Scaling description of the dielectric function near the mobility edge

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We give a full renormalization-group treatment of the electron density-density correlation function in a disordered system near the mobility edge. We extract the scaling behavior of the diffusion constant and conductivity in the critical region. Thus, the behavior of these quantities as a function of wave number and frequency is determined and the crossover between the high-frequency and large-wave-number domains is given explicitly.

I. INTRODUCTION

In this paper we describe the scaling properties of the dielectric function and the diffusion constant of a disordered noninteracting electronic system. We discuss the critical behavior, near the mobility edge, of the wave-number (q) and frequency (ω) dependence of these quantities. That is, we give a full renormalization-group calculation of the scaling function for the diffusion constant, or conductivity, including the multiplicative constants. Thus, results are obtained for the behavior for either the large-frequency or large-wave-number critical regimes. In addition, the crossover between them is given explicitly. The critical wave-number dependence of the diffusion constant, or conductivity, which is not obtainable from perturbation calculations of the usual response functions, has not been calculated before. We restrict our considerations to the absolute zero of temperature.

Let us consider the retarded density-density correlation function for a noninteracting disordered system,

$$\Pi(q,\omega)=i\int d\mathbf{r}\int d\mathbf{r}'\int dt\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}e^{-i\omega t}\{\langle[\rho(r,t),\rho(r',0)]\rangle\}_{\rm dis}\theta(t)\ . \tag{1.1}$$

Here, the square brackets denote the commutator, the angular brackets the quantum thermal average, and $\{\cdots\}_{\rm dis}$ the disorder average. This is the density response to a change in chemical potential μ, i.e., $\Pi(q,\omega)=d\rho(q,\omega)/d\mu(q,\omega)$. At $\omega=0$, and small q, Π is just the thermodynamic density of states $dn/d\mu$. For small (q,ω), we may write

$$\Pi=\Pi^0+\Pi^1\ , \tag{1.2}$$

where $\Pi^0=dn/d\mu$ and the correlation function Π^1 is the "retarded-advanced" piece of Π:

$$\Pi^1(q,\omega)=(i/2\pi)\int d\mathbf{r}\int d\mathbf{r}'\int_{-\omega}^{0}d\varepsilon\{G^R(\varepsilon+\omega,\mathbf{r},\mathbf{r}')G^A(\varepsilon,\mathbf{r}',\mathbf{r})\}_{\rm dis}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\ , \tag{1.3}$$

where $G^{R,A}$ are the standard single-electron Green's functions in the disordered system.

At $\omega=0$, $\Pi^1=0$ and it is convenient to write

$$\Pi^1=i\omega\mathcal{X}\ . \tag{1.4}$$

For weak disorder, standard perturbation theory yields the characteristic diffusion propagator form for $\mathcal{X}$,

$$\mathcal{X}=\mathcal{X}_0(q,\omega)=\frac{N_1}{Dq^2-i\omega}\ , \tag{1.5}$$

where $D=v_F^2\tau/d$ is the diffusion constant in d dimensions and N_1 is the single-particle density of states. Weak disorder is defined by $E_F\tau\gg 1$, where τ is the elastic scattering time. Here, v_F and E_F are the Fermi velocity and the Fermi energy.

The Einstein relation gives the conductivity σ in terms of D:

$$\sigma=e^2D\,dn/d\mu\ . \tag{1.6}$$

For noninteracting electrons, $dn/d\mu$ and N_1 are identical, and in the following we use N_1 everywhere.

A standard treatment[1-3] of the localization problem reveals that the localization transition is driven by interacting diffusion modes of the form of Eq. (1.5). To pursue this line in the present context, it is convenient to introduce, as the coupling constant, the variable t,

$$t=(N_1D)^{-1}\ , \tag{1.7}$$

which scales as L^{d-2} and so has canonical dimension[4] $2-d$. We redefine the frequency scale by introducing $\tilde{\omega}$,

$$\tilde{\omega}=N_1\omega\ , \tag{1.8}$$

which has dimension d. Thus, instead of Eq. (1.5), we have

$$\tilde{\chi}_0(q,\omega)=\frac{t}{q^2+h}\ , \tag{1.9}$$

where we have defined

$$\tilde{\chi}=\chi/N_1^2\ , \tag{1.10}$$

which has dimension $-d$, and

$$h=-i\tilde{\omega}t\ , \tag{1.11}$$

which has dimension 2.

Although h is imaginary, it is easier to consider it real. This would be the case in the physical situation at finite temperatures when inelastic scattering is present. Then $-i\omega$ is replaced by the inelastic scattering rate $1/\tau_{\rm in}$. In the present case, we may continue to imaginary h at the end. As pointed out by Schäfer and Wegner,[5] the h in Eq. (1.9) is analogous to the inverse transverse susceptibility in the ordinary O(n) spin model below T_c. In that case the transverse correlation function is given, at low temperature T, by

$$T\chi=\int\langle s_\perp(q,t)s_\perp(-q,0)\rangle dt =\frac{TM}{K_1q^2+H}\ ,$$

where M is the magnetization, K_1 is the spin-stiffness constant, and H is the external field. In the localization problem, t [Eq. (1.7)] and h [Eq. (1.10)] play the role of temperature and magnetic field. In fact, the form of the diffusion propagator in Eq. (1.9) is identified with the bare transverse propagator in standard treatments of the fixed-length spin O(n) model or nonlinear σ model.[6,7]

II. SCALING BEHAVIOR OF $\chi(q,\omega)$

We now assume that the one-parameter scaling[8,1–3] description of the localization problem is valid and deduce the behavior of the function $\chi(q,\omega)$. For the analogous O(n) model, the corresponding problem for the transverse correlation function has been worked out, for example, in Ref. 5, but only at $q=0$. In the localization problem, the scaling behavior of $\chi(q=0,\omega)$ has been considered by Wegner,[9] Hikami,[3] and Shapiro and Abrahams.[10] A heuristic discussion for finite q has been given by Imry, Gefen, and Bergman.[11] For small q, our discussion will be based on the momentum-shell recursion method of Wilson and Kogut[12] as applied by Nelson and Pelcovits[7] to the O(n) case. However, at larger q, the scaling behavior of χ is inaccessible by the usual technique and we are forced to consider the invariant correlation function as first discussed by Elitzur[13] and Jevicki.[14] This will be described in Sec. IV.

We begin with a system specified by t and h (or $\tilde{\omega}$) measured on the basis of some unit of length. The intrinsic momentum cutoff for the diffusive motion described in Eq. (1.5) is the inverse elastic mean free path. Therefore, it is convenient to choose the initial length unit as the elastic mean free path $l=v_F\tau$. We then change the length scale by a factor $b>1$ and integrate out all Fourier components $(bl)^{-1}<q<l^{-1}$. The new rescaled problem is characterized by a renormalized $t(b)$ which obeys the scaling equation

$$\frac{dt}{d\ln b}=\beta(t)\ . \tag{2.1}$$

The nontrivial content of Eq. (2.1) is that $\beta(t)$ is a function of t only, independent of l or b. The β function has been computed by perturbation theory[3,8,15] in t:

$$\frac{dt}{d\ln b}=\beta(t)=-\varepsilon t+2t^2+\mathcal{O}(t^5)\ , \tag{2.2}$$

where $\varepsilon=d-2$, and $\mathcal{O}(\cdots)$ indicates that the correction, if not zero, is of order higher than t^4. The unstable zero of $\beta(t)$ and the associated fixed-point value

$$t^*=\varepsilon/2 \tag{2.3}$$

signals the mobility edge.

The localization problem is different from the O(n) model in a number of respects. The former, as is well known,[3,5] corresponds to the $n\to0$ limit of a generalized nonlinear σ model of $n\times n$ matrix fields of noncompact symmetry. In this O(n,n) formulation, the density of states N_1 plays the role of the order parameter and exhibits no anomaly at the transition corresponding to the mobility edge. This has been discussed in detail by McKane and Stone.[16] The density of states per unit volume scales in a trivial manner,

$$N_1\to b^dN_1(b)\ , \tag{2.4}$$

corresponding to a critical exponent $\beta=0$. In the O(n) model, on the other hand, a nontrivial spin rescaling occurs[7] at each step of the renormalization-group process:

$$S_q\to\zeta(b)S_q(b)\ . \tag{2.5}$$

The spin-rescaling factor ζ is determined by the requirement[12] that the rescaled Hamiltonian have the same form as the original one. The result[7] is

$$\zeta=b^db^{-(d-2+\eta)/2}\ . \tag{2.6}$$

The factor b^d in Eq. (2.6) simply comes from the definition of the field S_q in momentum space and corresponds to the same factor in Eq. (2.4). We see that an additional critical index η is introduced in the spin problem. In localization, however, the scaling of Eq. (2.4), which gives $\zeta=b^d$, implies $\eta=2-d$. This makes the localization problem simpler in some respects than the O(n) model, where the nontrivial order-parameter- (spin-) rescaling factor ζ is responsible for anomalous dimensions in the model.

Turning now to the "transverse" correlation function $\tilde{\chi}$, we write down at once the homogeneity scaling relation it satisfies,

$$\tilde{\chi}(q,t,h)=b^d\tilde{\chi}(bq,t(b),h(b))\ . \tag{2.7}$$

In the O(n) model,[7] an extra factor ζ^2b^{-2d} appears on the right-hand side (rhs). Instead of $\tilde{\chi}$, it is useful to discuss the scaling of the generalized diffusion constant $\tilde{D}(q,\omega)$. It is defined from $\tilde{\chi}$ by

$$\tilde{\chi}^{-1}=\tilde{D}q^2-i\tilde{\omega}\ . \tag{2.8}$$

In zeroth order, $\tilde{D}=N_1D$. According to Eqs. (2.7) and (2.8), $\tilde{D}$ satisfies

$$\tilde{D}(q,t,h)=b^{-d+2}D(bq,t(b),h(b)) \, . \tag{2.9}$$

Thus, in general, near the mobility edge, the diffusion constant becomes *scale dependent*. This will be discussed further in Sec. V.

It is now convenient to replace the b and $t(b)$ dependences by an equivalent dependence on correlation length. This quantity arises naturally as follows: We integrate the β function to get $t(b)$ in terms of its *initial value*, t, at $b=1$. The result is

$$\frac{t^*}{t(b)}-1=\left[\frac{t^*}{t}-1\right]b^{\varepsilon} \, , \tag{2.10}$$

where $t^*=\varepsilon/2$. The correlation length ξ, in units of the mean free path, is given by the value of b for which $t^*/t(b)\approx 2$. That is, it is the scale characterizing the crossover from the critical region $t(b)\approx t^*$ to the metallic region where $t(b)$ scales as $b^{-\varepsilon}$, i.e., the scaling has reached the point where the first term in the β function [Eq. (2.2)] dominates. In units of the mean free path, $\xi(t)$ is given by

$$\xi(t)=\left|\frac{t^*-t}{t}\right|^{-1/\varepsilon} . \tag{2.11}$$

Thus, Eq. (2.10) becomes

$$t^*/t(b)=1+(b/\xi)^{\varepsilon} \, . \tag{2.12}$$

We now replace, in Eq. (2.9), the t dependence by the correlation-length dependence. Thus by setting $b=\xi$ in Eq. (2.9), we see that $\tilde{D}(q,\omega)$ must have the scaling form

$$\tilde{D}(q,t,h)=\frac{1}{t^*}\xi^{-\varepsilon}F(x,y) \, , \tag{2.13}$$

where F is dimensionless and

$$x=-i\tilde{\omega}\xi^{d}, \quad y=q\xi \tag{2.14}$$

are the dimensionless arguments of F.

We now give various limits of the scaling function $F(x,y)$. In the critical region, $\xi\to\infty$, $x,y\gg 1$. Here, $\tilde{D}$ must be independent of ξ. Thus,

$$F(x,0)=\alpha_1 x^{\varepsilon/d} \, , \tag{2.15}$$

$$F(0,y)=\alpha_2 y^{\varepsilon} \, . \tag{2.16}$$

The prediction of Eq. (2.15) that $\tilde{D}\to 0$ as $\omega^{\varepsilon/d}$ has been made before.[9,3,10] The crossover between these limiting behaviors must also be independent of ξ. This boundary is then given by

$$x=\alpha_3 y^{d} \, . \tag{2.17}$$

The general arguments we have given do not give us the proportionality constants α in Eqs. (2.15)—(2.17) nor the detailed crossover to the region x,y small. In what follows, we shall calculate these constants and discuss the behavior of $F(x,y)$ everywhere in $2+\varepsilon$ dimensions.

III. RENORMALIZATION-GROUP CALCULATION

We employ the usual renormalization-group strategy: For a system near its critical point we change the length scale until we leave the critical region and reach the weak-coupling region where perturbation theory is valid. We then use the scaling relations of Sec. II to obtain the critical behavior of quantities from their perturbation expansion.

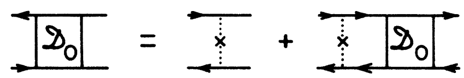

FIG. 1. Lowest-order particle-hole diffusion ladder.

We follow Nelson and Pelcovits[7] and employ the Wilson-Kogut[12] momentum-shell technique to derive recursion relations to effect the scaling. We can find $\chi\propto\{G^RG^A\}_{\rm dis}$ in perturbation theory by summation of the maximally crossed diagrams. The calculation is carried out, essentially following Hikami,[3] as follows. The bare diffusion [particle-hole (ph) or particle-particle (pp)] propagator $\mathscr{D}_0(q,\omega)$ is the sum of ladder diagrams as shown in Fig. 1. From

$$\mathscr{D}_0=[2\pi N_1\tau^2/(Dq^2-i\omega)]^{-1} \, ,$$

one obtains

$$\chi(q,\omega)=2\pi N_1\tau^2\mathscr{D} \, , \tag{3.1}$$

as in Fig. 2, by adding the single-particle Green's functions at the ends of $\mathscr{D}$ and adding an undressed bubble G^RG^A. Schematically, $\chi=G^RG^A+G^RG^A\mathscr{D}G^RG^A$. Here, $G^{R\,(A)}$ is the average retarded (advanced) single-particle Green's function. A perturbation procedure for $\mathscr{D}$ is established as in Ref. 3. The lowest-order self-energy $\Sigma(q,\omega)=\mathscr{D}^{-1}-\mathscr{D}_0^{-1}$ is found as in Fig. 3. The (q,ω)-dependent part of $\Sigma(q,\omega)$ is given by

$$\Sigma(q,\omega)=Dq^2\left[2\tau^2\sum_{q'}\frac{1}{D(q')^2-i\omega}\right] . \tag{3.2}$$

When this correction to $\mathscr{D}^{-1}$ is used in the determination of χ, we find

$$\tilde{\chi}^{-1}=\frac{1}{t}(q^2+h)+\frac{2}{\varepsilon}(h^{\varepsilon/2}-1)q^2 \, . \tag{3.3}$$

In deriving Eq. (3.3), we have evaluated the integral in Eq. (3.2) in $2+\varepsilon$ dimensions and used Eqs. (3.1) and (1.10) to pass to χ and $\tilde{\chi}$. The variables t,h are defined in Eqs. (1.7)—(1.11). In Eq. (3.2) we chose an ultraviolet cutoff Λ of order $1/l$, but have adjusted it to absorb the order unity phase-space factor. The "frequency" h is measured in units of the square of the cutoff. At $\varepsilon=0$ ($d=2$), Eq. (3.3) gives the usual $\ln\omega$ localization correction to the diffusion constant.

We remark from Eq. (3.3) that, at $\varepsilon\to 0$, no $\ln q$ terms occur, even at finite q. This is reminiscent of the $O(n)$ model, where $\ln q$ singularities do not occur anywhere in

FIG. 2. Density-density correlation function.

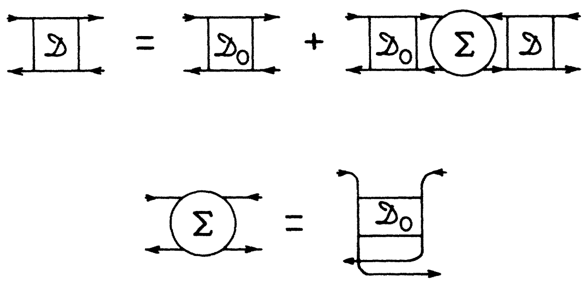

FIG. 3. Definition of self-energy Σ and its lowest-order graph. The $\mathscr{D}_0$ appearing in Σ is a particle-particle ladder.

the perturbation expansion of the transverse correlation function. This will be discussed further in the next section.

The diffusion constant $\tilde{D}(q,t,h)$ (recall $\tilde{D}=N_1 D$) is the coefficient of q^2 in $\tilde{\chi}^{-1}$. Its scaling is given in Eq. (2.9). From Eqs. (2.9) and (3.3), we find, by scaling with a factor b,

$$\tilde{D}(q,t,h)=b^{-\varepsilon}\left[\frac{1}{t'}+\frac{1}{t^*}[(h')^{\varepsilon/2}-1]\right], \tag{3.4}$$

where we have denoted $t(b),h(b)$ by t',b', and $t^*=\varepsilon/2$ is the zero of the β function Eq. (2.2). The rhs of Eq. (3.4) can be written in the form of Eq. (2.13) by expressing t' in terms of ξ/b according to Eq. (2.12). We find

$$F(x,y)=1+\left[\frac{\xi}{b}\right]^{\varepsilon}(h')^{\varepsilon/2}. \tag{3.5}$$

We next choose the appropriate scale factor b. We have to discuss several cases separately, depending on the relative magnitudes of ω,q, that is, x,y.

A. $q<\omega$

Here we follow Ref. 7. The expansion parameter for the problem at $d=2+\varepsilon$ is $t\ln h$. Although $t=O(\varepsilon)$, for small h, $t\ln h$ is not small. The strategy is to scale out of the critical region by $b=b_\omega$ such that $h(b_\omega)=1$. Then, from Eq. (3.5),

$$F(x,y)=1+(\xi/b_\omega)^{\varepsilon}. \tag{3.6}$$

The quantity ξ/b_ω in Eq. (3.6) may be expressed as a function of

$$x=-i\tilde{\omega}\xi^d=h\xi^d/t \tag{3.7}$$

only as follows: Since the frequency $\tilde{\omega}=N_1\omega$ scales as b^d [Eq. (2.4)], we have

$$1=h(b_\omega)=i\tilde{\omega}(b_\omega)t(b_\omega)=ib_\omega^d\tilde{\omega}t(b_\omega).$$

Combining this with Eqs. (3.7) and (2.12), we find

$$t^*x=(\xi/b_\omega)^d[1+(b_\omega/\xi)^{\varepsilon}]. \tag{3.8}$$

Thus, Eqs. (3.6) and (3.8) give $F(x,y)$ as a function of x only for small q. In particular, we require $q'=bq<1$ in the scaled system. Therefore, Eq. (3.6) is valid only if

$$qb_\omega=yb_\omega/\xi<1, \tag{3.9}$$

which, since b_ω/ξ is a function of x only [Eq. (3.8)], defines the boundary of region (i) in the x,y plane shown in Fig. 4. The boundary may be defined explicitly at large and small x: According to Eqs. (3.8) and (3.9), if $xt^*\gg 1$,

$$xt^*\approx(\xi/b_\omega)^d,\quad xt^*>y^d, \tag{3.10a}$$

and if $xt^*\ll 1$,

$$xt^*\approx(\xi/b_\omega)^2,\quad xt^*>y^2. \tag{3.10b}$$

From Eqs. (3.6) and (3.10), we find $F(x,y)$ explicitly in region (i): For $xt^*\gg 1$,

$$F=1+(xt^*)^{\varepsilon/d}, \tag{3.11a}$$

and for $xt^*\ll 1$,

$$F=1+(xt^*)^{\varepsilon/2}. \tag{3.11b}$$

Equation (3.11a) describes the critical region as in Eq. (2.15) and Refs. 9, 3, and 10. Equation (3.11b) gives the small frequency correction to the diffusion constant. From Eqs. (3.10) and (3.11) we may find two of the α constants we introduced in Eqs. (2.15)–(2.17). They are

$$\alpha_1=(\varepsilon/2)^{\varepsilon/d},\quad \alpha_3=2/\varepsilon. \tag{3.12}$$

It is interesting to note that our scaling result is identical to the self-consistent equation of Vollhardt and Wolfle.[17] The latter is obtained from the perturbation result for D (we use the units $e^2/h=1$):

$$D=D_0+\frac{1}{4\pi^2}\frac{\Lambda^{\varepsilon}}{N_1}\frac{2}{\varepsilon}\left[\left[\frac{\omega}{D_0\Lambda^2}\right]^{\varepsilon/2}-1\right], \tag{3.13}$$

where Λ is the ultraviolet cutoff of order $1/l$. Self-consistency is achieved by replacing D_0 on the rhs by D. Then by using Eq. (2.13) to express $\tilde{D}=N_1 D$ in terms of F, Eq. (2.11) for ξ in terms of t, and Eq. (2.14) for x, we find

$$F=1+(xt^*/F)^{\varepsilon/2}, \tag{3.14}$$

which is identical to our Eqs. (3.6) and (3.8). This result

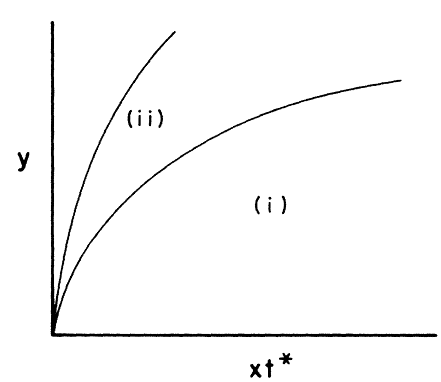

FIG. 4. Regions of the $x=\omega N_1\xi^d$, $y=q\xi$ plane. The scaling result of Eq. (3.11) is valid in region (i) and that of Eq. (3.17) is valid in region (ii).

has also been obtained by Hikami[18] by solution of the Callan-Simanzyk equation.

B. $q > \omega$

When q is sufficiently large that $qb_\omega > 1$, then we must stop the scaling at b_q such that $qb_q = 1$. If $t(b_q)|\ln h(b_q)|$ is small, then the perturbative result of Eqs. (3.4) and (3.5) is valid. We use steps similar to those which led to Eqs. (3.6) and (3.8). From Eq. (3.5), at $b_q = q^{-1}$, we have

$$F(x,y) = 1 + (q\xi)^{\varepsilon}(h')^{\varepsilon/2} = 1 + y^{\varepsilon}(h')^{\varepsilon/2} , \tag{3.15}$$

where

$$h' = h(b_q) = -i\tilde{\omega}(b_q)t(b_q) = xt^* y^{-d}(1+y^{-\varepsilon})^{-1} . \tag{3.16}$$

Here, we have used Eq. (2.12) again, but with $b = b_q = q^{-1}$. Thus, we obtain

$$F(x,y) = 1 + [xt^*/(y^{\varepsilon}+1)]^{\varepsilon/2} . \tag{3.17}$$

The boundary for this value of F is determined by

$$t'|\ln h'| = \frac{t^*}{1+y^{-\varepsilon}} \ln\left[\frac{xt^*}{y^d(1+y^{-\varepsilon})}\right] < 1 . \tag{3.18}$$

Condition (3.18) has the following limits: If $y \gg 1$,

$$xt^* > y^d e^{-1/t^*} , \tag{3.19a}$$

and if $y \ll 1$,

$$xt^* > y^2 e^{-1/t^* y^{\varepsilon}} , \tag{3.19b}$$

which defines the boundary of region (ii) on Fig. 4.

We have apparently extended the scaling result for F to a region of larger y. However, within region (ii), F is still independent of y since Eq. (3.16) differs from the y-independent result for F in region (i) [Eqs. (3.11)] only by terms of order $y^{\varepsilon^2/2}$, which is unity to the order of accuracy of our calculation. The present scaling theory, as in the O(n) case,[7] does not extend to obtaining the q dependence of the diffusion constant, particularly, in the region $y \gg 1$. This is a reflection of the fact, as mentioned before, that a $\ln q$ expansion does not occur in perturbation theory for χ.

IV. CRITICAL REGION

Recall the definitions $x = -iN_1\omega\xi^d$, $y = q\xi$. In the critical region $y \gg 1$, ordinary perturbation theory at $d=2$ leads to $\ln x$ singularities as $x \to 0$ in all terms. However, scaling arguments show that we expect $\chi \propto y^{-d}$, free of singularity at $x=0$. This same situation obtains in the O(n) case where the resolution[13] is based on the fact that in the critical region $y = q\xi \gg 1$, when the magnetic field h ($= xt\xi^{-d}$ in our case) is small, the magnetization is fluctuating so strongly that there is no distinction between the transverse and longitudinal correlation functions. The asymmetry is restored and one studies the invariant correlation function

$$\chi_I = \langle \mathbf{s}(r)\cdot\mathbf{s}(0)\rangle .$$

It turns out that the $\ln h$ singularities cancel in the invariant correlation function and for it a $\ln q$ expansion *is* obtained at $d = 2+\varepsilon$. Furthermore, one then succeeds in an explicit calculation of the exponent η. Informative discussions of this have been given by McKane and Stone[19] and by Amit and Kotliar.[20]

The perturbation theory for the interacting diffusion modes that we have used so far does not admit a simple formulation for the invariant correlation function since it deals only with the localization analogue of the transverse correlation function in the spin problem. It is the "retarded-advanced" piece of the density-density correlation function as in Eq. (1.3).

Fortunately, the field-theoretic formulation[16] of localization explicitly exhibits the full symmetry of the problem and the invariant correlation function has been calculated by McKane and Stone.[16] Their Eq. (5.11) is

$$\tilde{\chi}_I(r) = -n\pi[1 + 4nt'\ln r + 4n(n+1)(t')^2\ln^2 r - 2n\varepsilon t'\ln^2 r] . \tag{4.1}$$

The last term was omitted[21] in Ref. 16. The renormalized coupling t' is determined from the bare coupling t in Eqs. (5.8)–(5.10) of Ref. 16. The replica index n, eventually taken to zero, determines the symmetry of the model. Here, it is a noncompact O(n,n)/[O(n)×O(n)] realization of the localization problem. The β function[3,16] is

$$\beta(t) = -[\varepsilon t + 2(n-1)t^2 + \mathcal{O}(t^3)] . \tag{4.2}$$

At $\xi \to \infty$, $t \to t^* = \varepsilon/(2-2n)$, and the expansion, Eq. (4.1), for χ_I exponentiates as it should,

$$\tilde{\chi}_I = -n\pi r^{2-d-\eta}, \quad \eta = \frac{n+1}{n-1}\varepsilon . \tag{4.3}$$

Here, the previous scaling result $\eta = 2-d$ is recovered explicitly at $n \to 0$.

As we implied earlier in this section, the quantity of interest in the localization problem is the transverse piece of the invariant correlation function χ_I. In the O(n) nonlinear σ model, it is simple to extract the transverse piece. In the critical region $q\xi \to \infty$, the symmetry being restored, all components of χ_I are equivalent, so any single component may be found from χ_I by dividing by n.

For the present case of O(n,n) noncompact symmetry one could, in principle, extract the transverse piece from the invariant correlation function, Eq. (4.3), at finite n and then, after Fourier transformation, pass to the $n=0$ limit. In the absence of a rigorous recipe to extract the transverse part χ from χ_I in the critical region at finite n, we resort to the following argument: At finite n, all components of χ_I are massless near the critical point. Since χ_I is a trace, it has $(2n)^2$ equivalent components. However, in the limit $n \to 0$ the longitudinal part of χ_I becomes massive and has no critical behavior whatever.[16] Therefore, in the physical situation, at $n \to 0$, all singularities in χ_I belong to the transverse part we seek. On this basis we shall proceed from Eq. (4.3) by dividing $4n^2$ and taking the Fourier transform. The result for any one component of χ_I is

$$\tilde{\chi}(q) = \tfrac{1}{2}\pi^2\varepsilon/q^{2-\eta} . \tag{4.4}$$

Finally, at $n \to 0$, where $\eta = 2-d$, we find

$$\tilde{\chi}(q)=\tfrac{1}{2}\pi^2\varepsilon/q^d \,. \tag{4.5}$$

We can express this result in the notation introduced in Eqs. (2.8)–(2.14):

$$F(x,y)=\tfrac{1}{8}y^{\varepsilon}, \quad x \ll y \tag{4.6}$$

which is, of course, of the scaling form given in Eq. (2.16). We therefore have found the final α constant introduced in Eqs. (2.15)–(2.17), $\alpha_2=\frac{1}{8}$.

V. CONCLUSION

We have derived the critical behavior of the diffusion constant of a disordered noninteracting electronic system near its mobility edge. In particular, the diffusion constant becomes *scale dependent.* At short distances, $<\xi$ (i.e., $q\xi$ large), D is q dependent, going as q^{d-2}. This behavior is the equivalent in momentum space of the familiar[8,22] length-scale dependence,

$$\sigma \propto D \propto 1/L^{d-2} \,, \tag{5.1}$$

for $L<\xi$. Similarly, at $N_1\omega\xi^d$ large, the diffusion constant is frequency dependent according to $D\propto\omega^{(d-2)/d}$, a result[9] simply derived[10] from Eq. (5.1) by the substitution $L_\omega=\sqrt{D/\omega}$.

Some results of this work, particularly the scale dependence of the diffusion constant, have already been used in several places. Lee[23] discussed the screening and density of states near the mobility edge for interacting electrons. He used the behavior of the diffusion constant in the critical region to derive the perturbative interaction correction to the density of states. Anderson, Muttalib, and Ramakrishnan[24] showed how the critical q dependence of $D(q,\omega)$ can enhance the Coulomb pseudopotential and decrease the T_c of a disordered superconductor near the mobility edge. Kapitulnik and Kotliar[25] also discussed disordered superconductors. Using the scale-dependent diffusion constant, they have made explicit predictions for the critical-field behavior near the mobility edge. Some of these results have been summarized in the review of Lee and Ramakrishnan.[22]

One of the motivations for obtaining the critical behavior of the diffusion constant was to discuss electron screening and dielectric constants on either side of the mobility edge. This problem has been addressed by several authors.[11,23,26,27] Imry, Gefen, and Bergman,[11] in particular, were the first to discuss the behavior of the dielectric constant in the critical region for the various regions of q and ω. Our results enable the evaluation of the constants in their heuristically obtained interpolation formulas.

We now summarize our conclusions for the physically relevant quantities. Once the diffusion constant is obtained in the critical region, then the retarded-advanced correlation Π^1 of Eq. (1.3) is determined by means of Eqs. (1.4) and (1.5). The full polarizability $\Pi(q,\omega)$ [Eqs. (1.1) and (1.2)] is given by

$$\Pi(q,\omega)=-iN_1\frac{D(q,\omega)q^2}{D(q,\omega)q^2-i\omega} \,. \tag{5.2}$$

The dielectric function is obtained[28] as

$$\epsilon(q,\omega)=1+iV_q\Pi(q,\omega) \,, \tag{5.3}$$

where V_q is the Fourier transform in d dimensions of the Coulomb interaction e^2/r. Thus we obtain

$$\epsilon(q,\omega)=\frac{\kappa_d^{d-1}D(q,\omega)q^{3-d}}{D(q,\omega)q^2-i\omega} \,, \tag{5.4}$$

where

$$\kappa_d=\sqrt{\pi}[4N_1e^2\Gamma((d-1)/2)]^{1/(d-1)} \tag{5.5}$$

is the inverse Thomas-Fermi screening length in d dimensions. We see from Eq. (5.4) that the static screening is unaffected by critical fluctuations on the metallic side,

$$\epsilon(q,\omega)=1+(\kappa_d/q)^{d-1} \,, \tag{5.6}$$

as pointed out in Refs. 11 and 23. As emphasized by Lee,[23] the region of static screening $\omega<D(q,\omega)q^2$ shrinks to zero near the mobility edge.

The diffusion constant itself is given as follows: For $\omega N_1\xi^d>1$ and $q\xi<1$,

$$D(q,\omega)=(2/\varepsilon N_1)^{2/d}\omega^{\varepsilon/d} \,, \tag{5.7a}$$

for $q\xi>1$ and $\omega N_1\xi^d<1$,

$$D(q,\omega)=(1/4\varepsilon N_1)q^{\varepsilon} \,, \tag{5.7b}$$

and for $\omega N_1\xi d$ and $q\xi<1$,

$$D(q,\omega)=(2/\varepsilon N_1)\xi^{-\varepsilon} \,. \tag{5.7c}$$

Equation (5.7c) is obtained at once from Eq. (2.13) since $F(x,y)$ is unity outside the critical region [see, for example, Eq. (3.6)]. From Eqs. (5.4)–(5.7), the constants of the interpolation formulas of Ref. 11 are easily obtained.

In Ref. 11, $d=3$ and the imaginary part of the dielectric constant is identified with the conductivity. For general d, however, the conductivity is obtained on the metallic side from a generalized Einstein relation:

$$\sigma(q,\omega)=e^2(\omega/q^2)\Pi(q,\omega)=\frac{e^2N_1D(q,\omega)\omega}{\omega+iD(q,\omega)q^2} \,. \tag{5.8}$$

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[1]F. Wegner, Z. Phys. B **35**, 208 (1979).
[2]K. B. Efetov, A. I. Larkin, and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys.—JETP **52**, 568 (1980)].
[3]S. Hikami, Phys. Rev. B **24**, 2671 (1981).
[4]See, for example, P. Pfeuty and G. Toulouse, *Introduction to the Renormalization Group and Critical Phenomena* (Wiley, New York, 1977), Chap. 3.
[5]L. Schäfer and F. Wegner, Z. Phys. B **38**, 113 (1980).
[6]E. Brézin and J. Zinn-Justin, Phys. Rev. B **14**, 3110 (1976).
[7]D. R. Nelson and R. A. Pelcovits, Phys. Rev. B **16**, 2191 (1977).
[8]E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[9]F. Wegner, Z. Phys. B **25**, 327 (1976).
[10]B. Shapiro and E. Abrahams, Phys. Rev. B **24**, 4889 (1981); B. Shapiro, *ibid.* **25**, 4266 (1982).
[11]Y. Imry, Y. Gefen, and D. Bergman, Phys. Rev. B **26**, 3436 (1982).
[12]K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 77 (1974).
[13]S. Elitzur, Nucl. Phys. **212B** [Part.], 501 (1983).
[14]A. Jevicki, Phys. Lett. **71B**, 327 (1977).
[15]L. P. Gor'kov, A. I. Larkin, and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. Pis'ma Red. **30**, 248 (1979) [JETP Lett. **30**, 248 (1979)]; E. Abrahams and T. V. Ramakrishnan, J. Non-Cryst. Solids **35**, 15 (1979).
[16]A. J. McKane and M. Stone, Ann. Phys. (N.Y.) **131**, 36 (1981).
[17]D. Vollhardt, and P. Wölfle, Phys. Rev. B **22**, 4666 (1980).
[18]S. Hikami, in *Anderson Localization,* edited by Y. Nagaoka and H. Fukuyama (Springer, Berlin, 1982).
[19]A. McKane and M. Stone, Nucl. Phys. **163B**, 169 (1980).
[20]D. J. Amit and G. B. Kotliar, Nucl. Phys. **170B [FS1]**, 187 (1980).
[21]The term we have added to Eq. (5.11) of Ref. 16 comes from the ε expansion of the $x^{-\varepsilon}$ piece of the second term of their Eq. (5.5). Other differences arise from the following: Their n is twice ours, their g^2 is $4t$, our $\widetilde{\chi}$ is $\pi/2$ times their $G^{(2)}$, and they have divided out a factor $-(n_+ + n_-)$ in passing from Eq. (5.4) to (5.5).
[22]P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985), Sec. II.
[23]P. A. Lee, Phys. Rev. B **26**, 5882 (1982).
[24]P. W. Anderson, K. A. Muttalib, and T. V. Ramakrishnan, Phys. Rev. B **28**, 117 (1983).
[25]A. Kapitulnik and G. Kotliar, Phys. Rev. Lett. **54**, 473 (1985).
[26]W. L. McMillan, Phys. Rev. B **24**, 2739 (1981).
[27]A. Kawabata, J. Phys. Soc. Jpn. 53, 1429 (1984).
[28]B. L. Altshuler, A. G. Aronov, and P. A. Lee, Phys. Rev. Lett. **44**, 1288 (1980).