\theta\right].\qquad(56)$$

We see that the crucial point in obtaining the difference between $N_+(0)$ and $N_-(0)$ is that, because of twisting, the angle u of our pseudospin rotation is shifted slightly away from $u=\pi/4$ and the sign of this shift is odd in $\hat{\mathbf{k}}\cdot\hat{l}(0)$.

It is interesting to note that this odd contribution to the density of states is already present in gradient expansion, although in a singular way. Indeed, Gorkov's equations read[2]

$$\begin{bmatrix}\omega-\xi_k-\mathbf{k}\cdot\dfrac{\mathbf{p}}{m}-\dfrac{\mathbf{p}^2}{2m} & -\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}\cdot\mathbf{p}+\frac{1}{2}i\,\text{div}\mathbf{\Delta} \\[2ex] -\mathbf{\Delta}^*\cdot\mathbf{k}-\mathbf{\Delta}^*\cdot\mathbf{p}+\frac{1}{2}i\,\text{div}\mathbf{\Delta}^* & \omega+\xi_k+\mathbf{k}\cdot\dfrac{\mathbf{p}}{m}+\dfrac{\mathbf{p}^2}{2m}\end{bmatrix}g_k(\mathbf{r})=1\qquad(57)$$

[it is obtained from Eq. (30) by assuming a rapid spatial variation $e^{i\mathbf{k}\cdot\mathbf{r}}$]. To first order in the gradient, we can neglect $\mathbf{p}^2/2m$. If we set

$$g=g^0+g^1,\quad \mathcal{H}=\mathcal{H}_0+\mathcal{H}_1,\qquad(58)$$

where g^0 is zeroth order and g^1 first order in the gradient, and

$$\mathcal{H}_0=\begin{bmatrix}\xi_k & \mathbf{\Delta}\cdot\mathbf{k} \\ \mathbf{\Delta}^*\cdot\mathbf{k} & -\xi_k\end{bmatrix},\quad \mathcal{H}_1=\begin{bmatrix}\dfrac{\mathbf{k}\cdot\mathbf{p}}{m} & \mathbf{\Delta}\cdot\mathbf{p}-\frac{1}{2}i\,\text{div}\mathbf{\Delta} \\[2ex] \mathbf{\Delta}^*\cdot\mathbf{p}-\frac{1}{2}i\,\text{div}\mathbf{\Delta}^* & -\mathbf{k}\cdot\dfrac{\mathbf{p}}{m}\end{bmatrix},\qquad(59)$$

we have

$$g^0=(\omega-\mathcal{H}_0)^{-1},\quad g^1=g_0\mathcal{H}_1g_0.\qquad(60)$$

The density of states at zero energy is obtained from

$$N_k(0)=-\frac{1}{\pi}\text{Im}\,g^1_{11}(\omega=i\epsilon),\quad \epsilon\rightarrow0,\qquad(61)$$

where g^1_{11} is the upper left matrix element of g^1. After performing the algebra, we find two contributions to $\text{Im}\,g^1_{11}(i\epsilon)$. The first gives zero after integration over ξ_k, whether because the terms are odd with respect to ξ_k (we assume particle-hole symmetry) or because the explicit integration over ξ_k gives zero. The second is

$$N_k(0)=-\frac{\epsilon}{(\epsilon^2+\xi_k^2+|\Delta_k|^2)^2}\frac{\delta^2}{\pi k_F^2}(\mathbf{k}\cdot\hat{l})\hat{l}\cdot\nabla\times\hat{l}.\qquad(62)$$

For $\epsilon\rightarrow0$ we obtain a singular contribution from the node of the gap $\Delta_k=0$ and $\xi_k=0$, in contrast with the full solution which is regular, the contribution to $N(0)$ coming from a range of values for $\hat{\mathbf{k}}$ and ξ_k. Nevertheless, if we integrate Eq. (62) over ξ_k and $\hat{\mathbf{k}}$ with $\hat{\mathbf{k}}\simeq\hat{l}$, we find

$$N_+(0)=N_0\int d\xi_k\frac{d\Omega_k}{4\pi}N_k(0)=-\frac{N_0}{4k_F}\hat{l}\cdot\nabla\times\hat{l}\qquad(63)$$

and, similarly,

$$N_-(0)=\frac{N_0}{4k_F}\hat{l}\cdot\nabla\times\hat{l}\qquad(64)$$

for $\hat{\mathbf{k}}\simeq-\hat{l}$. Equations (63) and (64) are in agreement with the odd contribution in Eqs. (55) and (56). Naturally, Eqs. (63) and (64) make no sense by themselves since one of the density of states $N_+(0)$ or $N_-(0)$ is necessarily negative. One needs the even contribution to recover a physically meaningful result. However, this cannot be obtained from a gradient expansion since it is not analytic.

VI. CONCLUSION

In this paper we have studied the rather subtle effect of twisting in a ^{3}He-A texture. Broadly speaking, the effects which we have considered are of order δ/E_F compared to those produced by bending. However, they have a different symmetry, which makes it necessary to take them into account to understand the physics of ^{3}He-A. Our

basic result is that the density of states is no longer even with respect to **k** when twisting is taken into account. The odd contribution proportional to twisting is usually small compared to the even one. However, it has important effects.

The most important effect is the following. If in a fixed texture there is a density fluctuation, a corresponding fluctuation in the current appears to be carried by the excitations. This current fluctuation turns out to be exactly identical to the fluctuation of the C_0 term in the standard expression for the current. Therefore the fluctuations of the C_0 term in the current are carried by the normal fluid and not by the superfluid. It is very tempting to generalize and to view the C_0 term as being linked to the normal fluid and not the superfluid. It would be caused by the deformation of the excitation wave functions under the influence of the condensate. The condensate alone would give a current $\rho v_s + (1/4m)\nabla \times \rho \hat{l}$, but it corresponds to a highly excited state. One goes back to the ground state by removing excitations, but in this process the additional C_0 term in the current appears. This interpretation leads to an easy explanation of the difference between calculations of the angular momentum of ³He-A samples using the BCS wave function[9] directly and the other ones that make use of the current expression[6]: the former take only into account the condensate and miss the contribution of the normal liquid. Our explanation is a local one, in contrast with the global one recently proposed by Stone et al.,[17] where the difference is attributed to surface currents.

Our results allow us to understand physically the origin of the term in the Josephson equation containing v_n at $T=0$: one can calculate microscopically the density fluctuation due to the excitations produced by the appearance of a normal velocity v_n. The result is precisely the term required by the Josephson equation, and its expression is directly proportional to the odd term in the density of states. In the same way we have been able to rewrite the momentum conservation law as a rather simple equation of evolution for the normal velocity. This equation lends itself to a simple physical interpretation: at $T=0$ the excitation distribution is frozen and the equation for $\partial_t v_n$ is a simple translation of the adiabatic evolution of an excitation energy under the influence of the moving superfluid.

In order to obtain the above results, one needs a precise knowledge of the excitation wave functions. This has been obtained by an approximate method where the spatial variation of the order parameter are linearized. This approximation is well suited to the local study of these wave functions. The problem turns out to be very similar to that of a charged particle with an anisotropic mass tensor in a magnetic field parallel to curl$\hat{l}$. Accordingly, the motion of an excitation is free along curl$\hat{l}$ and the group velocity is along curl$\hat{l}$.

ACKNOWLEDGMENT

APPENDIX A

In the collisionless regime we have to solve

$$i\partial_t G_c(t,t') - \mathscr{H}(t)G_c(t,t') = \delta(t-t') \qquad \text{(A1)}$$

for the matrix Green's function G_c; $\mathscr{H}(t)$ is given by

$$\mathscr{H}(t) = \begin{bmatrix} H_c - \mu & \Delta e^{i\phi(t)} \\ \Delta^* e^{-i\phi(t)} & -(H_c-\mu) \end{bmatrix} \qquad \text{(A2)}$$

and H_c is the kinetic energy. The phase factor $e^{i\phi(t)}$ can be removed by the transformation

$$G_c(t,t') = \begin{bmatrix} e^{i\phi(t)/2} & 0 \\ 0 & e^{-i\phi(t)/2} \end{bmatrix}$$

$$\times G'(t,t') \begin{bmatrix} e^{-i\phi(t')/2} & 0 \\ 0 & e^{i\phi(t')/2} \end{bmatrix} . \qquad \text{(A3)}$$

One finds that G' satisfies Eq. (A1), except that the phase factor has disappeared and μ is replaced by $\mu - \partial_t\phi/2$. We linearize the time dependence of $\phi(t)$ since only $\partial_t\phi$ is of interest in hydrodynamics: $\phi(t) \simeq t\partial_t\phi$. After Fourier transformation with respect to $t-t'$, we find, for the particle-particle component G_c^{11} of the matrix Green's function,

$$G_c^{11}(\omega,\mu) = \mathscr{G}^{11}(\omega+\tfrac{1}{2}\partial_t\phi, \mu - \tfrac{1}{2}\partial_t\phi) , \qquad \text{(A4)}$$

where $\mathscr{G}$ satisfies Eq. (A1) without the phase factor. On the other hand, in the hydrodynamic regime we perform a gauge transformation on the Hamiltonian and this leads—for the corresponding Green's function G_h—to Eq. (A1) with μ replaced by $\mu - \partial_t\phi/2$ and without the phase factor. In other words,

$$G_h^{11}(\omega,\mu) = \mathscr{G}^{11}(\omega, \mu - \tfrac{1}{2}\partial_t\phi) . \qquad \text{(A5)}$$

Therefore,

$$m(g_h - g_c) = \sum_k \int \frac{d\omega}{2i\pi} k[G_h^{11}(\omega,\mu) - G_c^{11}(\omega,\mu)] \qquad \text{(A6)}$$

$$\simeq \sum_k \int \frac{d\omega}{2i\pi} k[\mathscr{G}^{11}(\omega) - \mathscr{G}^{11}(\omega+\tfrac{1}{2}\partial_t\phi)] \qquad \text{(A7)}$$

to first order in $\partial_t\phi$. The integral over ω is not zero because in the change of variable $\omega + \tfrac{1}{2}\partial_t\phi \to \omega$ the position of the cuts of the (time-ordered) Green's function $\mathscr{G}$ is also shifted. This leads to

$$m(g_h - g_c) = \sum_k \int_C \frac{d\omega}{2i\pi} k\mathscr{G}^{11}(\omega) , \qquad \text{(A8)}$$

where the clockwise contour C encircles the real axis and cuts it at $\omega=0$ and $\omega = \tfrac{1}{2}\partial_t\phi$. The result is expressed in terms of the density of states

$$N_k(0) = -\frac{1}{\pi}\text{Im}\,\mathscr{G}_k^{11}(0+i\delta) \qquad \text{(A9)}$$

in the following way:

$$m(g_h - g_c) = -\tfrac{1}{2}\partial_t\phi \sum_k kN_k(0) . \qquad \text{(A10)}$$

Taking into account

$$N_\pm(0) = \sum_{k \simeq \pm k_F \hat{l}} N_k(0) \ , \qquad \text{(A11)}$$

we obtain Eq. (10).

In the same way, if we consider the effect of a phase gradient in the Josephson equation, we can handle it by a gauge transformation, which amounts to $\omega \to \omega - \mathbf{k} \cdot \mathbf{v}_s$.

Therefore a superfluid velocity produces a density fluctuation

$$\delta\rho = \sum_k \int \frac{d\omega}{2i\pi} [\mathscr{G}^{11}(\omega - \mathbf{k} \cdot \mathbf{v}_s) - \mathscr{G}^{11}(\omega)]$$

$$= -\mathbf{v}_s \cdot \sum_k \mathbf{k} N_k(0) \ , \qquad \text{(A12)}$$

which agrees with Eq. (16).

APPENDIX B

We sketch here the steps which lead from Eq. (21) to Eq. (23). We restrict ourselves to second-order terms in the stress tensor, so the modifications[18] to the hydrodynamics of Hu and Saslow[11] are negligible and we make use of their equations and notations. The time derivative of the current equation (7) is calculated from ($\hbar = 1$)

$$\dot{\rho} + \nabla \cdot \mathbf{g} = 0, \quad v_i^s = -\partial_i \mathscr{f}_\phi - \frac{1}{2m} \hat{l} \cdot \partial_i \hat{l} \times \hat{l} \ , \ $$

$$\mathscr{f}_\phi = \mu + \mathbf{v}^n \cdot \mathbf{v}^s + \frac{1}{4m} \hat{l} \cdot \nabla \times \mathbf{v}^n \ , \qquad \text{(B1)}$$

$$-\frac{1}{2m} \dot{\hat{l}}_i = \frac{1}{2m} \mathbf{v}^n \cdot \nabla l_i + \beta \hat{l} \times \mathbf{\Psi} - (\alpha_1 \delta_{ij}^T l_k + \alpha_2 \delta_{ik}^T l_j) l_j v_k^n \ , $$

where $\delta_{ij}^T = \delta_{ij} - l_i l_j$. In the mass conservation law, we can take Eq. (7) for g_i because the ρ_n contribution would give higher-order terms. For these reasons, terms containing $\partial_i C_0 \simeq \partial_i \rho$ will be discarded.

Making use of Eqs. (B1) and (22) and trying to express the result as a divergence of a stress tensor, one obtains, after some algebra,

$$\dot{g}_i^s + \partial_j \tilde{\sigma}_{ij} = -\frac{3C_0}{2m} l_i (\hat{l} \cdot \nabla \times \hat{l}) - \frac{\dot{C_0}}{2m} l_i (\hat{l} \cdot \nabla \times \hat{l}) + \frac{C_0}{2m} \{\partial_i \hat{l} \cdot \hat{l} \times [(\hat{l} \cdot \nabla) \mathbf{v}^n + (\mathbf{v}^n \cdot \nabla) \hat{l}] - (\hat{l} \partial_i \mathbf{v}^n)(\hat{l} \cdot \nabla \times \hat{l})\} \ , \qquad \text{(B2)}$$

where

$$\tilde{\sigma}_{ij} = P\delta_{ij} + \frac{1}{2m} \phi_{kj} \partial_i l_k + v_i^s g_j + \frac{1}{4m} \epsilon_{jkp} l_p \partial_i v_k^n - \frac{C_0}{2m} l_i \bar{\phi}_j - \frac{1}{4m} \epsilon_{ijk} \partial_t (\rho l_k) \ , \qquad \text{(B3)}$$

with $\bar{\phi} = \hat{l} \times \dot{\hat{l}}$. The pressure P is defined by

$$P = \mu\rho + \mathbf{v}^n \cdot \mathbf{g} - \epsilon \ , \qquad \text{(B4)}$$

where ϵ is the energy density. This leads to the Gibbs-Duhem relation,

$$\partial_i P = \rho \partial_i \mu + \mathbf{g} \partial_i \mathbf{v}^n - \lambda^s \partial_i \mathbf{v}^s - \frac{1}{2m} \psi_j \partial_i l_j - \frac{1}{2m} \phi_{kj} \partial_i (\partial_j l_k) \ , \qquad \text{(B5)}$$

from the defining relation

$$d\epsilon = \mu \, d\rho + \mathbf{v}^n \cdot d\mathbf{g} + \lambda^s \cdot d\mathbf{v}^s + \frac{1}{2m} \psi_j \, dl_j + \frac{1}{2m} \phi_{kj} d(\partial_j l_k) \ , \qquad \text{(B6)}$$

where

$$\lambda^s = \mathbf{g} - \rho \mathbf{v}^n, \quad \Psi_i = \psi_i - \partial_j \phi_{ij} - \epsilon_{ijk} \lambda^s \cdot \nabla l_k \ . \qquad \text{(B7)}$$

When $\tilde{\sigma}_{ij}$ is closely compared with the stress tensor of Hu and Saslow, one finds

$$\sigma_{ij} = P\delta_{ij} + \frac{1}{2m} \phi_{kj} \partial_i l_k + g_i v_j^n + v_i^s \lambda_j^s - (\alpha_1 l_i \Psi_j + \alpha_2 l_j \Psi_i)$$

$$+ \frac{1}{4m} \epsilon_{ijk} l_k \nabla \cdot \lambda^s + [\gamma_{il}^{(1)} \epsilon_{jpq} + \gamma_{jq}^{(2)} \epsilon_{ipl} + \gamma_{ji}^{(3)} \epsilon_{ipq} + \gamma_{iq}^{(3)} \epsilon_{jpl}] l_p \partial_q v_l^n \ , \qquad \text{(B8)}$$

one finds them rather similar. If we forget momentarily the terms including $\partial_q v_l^n$, one obtains that $\tilde{\sigma}_{ij} - \sigma_{ij}$ is merely given by

$$\frac{C_0}{2m} l_i \{v_j^n \hat{l} \cdot \nabla \times \hat{l} + [\hat{l} \times (\mathbf{v}^n \cdot \nabla) \hat{l}]_j\} \equiv B_{ij} \ . \qquad \text{(B9)}$$

Now since $\mathbf{g}-\mathbf{g}^s$ is Galilean invariant, there is an identity obtained by comparison of $\partial_j(\tilde{\sigma}_{ij}-\sigma_{ij})$ and the rhs of Eq. (B2) under Galilean transformation. One finds

$$\partial_j(l_i\hat{l}\cdot\nabla\times\hat{l})+\nabla\cdot(l_i\hat{l}\times\partial_j\hat{l})-3l_i\partial_j\hat{l}\cdot\nabla\times\hat{l}+\hat{l}\cdot\partial_i\hat{l}\times\partial_j\hat{l}=0 \;, \tag{B10}$$

which can be checked directly. This allows us to rewrite Eq. (B2) as

$$\dot{g}^s_i+\partial_j(\tilde{\sigma}_{ij}+A_{ij}-B_{ij})=-\frac{3C_0}{2m}l_i[\dot{\hat{l}}+\nabla(\mathbf{v}^n\cdot\hat{l})]\cdot\nabla\times\hat{l}-\frac{\dot{C}_0}{2m}l_i\hat{l}\cdot\nabla\times\hat{l} \;, \tag{B11}$$

where

$$A_{ij}=\frac{C_0}{2m}l_il_k(\nabla v^n_k\times\hat{l})_j \tag{B12}$$

and

$$\frac{2m}{C_0}\partial_jA_{ij}=l_i\hat{l}\cdot(\partial_k\hat{l}\times\nabla v^n_k+\nabla\times\hat{l}\partial_kv^n_k)-3l_il_k\nabla v^n_k\cdot\nabla\times\hat{l}+l_k\partial_iv^n_k\hat{l}\cdot\nabla\times\hat{l}-\partial_i\hat{l}\cdot\hat{l}\times l_k\nabla v^n_k \;. \tag{B13}$$

If we identify $\tilde{\sigma}_{ij}+A_{ij}-B_{ij}$ with σ_{ij},

$$\sigma_{ij}=\tilde{\sigma}_{ij}+A_{ij}-B_{ij} \;, \tag{B14}$$

Eq. (B11) leads to Eq. (23). When the terms with $\partial_q v^n_l$ are compared in Eq. (B14), they are found to coincide and this leads to the explicit expressions for the γ's of Hu and Saslow:

$$\gamma_1^{(1)}+\gamma_1^{(3)}=\gamma_1^{(2)}+\gamma_1^{(3)}=-\frac{\hbar\rho}{4m} \;,$$

$$\gamma_{||}^{(1)}=\gamma_{||}^{(2)}=\gamma_{||}^{(3)}=-\frac{\hbar\rho}{8m}\frac{\rho}{\rho-C_0} \;, \tag{B15}$$

in agreement with their general relations between the γ's.

APPENDIX C

In this appendix we study how the excitation energies are affected by a motion of $\hat{l}$ and justify the result given in Eq. (27).

Since we are interested in the behavior of excitations for small time intervals, we shall treat the temporal variation of $\hat{l}$ to first order and write near the origin $\mathbf{r}=0$, $t=0$,

$$\hat{l}(\mathbf{r},t)\simeq\hat{l}(0)+(\mathbf{r}\cdot\nabla)\hat{l}+t\partial_t\hat{l} \;, \tag{C1}$$

where $\partial_t\hat{l}$ has only two components in the plane perpendicular to $\hat{l}(0)$, ∂_tl_x and ∂_tl_y.

In hydrodynamics, the motion of $\hat{l}$ is by definition very slow, so that at each time local equilibrium can be reached. We may therefore consider that the excitation wave functions follow adiabatically the motion of $\hat{l}$ and are determined as the instantaneous solutions of the Bogoliubov equations:

$$\begin{bmatrix} v_Fp_z & \Delta(0)\cdot[\mathbf{p}-k_F\hat{l}(\mathbf{r},t)] \\ \Delta^*(0)\cdot[\mathbf{p}-k_F\hat{l}(\mathbf{r},t)] & -v_Fp_z \end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}=\omega\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} \;, \tag{C2}$$

where $\hat{l}(\mathbf{r},t)$ is given by (C1) and we have taken the wave vector of the excitation $\mathbf{k}\simeq k_F\hat{l}(0)$.

Now performing the set of transformations leading in the static case to Eq. (40), we get from (C2) an equation with essentially the same structure as (40) (here we use the notations introduced in Sec. V),

$$\left[\left[\omega_0+t\frac{\cos\theta}{\gamma}\delta\partial_tl_x\right]\sigma_3-\left[\gamma v_F(p_X-p_0)-t\frac{\delta^2}{2E_F}\frac{\sin\theta}{\gamma}\partial_tl_x\right]\sigma_1+(\delta BX'+t\delta\partial_tl_y)\sigma_2\right]X'=\omega X' \;. \tag{C3}$$

We see that the component of $\partial_t\hat{l}$ perpendicular to $\mathbf{B}$ (∂_tl_y with our choice of axis) may be eliminated by a shift of the X' coordinate. This corresponds physically to a translation of the wave function's center along X' at velocity $-\partial_tl_y/B$. In the same way, the t-dependent term involving $\partial_tl_x\sigma_1$ may be eliminated by a redefinition of p_0 which is of no importance for the energy. Finally, only $\partial_tl_x\sigma_3$ is able to modify the energy value and we find, from a comparison with Eq. (46),

$$\frac{\partial E_k}{\partial t}=\delta\frac{\cos\theta\partial_tl_x}{\gamma}=\frac{\delta}{\gamma B}(\partial_t\hat{l}\cdot\nabla\times\hat{l}) \;. \tag{C4}$$

For $\mathbf{k}\simeq-k_F\hat{l}(0)$ we would obtain the opposite quantity. Therefore our result is actually

$$\frac{\partial E_k}{\partial t}=\frac{\delta(\hat{\mathbf{k}}\cdot\hat{l})}{\gamma B}(\partial_t\hat{l}\cdot\nabla\times\hat{l}) \;, \tag{C5}$$

and thus takes the form anticipated by the qualitative argument presented in Sec. IV [see Eq. (27)]. If we keep γB in Eqs. (27) and (28) instead of using the approximate result $\gamma B\simeq B_\perp$, we obtain Eq. (29) with $|(\hat{l}\cdot\nabla)\hat{l}|$ replaced by γB. From expression (19) for ρ_n, this is just what is needed to agree with Eq. (23), even for the exceptional situation $B_\perp\lesssim B_{||}\mathcal{S}/E_F$.

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