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Fractal dimensionality of wave functions at the mobility edge: Quantum fractal in the Landau levels

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Wave functions at the critical energy separating localized and extended states in disordered electron systems are shown to have fractal structure. We present results for the fractal dimensionality numerically calculated for a two-dimensional system with Landau quantization in strong magnetic fields together with a turbulent structure when the phase of the wave function is considered. Implications for transport properties are discussed.

The current interest in fractal geometry[1] in solid-state physics leads naturally to the question of how this concept manifests itself in quantum systems. In fact, the analysis of the fractal systems so far, which includes infinite clusters at percolation threshold,[2] diffusion-limited aggregates,[3] and colloidal emulsion,[4] has primarily concentrated on classical systems, and a clear picture of a quantum fractal is desired. I was the first to show that wave functions in electron systems at the Anderson transition, at which states become localized due to disorder, have a fractal nature.[5] In this paper the fractal dimensionality of the wave functions is calculated for systems in a continuous space, and implications of the fractal wave function for transport properties are explored.

Before presenting results let us briefly review the discussions leading to fractal wave functions. In an electron system with disorder, a mobility edge E_c separates localized states from extended states on the energy spectrum. The question is, what is the wave function at E_c like. Starting from Wegner's argument[6] for the participation ratio p, which is the reciprocal of the mean fourth power of the wave function (ψ), I point out that the critical behavior of $p \rightarrow 0$ as $E \rightarrow E_c$ implies that the wave function at $E = E_c$ has to occupy an infinitesimal fraction of the total volume of L^d in the space of dimensionality d.[5] Since ψ at E_c should be extended as well, the only way to accomplish this is to have a wave function which is filamentary and extends over the whole system as a network. The nature of this structure is suggested by a scaling argument. The real-space renormalization approach[7] to Anderson localization shows that E_c corresponds to a fixed point in the renormalization, at which the renormalized Hamiltonian has a scale invariance. This is reminiscent of the fluctuation at a phase transition, at which the critical fluctuation is similar on every length scale.[8] Hence the picture emerges that the wave function at a mobility edge has a fractal nature with a scale-invariant network structure. The range of length scale over which ψ may be regarded as self-similar depends on the physical system considered. In short, a wave function at E_c has large, but special (self-similar) fluctuations.[9]

One should note that, although percolation with fractal critical clusters[2] is sometimes regarded as a classical counterpart to localization, the physics involved in Anderson localization is quite different from that involved in percolation, in that the former is essentially a quantum effect governed by quantum tunneling and interference.

We encounter a self-similar wave function in other physical systems with a typical example being the almost periodic system.[10,11] In a tight-binding system in an external periodic potential of periodicity incommensurate with the lattice constant, wave functions have self-similar structure with a Kantor-set energy spectrum. One can develop an exact scaling property by a renormalization-group method in this case, while the wave functions at the randomness-driven transition discussed here has a statistical scale invariance.

To explore the nature of wave functions at E_c, we use here the two-dimensional (2D) electron system with random potentials with Landau quantization in a magnetic field. This system is of particular interest, since, apart from the quantum Hall effect[12] realized in this system, we can work on a continuous space rather than a tight-binding system, and, second, the position of the mobility edge has been worked out in detail.[13]

When a magnetic field is applied in a two-dimensional electron system, the density of states collapses into a series of Landau levels. In the presence of random potentials, each Landau level broadens with a width Γ. Although we can take Gaussian functions as a basis in a magnetic field, the envelope of a localized state is shown to be exponential.[13] The inverse localization length $\alpha(E)$ is shown[13] to be a continuous function of E, touching the $\alpha = 0$ axis at the center E_N of each Landau level of Landau index N, which may be identified as E_c.

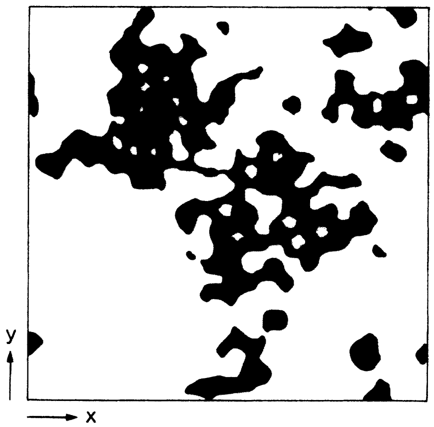

FIG. 1. A typical wave function, whose eigenenergy ($=E_0 - 0.003\Gamma$) is closest to $E_c = E_0$ for the system of size $L=(2\pi N_L)^{1/2}l$ with $N_L = 256$ being the total number of states for $N=0$. We depict the region where the amplitude of ψ is appreciable ($|\psi| \gtrsim 0.025$ for the normalized ψ with $l=1$), outside of which ψ damps off exponentially.

An example of ψ at E_N is shown in Fig. 1 for the $N=0$ (lowest) Landau level. This is a typical eigenstate with an eigenenergy closest to E_0 for a large but finite [$40l \times 40l$ with $l=(c\hbar/eH)^{1/2}$ being the cyclotron radius] system in periodic boundary conditions with dense ($c_i = 10$), short-range scatterers distributed at random. The expected network structure for ψ is clearly exhibited. We have actually calculated the fractal dimensionality D of ψ. The fractal dimensionality of an object is its Hausdorff dimensionality.[1] Its original definition is that, when an object is covered by elements of linear dimension ΔL, the number n of elements required behaves as $n \sim \Delta L^{-D}$. Here we adopt the original definition, since this definition does not depend on the choice of origin, so that it is applicable to any (random and/or finite) systems. In Fig. 2 an example is shown of the log-log plot of the number of elements required to cover the regions where $|\psi|$ has appreciable amplitude, versus the size of each element. Since a wave function is a linear combination of the Gaussian Landau wave function of size l, the lower limit of the length scale for the fractal is naturally of $O(l)$. The final D is obtained by an ensemble average over 80 samples, in which 127 states with eigenenergies lying within 0.008Γ of the level center are taken. The result shows that $D = 1.57 \pm 0.03$.

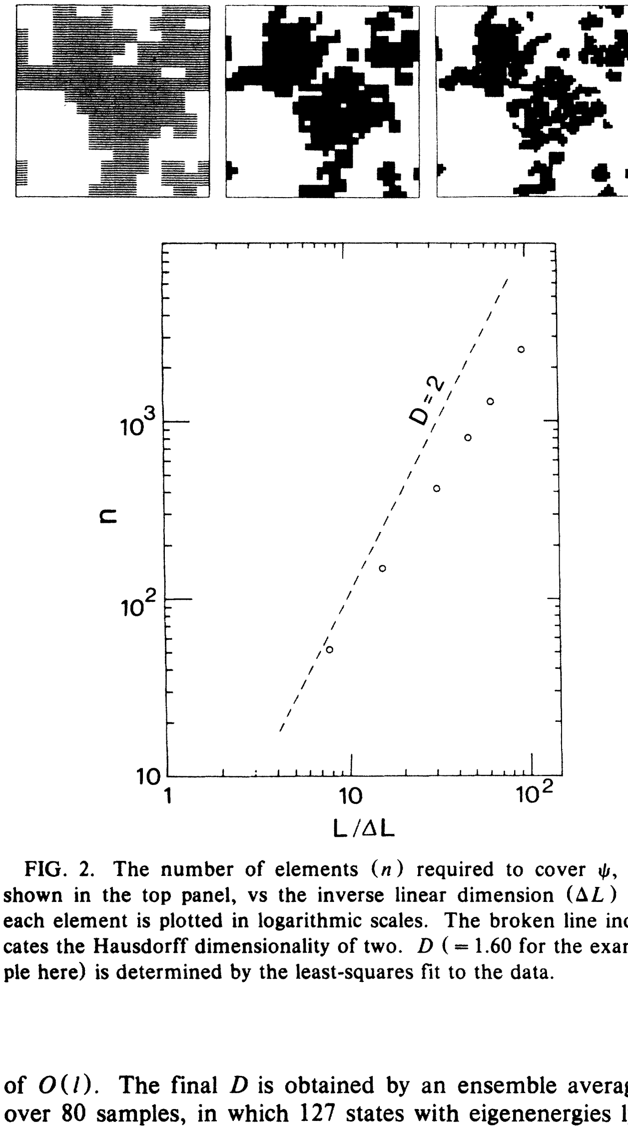

FIG. 2. The number of elements (n) required to cover ψ, as shown in the top panel, vs the inverse linear dimension (ΔL) of each element is plotted in logarithmic scales. The broken line indicates the Hausdorff dimensionality of two. D ($=1.60$ for the example here) is determined by the least-squares fit to the data.

Now let us comment on this result. First, in a 2D random system in strong magnetic fields considered here, the

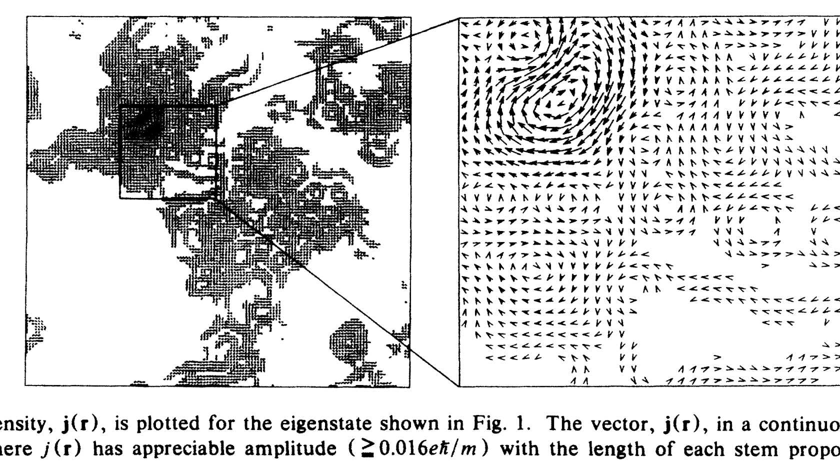

FIG. 3. The current density, $\mathbf{j}(\mathbf{r})$, is plotted for the eigenstate shown in Fig. 1. The vector, $\mathbf{j}(\mathbf{r})$, in a continuous 2D space is represented by arrows in a region where $j(\mathbf{r})$ has appreciable amplitude ($\gtrsim 0.016e\hbar/m$) with the length of each stem proportional to $j(\mathbf{r})$. The right panel is an enlargement.

electronic structure depends strongly on the nature of random potential.[14] In the above example, we consider the limit of short-range scatterers, in which the quantum jump of cyclotron motion is maximal, so that the wave function may be called a quantum fractal. In the opposite limit of a slowly varying potential, on the other hand, the motion of an electron reduces to a drift along equipotential lines. Percolation theory[15] for a random potential in a continuous 2D space shows that there is always a percolating path at $E=E_N$, so that the state at E_N in this opposite limit corresponds to a critical path, which is again a fractal.

It is noted that the obtained fractal dimensionality of 1.6 is similar to that of the geometrical pattern called a Sierpiński gasket (or arrowhead) in the literature.[1] If we go to 3D systems, the wave function at E_c is expected to have a fractal dimensionality less than three, which may possibly be similar to that of a fractal skewed web, a 3D analog of the Sierpiński gasket. It may be interesting to note that the critical percolation clusters[2] in 2D and the diffusion-limited aggregates[3] in 2D also have a fractal dimensionality close to 1.6, although the existence of some universal fractal dimensionality is unlikely.

Another point of much importance is specific to systems in magnetic fields. In a magnetic field, a wave function has to be complex with $\psi(\mathbf{r})=|\psi(\mathbf{r})|\exp[i\phi(\mathbf{r})]$. It has been shown[16] that there is an extra structure in ψ related to the behavior of the phase. If we look at the gauge-invariant current density, $\mathbf{j}(\mathbf{r})=(e\hbar/m)|\psi|^2(\nabla\phi-e\mathbf{A}/c\hbar)$, ψ is seen to have a striking turbulence structure as shown in Fig. 3. The current, which is divergence free for a stationary state due to the equation of continuity, exhibits a number of vortices. At $E=E_c$, the wave function in a magnetic field should thus in fact be called a turbulent fractal.[17]

The concept of a quantum fractal can be a key concept in understanding some of the basic properties in disordered systems. Here I comment on its relevance to the transport property. In the localized regime below E_c, each state is confined within a finite region. However, if we consider the conduction paths along which low-temperature, phonon-assisted hopping [variable-range hopping (VRH)[18]] takes place, the paths extend over the whole system. We can consider this type of network woven out of states with energies within ΔE (optimized for a given T in the VRH) of a given E as shown in Fig. 4 to ask ourselves what their fractal dimensionality is. The dimensionality of the network should again be lower than that of the host system ($D=1.75$ for the example shown in Fig. 4 for a 2D system). Now, in standard VRH theory, the total number of states involved in the hopping conduction is assumed to be L^d times the density of states per unit volume in a region of linear dimension L, from which the conductivity, $\sigma\sim\exp[-(T_0/T)^{1/(d+1)}]$, results. The above consideration shows, however, that the relevant states form a fractal as a whole with the number of the states behaving like L^D instead. This is quite physical, since the conduction takes

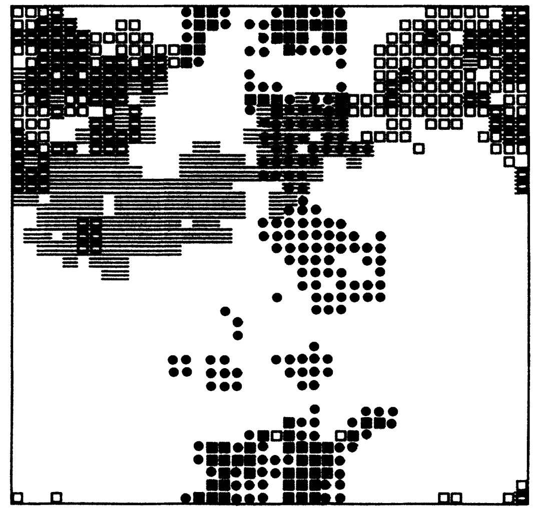

FIG. 4. An example of the set of localized wave functions forming an extended network with their eigenenergies close to a given energy. Here three states are shown with $(E-E_0)/\Gamma=-0.348$ (shaded), -0.332 (●), and -0.317 (□) by a similar plot as in Fig. 1.

place along a network, in which only a fraction of the states participate.[19] The hopping conductivity should then be replaced by $\sigma\sim\exp[-(T_0/T)^{1/(D+1)}]$, i.e., the power ($\nu$) of T is predicted to be larger than $\frac{1}{4}$ ($\frac{1}{3}$) in 3D (2D).

In the critical-percolation-path method,[20] which is one of the standard theories used to derive the VRH, the hopping conduction is related to a classical percolation path, so that the critical hopping path always has the structure of a critical percolation cluster. The fractal dimensionality, D_p, of the critical cluster at the classical percolation has been extensively studied,[21] but the results for D_p vary according to the definition or the method used to calculate the quantity. Hyperscaling gives $D_p=d/(1+1/\delta)$, where the index δ is estimated to be 5 (18) in 3D (2D), but this result deviates from the fractal dimensionality of about 1.6 for the backbone of the critical cluster in 2D obtained by Kirkpatrick.[2] Thus the determination of the precise values and relation of D and D_p remains a future problem.

The experimental results for the VRH for 2D electron systems in strong magnetic fields ranges from $\nu=\frac{1}{3}$ (Ref. 22) to $\frac{1}{2}$.[23,24] We should, however, note that (i) the experimental results quoted above are obtained for E_F in the tail region (minimum of the density of states between adjacent Landau levels), where localization is the strongest and a localized state can be almost Gaussian rather than exponentially localized, and (ii) the electron-electron interaction, which is neglected here, can give rise to effects such as a Coulomb gap.

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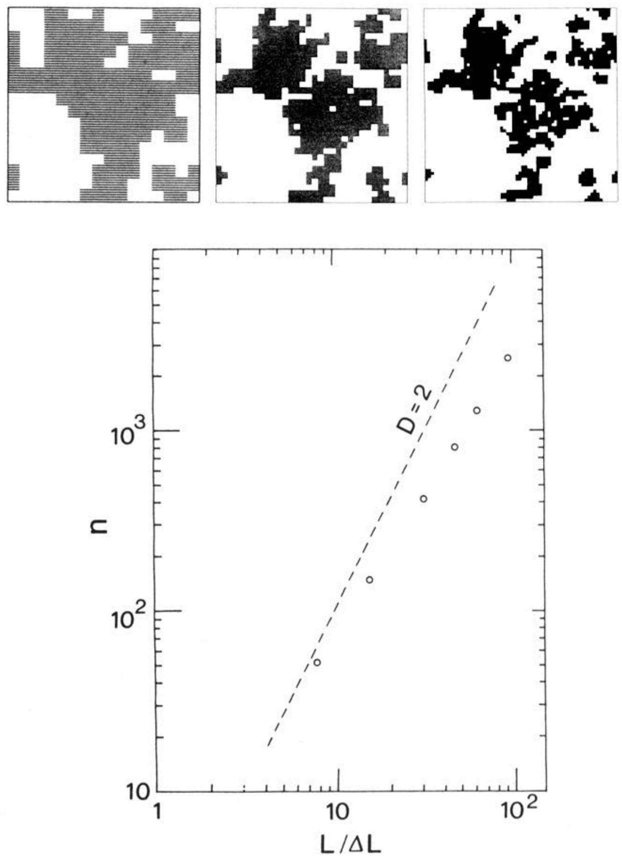

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