Phonon decay in two-dimensional liquid ^{4}He

C. I. Um, H. K. Oh, and W. H. Kahng
Department of Physics, Korea University, Seoul, Korea

A. Isihara
Statistical Physics Laboratory, Department of Physics, State University of New York at Buffalo, Buffalo, New York 14260

(Received 10 December 1985)

The threshold for the phonon decay in two-dimensional liquid ^{4}He is calculated. The decay threshold q^* is slightly larger than the critical wave number q_c and is given by $q^*=1.15q_c$. The decay rate has a q^3 dependence. The decay rate and the threshold q^* decrease as the pressure increases.

I. INTRODUCTION

Since Landau's well-known work, there has been continuous interest in the elementary excitations of liquid ^{4}He in both bulk and film forms at low momenta and temperatures.[1] In the bulk case, the interest is in whether the excitation spectrum is slightly concave up or convex down in deviating from linearity at low momenta (0—0.6 Å^{-1}). This question is important because, if the spectrum is concave up, three-phonon processes (3PP) represent the lowest, while if it is convex down, four-phonon processes are the lowest.[2]

It has been revealed experimentally[3] that there exists a cutoff momentum q_c and energy E_c above which no phonon decay takes place. However, the cutoff momentum q_c derived by Jackle and Kehr[4] is slightly smaller than that of Dynes and Narayanamurti[3] for the same excitation spectrum.

It is the purpose of the present article to evaluate the decay threshold, decay rate, and dependence of the decay rate on pressure for two-dimensional (2D) liquid ^{4}He. For this purpose, we shall use the dispersion relation which we derived earlier[5] and extend the treatments[6-8] for bulk helium.

Our basic approach differs from those based on variational principles in the sense that the internal energy of liquid helium is evaluated at low but finite temperatures, in consideration of the collective couplings between helium particles. In the ring and exchange-ring approximation, the energy can be expressed in terms of quasiparticles in the Landau fashion. For small momenta, the dispersion relations for two and three dimensions have been evaluated explicitly, adopting a soft potential as follows:

$$\varepsilon(q)=c_0q(1+\gamma q^2-\delta q^4+\cdots)\quad(2\text{D}), \tag{1.1}$$

$$\varepsilon(q)=c_0q(1+\delta_1q^2-\delta_2q^3+\delta_3q^4+\cdots)\quad(3\text{D}). \tag{1.2}$$

Note that the 3D disperson relation has a cubic term, while the 2D case does not. The sound velocity c_0 and the coefficients γ, δ, etc., have been evaluated explicitly as functions of the potential parameters.[5]

In Sec. II, we describe the phonon-phonon interactions through second quantization in 2D liquid ^{4}He. We evaluate the decay rate for 3PP by using Eq. (1.1) in Sec. III. Finally, the results and discussions are presented in Sec. IV.

II. PHONON-PHONON INTERACTIONS IN THE 2D QUANTUM LIQUID

In this section, we consider the phonon-phonon interaction in the 2D quantum liquid. We introduce the density operator $\hat{\rho}(r,t)$ and the scalar velocity operator $\hat{\phi}(r,t)$ given as

$$\hat{\rho}(\mathbf{r},t)=\hat{\rho}_0+\hat{\rho}'(\mathbf{r},t), \tag{2.1}$$

$$\hat{v}(\mathbf{r},t)=\nabla\hat{\phi}(\mathbf{r},t), \tag{2.2}$$

where ρ_0 and $\hat{\rho}'(\mathbf{r},t)$ are the constant and variable parts, respectively, of the density operator, and $\hat{v}(\mathbf{r},t)$ is the velocity operator. We also define the field operators:

$$\hat{\phi}(\mathbf{r},t)=\sum_{\mathbf{q}}\left[\frac{\hbar c_0}{2S\rho q}\right]^{1/2}(a_{\mathbf{q}}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}+a_{\mathbf{q}}^{\dagger}e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}), \tag{2.3}$$

$$\hat{\rho}(\mathbf{r},t)=\hat{\rho}_0+i\sum_{\mathbf{q}}\left[\frac{\rho\hbar q}{2Sc_0}\right]^{1/2}(a_{\mathbf{q}}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}+a_{\mathbf{q}}^{\dagger}e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}), \tag{2.4}$$

where S is the area of the system, c_0 is the sound velocity in the liquid, ρ is the mass density of liquid and q is the wave vector. We can confirm that $\hat{\rho}'(\mathbf{r},t)$ and $\hat{\phi}(\mathbf{r},t)$ are canonically conjugate generalized coordinate and momentum operators, respectively. Annihilation operator $a_{\mathbf{q}}$ and creation operator $a_{\mathbf{q}}^{\dagger}$ satisfy the Bose commutation relations:

$$[a_{\mathbf{q}},a_{\mathbf{q}'}^{\dagger}]=\delta_{\mathbf{q}\mathbf{q}'}, \tag{2.5}$$

$$[a_{\mathbf{q}},a_{\mathbf{q}'}]=[a_{\mathbf{q}}^{\dagger},a_{\mathbf{q}'}^{\dagger}]=0. \tag{2.6}$$

Using the fact that Bessel function $J_0(r)$ is even, we can easily show the following commutation relations between

$\hat{\phi}(\mathbf{r},t)$ and $\hat{\rho}'(\mathbf{r},t)$:

$$[\hat{\phi}(\mathbf{r},t),\hat{\rho}'(\mathbf{r},t)] = -i\hbar\delta(\mathbf{r},\mathbf{r}')\,, \tag{2.7}$$

$$[\hat{\phi}(\mathbf{r},t),\hat{\phi}(\mathbf{r}',t)] = [\hat{\rho}(\mathbf{r},t),\hat{\rho}(\mathbf{r}',t)] = 0\,. \tag{2.8}$$

In a continuum model[9] for the superfluid liquid ^{4}He, Hamiltonian $\hat{H}$, internal energy density $E(\rho)$, and chemical potential μ are given by

$$\hat{H} = \int [\tfrac{1}{2}\hat{\mathbf{v}}\cdot\hat{\rho}\hat{v} + E(\rho)]d^2r\,, \tag{2.9}$$

$$E(\rho) = \int n\varepsilon(q)d^2q + E(\rho_0)\,, \tag{2.10}$$

$$\mu = \mu_0 + \int n\frac{\partial\varepsilon(q)}{\partial q}a^2q\,, \tag{2.11}$$

where n is the number density of excitations, $\varepsilon(q)$ is the energy in the reference frame in which the superfluid component is at rest, and μ_0 is the chemical potential at zero temperature. Using Eqs. (2.9)–(2.11) and the thermodynamic relation of $\partial\mu/\partial\rho = c_0^2/\rho$, we can easily derive the following:

$$\hat{H} = \hat{H}_0 + \hat{H}_3 + \hat{H}_4 + \cdots\,, \tag{2.12}$$

$$\hat{H}_0 = \int \left[\frac{1}{2}\rho_0\hat{\mathbf{v}}\cdot\hat{\mathbf{v}} + \frac{1}{2}\frac{c_0^2}{\rho}(\hat{\rho}')^2\right]d^2r\,, \tag{2.13}$$

$$\hat{H}_3 = \int \left[\frac{1}{2}\hat{\mathbf{v}}\cdot\hat{\rho}'\hat{v} + \frac{1}{3!}\frac{\partial}{\partial\rho}\left[\frac{c_0^2}{\rho}\right](\hat{\rho}')^3\right]d^2r\,, \tag{2.14}$$

$$\hat{H}_4 = \int \left[\frac{1}{4}\frac{\partial^2}{\partial\rho^2}\left[\frac{c_0^2}{\rho}\right](\hat{\rho}')^4\right]d^2r\,. \tag{2.15}$$

In terms of $a_{\mathbf{q}}$ and $a_{\mathbf{q}}^\dagger$, $\hat{H}_0$, $\hat{H}_3$, and $\hat{H}_4$ can be expressed by

$$\hat{H}_0 = \sum_{\mathbf{q}} c_0\hbar q(a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \tfrac{1}{2})\,, \tag{2.16}$$

$$\hat{H}_3 = -2\pi^2 i\left[\frac{\hbar}{2S}\right]^{3/2}\sum_{\mathbf{q},\mathbf{q}_1,\mathbf{q}_2}\left[\left[\frac{c_0q_1}{\rho q q_2}\right]^{1/2}(\mathbf{q}\cdot\mathbf{q}_2) + \frac{\rho_0^2}{3c_0^2}\left[\frac{c_0 q q_1 q_2}{\rho_0}\right]^{1/2}\frac{\partial}{\partial\rho}\left[\frac{c_0}{\rho}\right]^2\right]$$
$$\times[a_{\mathbf{q}}a_{\mathbf{q}_1}a_{\mathbf{q}_2}\delta(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)e^{-i(\omega+\omega_1+\omega_2)t}$$
$$+ a_{\mathbf{q}}a_{\mathbf{q}_1}^\dagger a_{\mathbf{q}_2}^\dagger\delta(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)e^{-i(\omega-\omega_1-\omega_2)t} + \cdots]\,, \tag{2.17}$$

$$\hat{H}_4 = \frac{(2\pi)^2}{4!}\left[\frac{\rho_0\hbar}{2Sc_0}\right]^2\frac{\partial^2}{\partial\rho^2}\left[\frac{c_0^2}{\rho}\right]\sum_{\mathbf{q},\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}[a_{\mathbf{q}}a_{\mathbf{q}_1}a_{\mathbf{q}_2}a_{\mathbf{q}_3}\delta(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)$$
$$\times e^{-i(\omega+\omega_1+\omega_2+\omega_3)t} + a_{\mathbf{q}}a_{\mathbf{q}_1}^\dagger a_{\mathbf{q}_2}^\dagger a_{\mathbf{q}_3}\delta(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2+\mathbf{q}_3)$$
$$\times e^{-i(\omega-\omega_1-\omega_2+\omega_3)t} + \cdots]\,, \tag{2.18}$$

where the other six terms to be included in Eq. (2.17) are given by combinations of $a_{\mathbf{q}}, a_{\mathbf{q}_1}, a_{\mathbf{q}_2}$ and $a_{\mathbf{q}}^\dagger, a_{\mathbf{q}_1}^\dagger, a_{\mathbf{q}_2}^\dagger$. Fourteen additional terms with the combinations of $a_{\mathbf{q}}, a_{\mathbf{q}_1}, a_{\mathbf{q}_2}, a_{\mathbf{q}_3}$ and $a_{\mathbf{q}}^\dagger, a_{\mathbf{q}_1}^\dagger, a_{\mathbf{q}_3}^\dagger$ appear in Eq. (2.18), although not listed. The nonzero matrix elements of $a_{\mathbf{q}}$ and $a_{\mathbf{q}}^\dagger$ are

$$\langle n_{\mathbf{q}}-1|a_{\mathbf{q}}|n_{\mathbf{q}}\rangle = -i(n_{\mathbf{q}})^{1/2}e^{i\omega t}\,, \tag{2.19}$$

$$\langle n_{\mathbf{q}}|a_{\mathbf{q}}^\dagger|n_{\mathbf{q}}-1\rangle = i(n_{\mathbf{q}})^{1/2}e^{-i\omega t}\,. \tag{2.20}$$

The matrix element $\langle n_{\mathbf{q}}|\hat{H}_0|n_{\mathbf{q}}\rangle$ is $\sum_{\mathbf{q}}(n_{\mathbf{q}}+\frac{1}{2})\hbar\omega$, where $\omega = c_0 q$, so that H_0 is the Hamiltonian of a simple harmonic oscillator. $\hat{H}_3$ and $\hat{H}_4$ are then the anharmonic perturbative Hamiltonians. Transitions between different states occur due to $\hat{H}_3$, $\hat{H}_4$, and other higher-order perturbed Hamiltonians. The transition matrix element from state $|i\rangle$ to state $|f\rangle$ in the second-order perturbation is given by

$$\langle f|\hat{H}'|i\rangle = \langle f|\hat{H}_3|i\rangle + \langle f|\hat{H}_4|i\rangle + \sum_j\frac{\langle f|\hat{H}_3|j\rangle\langle j|\hat{H}_4|i\rangle}{E_j - E_i}\,. \tag{2.21}$$

III. PHONON DECAY IN THE 2D LIQUID ^{4}He

A finite lifetime of an elementary excitation in a quantum liquid originates from two processes. One is the collision between elementary excitations, and the other is the spontaneous decay of a phonon into two, three, or more phonons. Since the collision probability tends to zero at very low temperatures, the collision process will become less important and negligible. At very low temperatures below 0.6 K, the main excitations in liquid ^{4}He are phonons. Therefore, we may consider the decay of the phonon into two phonons, only, i.e., the 3PP. When a single phonon with the energy $\varepsilon(q)$ is excited in liquid ^{4}He at zero temperature, a spontaneous 3PP may take place under the following conditions:

$$\mathbf{q}=\mathbf{q}_1+\mathbf{q}_2,\quad \varepsilon(q)=\varepsilon(q_1)+\varepsilon(q_2),\qquad q_1+q_2>q>|\mathbf{q}_1-\mathbf{q}_2| \;. \tag{3.1}$$

Let C, C_1, and C_2 be the velocities of phonons with wave vectors $\mathbf{q}$, $\mathbf{q}_1$, and $\mathbf{q}_2$, respectively. If C_1 and C_2 are both larger than C, the phonon with q is stable against decay into two phonons, while if C_1 and C_2 are both smaller than C, the phonon will decay into two phonons.[10] The decay angle θ between q and q_1 as one of the decay products is given by

$$\theta^2=6\gamma(q-q_1)^2-10\delta(q-q_1)^2(q^2+q_1^2-qq_1) \;. \tag{3.2}$$

If δ is neglected, the maximum decay angle becomes $\theta_m=\sqrt{6\gamma}q$ for a given q when q_1 approaches zero. But for large q, the third term in Eq. (1.1) contributes significantly, and then for a given q at $q_1=0$, the maximum decay angle θ_m can be obtained at $q=\sqrt{3\gamma/10\delta}=q_c/\sqrt{2}$ as

$$\theta_m=(9\gamma^2/10\delta)^{1/2}, \tag{3.3}$$

where $q_c=\sqrt{3\gamma/5\delta}$ is called the critical wave vector. Jackle and Kehr[4] have pointed out that beyond q_c, the 3PP are not allowed. If we solve Eq. (3.2) for q, we can obtain the maximum value of $q=q^*$ and the range of q_1 as follows:

$$0\le q_1\le q,\quad q\le q_c,\quad \tfrac{1}{2}(q-\{3[(q^*)^2-q^2]\}^{1/2})\le q_1\le \tfrac{1}{2}(q+\{3[(q^*)^2-q^2]\}^{1/2}),\quad q_c\le q\le q^* \;. \tag{3.4}$$

Here, the maximum value q^* $(=\sqrt{4\gamma/5\delta})$ is called the decay threshold wave vector. For $q>q^*$, the 3PP do not occur and at q^*, a phonon decays into two identical collinear phonons with $q^*/2$ with the energy conservation $\varepsilon(q^*)=2\varepsilon(q^*/2)$.

In the 3PP, the decay angle for the low-momentum phonon is small. Therefore, the transition matrix element of $\hat{H}_3$ can be written approximately as

$$\langle f|\hat{H}_3|i\rangle=8\pi^2(u+1)^2\left[\frac{\hbar}{2S}\right]^{3/2}\left[\frac{c_0}{\rho_0}qq_1q_2\right]^{1/2}\times\delta(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2), \tag{3.5}$$

where $u=(\rho_0/c_0)(\partial c_0/\partial\rho_0)$ is the Grüneisen constant. The differential decay rate per unit time is given by

$$d\omega=\frac{2\pi}{\hbar}|\langle f|\hat{H}_3|i\rangle|^2\delta(\varepsilon_F-\varepsilon_i)S^2\frac{d^2q_1d^2q_2}{(2\pi)^2} \;. \tag{3.6}$$

Substituting Eq. (3.5) and the relations

$$\delta^2(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)=\frac{S}{(2\pi)^2}\delta(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2), \tag{3.7}$$

$$|\mathbf{p}-\mathbf{q}|\cong(p-q)\left[1+\frac{pq\theta^2}{2(p-q)^2}\right] \tag{3.8}$$

into Eq. (3.6) and performing the $\mathbf{q}_2$ and θ integrals in Eq. (3.6), we get

$$w=\frac{\hbar^2c_0}{8\pi\rho_0}(u+1)^2I, \tag{3.9}$$

$$\begin{aligned}I&=\frac{1}{\sqrt{2}\hbar c_0}\int dq_1\frac{q_1(q-q_1)[1+(3\gamma+5\delta qq_1)(q^2-qq_1+q_1^2)-5\delta(q-q_1)^4]}{[3\gamma-5\delta(q^2-qq_1+q_1^2)]^{1/2}[1+3\gamma(q-q_1)^2-5\delta(q-q_1)^4]^{3/2}}\\&=\frac{1}{\sqrt{6\gamma}\hbar c}\int dq_1q_1(q-q_1)\left[1+\left[\frac{5\delta}{6\gamma}-\frac{3}{2}\gamma\right](q-q_1)^2+\left[\frac{5\delta}{6\gamma}+3\gamma\right]qq_1\right.\\&\qquad\left.+\tfrac{4}{5}\delta(q-q_1)^4-\tfrac{5}{2}\delta q^2q_1^2-\tfrac{15}{4}\delta qq_1(q-q_1)^2\right] \;.\end{aligned} \tag{3.10}$$

If we integrate I in the two regions given by Eqs. (3) and (4), we obtain the total decay rate:

$$w=\frac{(u+1)^2\hbar}{48\pi\sqrt{6\gamma}\rho_0}q^3\left[1+\left[\frac{21}{20}\gamma+\frac{2}{3}\frac{\delta}{\gamma}\right]q^2-\frac{53}{56}\delta q^4\right],\quad 0\le q\le q_c \tag{3.11}$$

$$\begin{aligned}w=\frac{(u+1)^2\hbar}{32\pi\sqrt{6\gamma}\rho_0}\{3[(q^*)^2-q^2]\}^{1/2}&\left[[2q^2-(q^*)^2]+\frac{1}{24}\frac{\delta}{\gamma}[16q^4+8q^2(q^*)^2-9(q^*)^4]\right.\\&+\tfrac{3}{40}\gamma[44q^4-38q^2(q^*)^2+9(q^*)^4]\\&\left.-\tfrac{1}{84}\delta[47q^6+\tfrac{507}{2}q^4(q^*)^2-\tfrac{621}{2}q^2(q^*)^4+\tfrac{1485}{16}(q^*)^6]\right],\quad q_c\le q\le q^* \;.\end{aligned} \tag{3.12}$$

At very low momentum (0.6 Å $^{-1}$) we can express the total decay rate approximately as

$$w \cong \frac{(u+1)^2\hbar}{48\pi\sqrt{6\gamma}\rho_0}q^3, \quad 0 \le q \le q_c \tag{3.13}$$

$$w \cong \frac{(u+1)^2\hbar}{32\pi\sqrt{6\gamma}\rho_0}[2q^2-(q^*)^2]\{3[(q^*)^2-q^2]\}^{1/2}, \quad q_c \le q \le q^* . \tag{3.14}$$

We notice that Eqs. (3.11) and (3.12) reduce to the identical form at $q=q_c$.

To evaluate the decay threshold q^*, we can also make use of the method developed by Sluckin and Bowley.[7] The lifetime Γ_p, obtained from the imaginary part of the self-energy by using renormalized bubble diagrams for the forward scattering, is given by

$$\Gamma_p = \frac{(u+1)^2 c}{4S\rho\hbar^3}\sum_{\mathbf{q}} \frac{pq\,|\mathbf{p}-\mathbf{q}|(\Gamma_{\mathbf{p}-\mathbf{q}}+\Gamma_{\mathbf{q}})}{(E_{\mathbf{p}}-E_{\mathbf{p}-\mathbf{q}}-E_{\mathbf{q}})^2+(\Gamma_{\mathbf{p}-\mathbf{q}}+\Gamma_{\mathbf{q}})^2}, \tag{3.15}$$

where u is the Grüneisen constant, ρ the density, c the sound velocity, and S the surface area. Substituting Eq. (1.1) and the relation

$$|\mathbf{p}-\mathbf{q}| \cong (p-q)\left[1+\frac{pq}{(p-q)^2}(1-\cos\theta)\right], \tag{3.16}$$

and performing the angular integral and finally changing the variables p and q into $x=p/q_c$ and $y=q_c$, we obtain, for the lifetime Γ_p,

$$\Gamma_p = \frac{(u+1)^2 q_c^2}{64\sqrt{5c}\,\pi\rho\hbar^3}F(x), \tag{3.17}$$

$$F(x)=\int_0^x dy\, y(x-y)(1-x^2+xy-y^2)^{-1/2}\left\{\left[1+\left[\frac{F(x)+F(x-y)}{Axy(x-y)(1-x^2+xy-y^2)}\right]^2\right]^{1/2} + \left[1+\left[\frac{F(x)+F(x-y)}{Axy(x-y)(1-x^2+xy-y^2)}\right]^2\right]^{-2}\right\}^{1/2}, \tag{3.18}$$

where

$$A=\frac{192\pi\rho\hbar^3(3\gamma^3)^{1/2}}{(u+1)^2} .$$

IV. RESULTS AND DISCUSSIONS

In the preceding sections, the excitation spectrum given by Eq. (1.1) was used for the calculation of the decay threshold, the decay angle, and the decay rate. The derivation of Eq. (1.1) can be found in Ref. 5. As for the parameters γ and δ in the equation, those which we have obtained recently by analyzing third sound in thin helium films can be used.[11] We shall use these parameters determined for several atomic layers to explain the dependence of the decay rate on the pressure and for comparing our results with those of Pitayevski and Levinson.[12]

In Fig. 1 we have shown two decay regions for 3PP. Region A corresponds to the decay area pointed out by Jackle and Kehr[4] and region B to the extended area of Dynes and Narayanamurti. Obviously the decay threshold momentum q^* is slightly larger than the critical wave vector q_c. One of the produced phonons may have the same vector as that of the original phonon, and then the other phonon is at rest. In the case of $q>q_c$ the decay products are not at rest. The maximum wave vector that

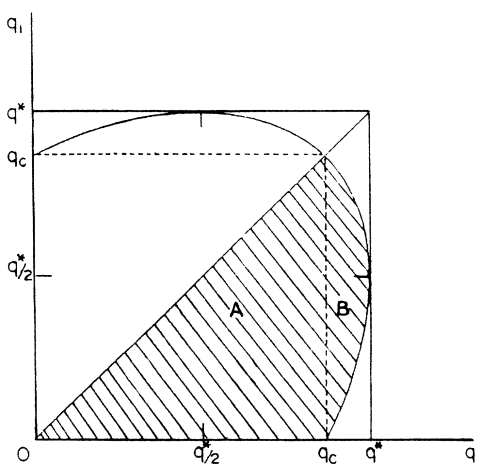

FIG. 1. Decay region determined from Eq. (3.4). Region A corresponds to the decay area pointed out by Jackle and Kehr, and region B to the extended area of Dynes and Narayanamurti.

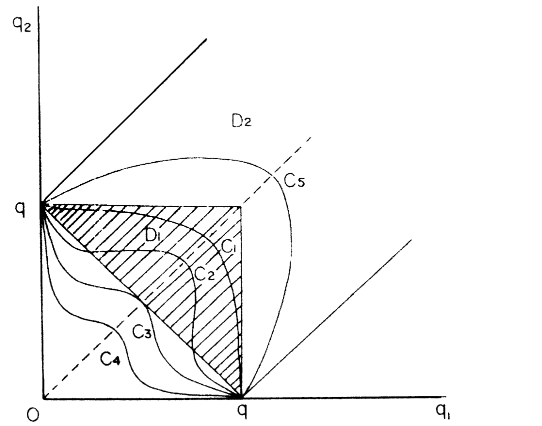

FIG. 2. The domain representing the decay process defined by condition (3.1). C_1, C_2, C_3, and C_4 are in the range of various q, i.e., $q<q_c$, $q_c<q<q^*$, $q=q^*$, and $q>q^*$, respectively.

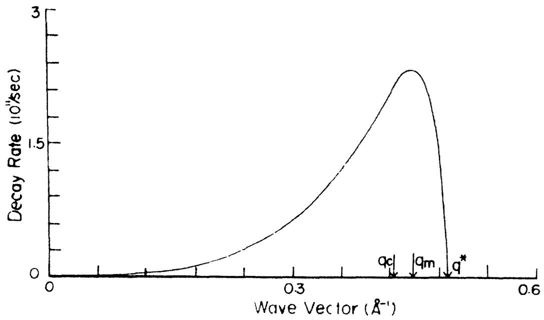

FIG. 3. Decay rate of a thin helium film at the density of 1.77 atomic layers. q_c, q_m, and q^* are the critical, maximum, and threshold wave vectors, respectively.

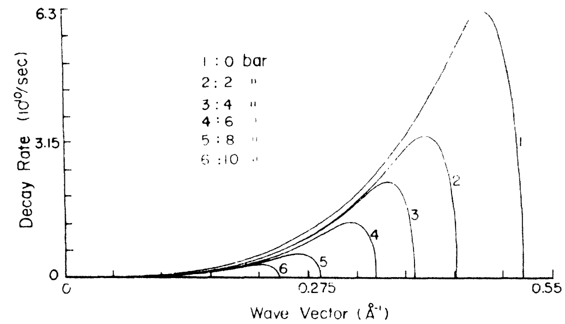

FIG. 4. Pressure dependence of the decay rate. The numerical parameters γ and δ are listed in Table I.

the decay products can have is $\hbar q_c$.

In Fig. 2, we have illustrated the domain for the 3PP. The second condition in Eq. (3.1) gives various curves C_s in the (q_1,q_2) plane, which are symmetric with respect to the line $q_1=q_2$, and the third condition determines the domain D_1+D_2. To obtain θ_m and q^* from Eq. (3.2), we have assumed that the decay angle is small. In small-angle decay processes for the small-momentum phonons, the decay products cannot have momentum larger than that of the original phonon. Therefore, the domain of the small-angle decay is limited to the region D_1. In this sense, our domain is different from that of Pitayevski and Levinson.

In Fig. 3, the decay rate [Eqs. (3.11) and (3.12)] is illustrated as a function of the wave vector at the density of 1.77 atomic layers. In computer plotting, the parameter u in Eqs. (3.11) and (3.12) for two dimensions is assumed to be 1.8, which is the value used by previous workers[13,14] for the bulk case. The decay rate increases as the wave vector increases. It passes the maximum value and then decreases rapidly to zero. We find that the maximum decay does not occur at $q=q_c$, but at $q=q_m=1.06q_c$. Sluckin and Bowley[7] pointed out that the maximum in the ultrasonic attenuation occurs at $1.1q_c$. However, we find that the decay rate becomes zero at $q^*=1.15q_c$, which is the true decay threshold. When we evaluate the lifetime Γ_p of the phonon obtained from the imaginary part of the self-energy by using renormalized bubble diagrams for the forward scattering, we find that Γ_p has a maximum value at $q=1.15q_c$. These two different methods give the same

TABLE I. Dependence of γ, δ, q_c, q_m, and θ_m on pressure (SVP denotes saturated vapor pressure and APP denotes Aldrich, Pethick, and Pines, Ref. 1).

Density[a]	Pressure (bars)	γ (Å^2)	δ (Å^4)	q_c (Å^{-1})	q_m/q_c	θ_m (deg)	Reference
	SVP	0.46	1.51	0.428	1.060	20.35	
	4.8	0.46	4.79	0.240		11.42	10
	10	0.40	16.3	0.121		5.39	
	14	0.40	34.9	0.083		3.68	
	5	0.368	2.70	0.286		12.20	APP
	10	0.270	4.31	0.194		7.10	APP
	15	0.240	12.06	0.109		3.76	
	10	0.136	1.87	0.209		5.41	14
0.0273	SVP	0.404	1.720	0.375	1.055	16.74	
0.0279	SVP	0.496	1.279	0.436	1.060	19.51	
0.0399	SVP	0.316	1.056	0.424	1.055	16.71	Present work
0.0419	SVP	0.341	1.173	0.418	1.056	17.11	
1.77 atomic layers	SVP	0.304	5.302	0.185	1.062	7.18	
0.0279	0	0.12	0.36	0.447	1.060	10.87	
0.0279	2	0.17	0.69	0.384	1.059	11.12	
0.0279	4	0.19	0.97	0.343	1.059	10.49	Present work
0.0279	6	0.29	1.89	0.303	1.058	11.47	
0.0279	8	0.49	4.66	0.251	1.059	12.34	
0.0279	10	0.58	7.86	0.210	1.060	11.25	

[a]All values in Å^{-2} unless otherwise indicated.

q^*. Since Eq. (1.1) is identical to the 3D spectrum, if we change the sign of γ in Eq. (1.3), the generalization of the three-phonon processes into n-phonon processes will give the same results. Therefore, the decay threshold q_n^* for the decay into n collinear and identical phonons can be given by

$$q_n^* = (1+n^{-2})^{-1/2} q_\omega^* = \sqrt{5/3}\, q_c \ .$$

The pressure dependence of the decay rate is shown in Fig. 4. To investigate the pressure dependence we have taken 0.0279 Å^{-2} for the density and the 3D numerical parameters γ and δ given by Dynes and Narayanamurti, because there are no experimental data for 2D liquid ^{4}He. However, we have found that the 2D numerical parameters do not deviate seriously from those of 3D. We have also observed that the decay rate and threshold are reduced gradually as presure increases. This is due to the fact that δ increases more rapidly than γ as the pressure increases. However, Jackle and Kehr have shown that γ decreases while δ increases with increasing pressure. Concerning the angular spreading θ_m, Sherlock *et al.*[10] used phonon beams of energy $\varepsilon(q)/k_B < 12$ K in bulk liquid ^{4}He at several pressures up to 24 bars and found that θ_m decreases roughly linearly as pressure increases. In the present analysis, we have found that θ_m is constant below 10 bars. Our theoretical values of γ, δ, q_c, q_m/q_c, and θ_m are listed in Table I. With these values of γ and δ, which were obtained from third-sound analyses, we observe that q_c decreases slowly as the density increases, while q_m/q_c and θ_m stay roughly constant. As in Eq. (3.17), the lifetime Γ_p is proportional to $F(p/q_c)$ which is illustrated in Fig. 5.

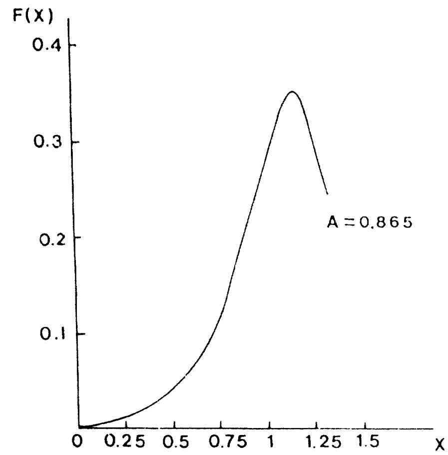

FIG. 5. $F(x)$ versus x for $A=0.865$.

ACKNOWLEDGMENTS

This work was supported in part by a grant to Korea University from the Korean Science and Engineering Foundation and in part by U.S. Office of Naval Research Contract No. N00014-84-K-0387 with the State University of New York at Buffalo.

[1]L. D. Landau, J. Phys. (Moscow) **11**, 91 (1947); R. P. Feynman and M. Cohen, Phys. Rev. **102**, 1189 (1956); A. D. B. Woods and R. A. Cowley, Rep. Prog. Phys. **36**, 1135 (1973); C. J. Aldrich, C. J. Pethick, and D. Pines, J. Low Temp. Phys. **25**, 677 (1976); W. G. Stirling, in Proceedings of the International Congress on Low Temperature Physics—LT-15, Grenoble [J. Phys. (Paris) Colloq. **39**, C6-1334 (1978)].

[2]H. J. Maris, Rev. Mod. Phys. **49**, 341 (1977).

[3]R. C. Dynes and V. Narayanamurti, Phys. Rev. B **12**, 1720 (1975).

[4]J. Jackle and K. W. Kehr, Phys. Rev. A **2**, 1757 (1974).

[5]A. Isihara and C. I. Um, Phys. Rev. B **19**, 5725 (1979); A. Isihara, S. T. Choh, W. H. Kahng, and C. I. Um, Physica **100B**, 74 (1980); A. Isihara and T. Samulski, Phys. Rev. B **16**, 1969 (1977).

[6]C. I. Um, A. Isihara, W. H. Kahng, and S. T. Choh, Physica **114B**, 191 (1982).

[7]T. J. Sluckin and R. M. Bowley, J. Phys. C **7**, 1779 (1974); C. I. Um, S. T. Choh, and A. Isihara, Phys. Rev. B **23**, 4498 (1981).

[8]A. Isihara, S. T. Choh, and C. I. Um, Phys. Rev. B **20**, 4482 (1979); Hyo-Jin Kim, C. I. Um, W. H. Kahng, H. K. Oh, and S. T. Choh, J. Korean Phys. Soc. **17**, 129 (1984).

[9]R. N. Bhatt and W. L. McMillan, Phys. Rev. A **10**, 1591 (1974); J. Wilks, *The Properties of Liquid and Solid Helium* (Oxford University Press, London, 1967).

[10]R. A. Sherlock, N. G. Mills, and F. G. Wyatt, J. Phys. C **8**, 2575 (1975).

[11]A. Isihara, C. I. Um, W. H. Kahng, H. G. Oh, and S. T. Choh, Phys. Rev. B **28**, 2509 (1983).

[12]L. P. Pitayevski and Y. B. Levinson, Phys. Rev. B **14**, 263 (1976).

[13]K. K. Singh and J. Prakash, Phys. Rev. B **17**, 1253 (1978); C. I. Um, W. H. Kahng, K. H. Yeon, S. T. Choh, and A. Isihara, *ibid.* **29**, 5203 (1984).

[14]R. Jones and P. D. Taylor, J. Phys. C **15**, 6709 (1982).