PHYSICAL REVIEW B VOLUME 33, NUMBER 9 1 MAY 1986

Anderson localization and breakdown of hydrodynamics in random ferromagnets

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(Received 10 February 1986)

The dynamic structure factor of Heisenberg magnets with weak randomness is computed. Under circumstances which are explained in detail, we find failure of hydrodynamic theory in the longitudinal structure factor due to localization of spin waves. Localization induces a power-law dependence on q and ω for the neutron scattering line shape near magnetic Bragg spots. The exponent describing the power law is related to the correlation-length exponent of Anderson localization. Random anisotropy magnets appear to be promising candidates for experimental investigations.

The breaking of a continuous symmetry is usually associated with corresponding low-frequency excitations of the system. For example, phonons and magnons are associated with broken translational and rotational invariance, respectively. For low frequencies and long wavelengths the dynamic properties of these systems can be deduced from hydrodynamic assumptions involving only the conservation laws of the Hamiltonian and the symmetry of the ordered state.[1] Anisotropy induces a gap in the excitation spectrum, but the modes remain qualitatively similar.

The addition of impurities may change the low-frequency modes qualitatively. For instance, it has been shown that impurities cause phonons to localize at all frequencies in two or fewer dimensions. However, low-frequency phonons in three dimensions do not localize.[2] Loosely speaking, if the impurities destroy the continuous symmetry operation that transforms between different ground states, then localization of the low-frequency modes is enhanced. In this paper we investigate the relevance of spin-wave localization in three-dimensional magnets and the implications for inelastic neutron scattering experiments.

We discuss the dynamics of Heisenberg ferromagnets (with conserved magnetization) in the harmonic approximation with moments, exchange, and Ising-Heisenberg uniaxial anisotropy which have random contributions. This problem can be related to that of the motion of an electron in a random medium.[3] Because electron localization has been studied extensively, a great body of theoretical results is available[4] that can be used to make detailed predictions for the spin system. Thus we can investigate the conditions needed to obtain localization of very low-frequency spin waves in three dimensions. We find that one must break the rotational invariance of the system to obtain localization of the low frequency modes, so the low-energy spin waves of a weakly diluted magnet are not localized. Two systems that display localization of these modes are proposed— the first is a system with random moments in a large magnetic field, and the second is a material with random Ising-Heisenberg anisotropy. It appears that random moment induced localization should only be relevant for very low temperatures ($\lesssim 1\ \mu$K), but random-anisotropy magnets are promising candidates for experiments.

We also use the analogy to calculate line shapes expected for neutron scattering experiments when localization effects are important. The scattering cross section for neutrons polarized perpendicular to the magnetization depends on the spin-wave dispersion $\omega(\mathbf{k})$ and the damping term $\tau^{-1}(\omega)$, while longitudinally polarized neutrons can be used to measure the spin-wave diffusion constant $D(\omega)$. Our main result is that localization results in power-law divergences in q and ω of the longitudinal structure factor $S_{zz}(\mathbf{q},\omega)$, with the power laws describing the divergences determined by the correlation length exponent ν for Anderson localization. The transverse structure factor $S_{xx}(\mathbf{q},\omega)$ remains basically unaffected.

Observation of localization using neutron scattering is advantageous for several reasons. First, both the one-particle and two-particle correlation functions are directly accessible experimentally, so one can check the markedly different theoretical predictions for their behaviors. Second, spin-wave interactions are extremely weak at low temperatures:[5] unlike electronic systems, there are no Coulomb interactions, and complications arising from inelastic scattering can be made very small by lowering the temperature.

Within the harmonic approximation used here, spin waves are scattered elastically by impurities. The validity of this approximation is the chief limitation on the approach presented. Anharmonicity leads to the appearance of a magnon inelastic mean-free-path λ_{in} and lifetime τ_{in} which diverge rapidly as $T \to 0$. Also, neither magnetization nor energy are strictly conserved, but rather they decay with associated relaxation times T_2 and T_1, respectively. Our results only hold for length scales less than λ_{in} and for frequencies greater than T_2^{-1}, T_1^{-1}, and τ_{in}^{-1}.

Our starting Hamiltonian is

$$H = -\sum_{\mathbf{R},\boldsymbol{\delta}} J(\mathbf{R},\mathbf{R}+\boldsymbol{\delta})\mathbf{S}(\mathbf{R})\cdot\mathbf{S}(\mathbf{R}+\boldsymbol{\delta}) - g\sum_{\mathbf{R}}\mu(\mathbf{R})S_z(\mathbf{R})H - \tfrac{1}{2}\sum_{\mathbf{R},\boldsymbol{\delta}} D(\mathbf{R})\{S_z^2(\mathbf{R}) - \tfrac{1}{2}[S_x^2(\mathbf{R}) + S_y^2(\mathbf{R})]\}\ , \tag{1}$$

where $\mathbf{S}(\mathbf{R})$ is an operator of a Heisenberg spin whose magnitude $S_0(\mathbf{R})$ varies from site to site because of dilution or mixing. The index $\boldsymbol{\delta}$ runs over the z nearest neighbors of $\mathbf{R}$. We assume that the exchange coupling $J(\mathbf{R},\mathbf{R}+\boldsymbol{\delta})$, the moment $\mu(\mathbf{R})$, the uniaxial anisotropy $D(\mathbf{R})$, and the spin

magnitude $S_0(\mathbf{R})$ carry random components

$$J(\mathbf{R},\mathbf{R}+\boldsymbol{\delta})=J_0+\delta J(\mathbf{R},\mathbf{R}+\boldsymbol{\delta}),\quad \mu(\mathbf{R})=\mu_0+\delta\mu(\mathbf{R})$$
$$D(\mathbf{R})=D_0+\delta D(\mathbf{R}),\quad S_0(\mathbf{R})=S_{00}+\delta S(\mathbf{R}).$$

The variables δJ, $\delta\mu$, δD, and δS are assumed to be random, uncorrelated, and Gaussian with small variances. We exclude negative values of J, S, and D so the ground state is uniformly ferromagnetically ordered along the z axis at zero temperature, as is appropriate for a mixture of Ising and Heisenberg spins.

For the Hamiltonian (1), the order parameter remains a constant of the motion, as for the pure case. If we look for small fluctuations about the ground state, then in the harmonic approximation the equation of motion for helical polarized spin waves described by $S^{\dagger}(\mathbf{R})=S_x(\mathbf{R})+iS_y(\mathbf{R})$ is, in the Heisenberg representation,

$$-i\hbar\frac{\partial}{\partial t}S^{\dagger}(\mathbf{R})=\sum_{\boldsymbol{\delta}}J(\mathbf{R},\mathbf{R}+\boldsymbol{\delta})[S^{\dagger}(\mathbf{R})S_0(\mathbf{R}+\boldsymbol{\delta})-S^{\dagger}(\mathbf{R}+\boldsymbol{\delta})S_0(\mathbf{R})]+g\mu(\mathbf{R})HS^{\dagger}(\mathbf{R})+\tfrac{3}{2}D(\mathbf{R})S_0(\mathbf{R})S^{\dagger}(\mathbf{R})\ . \tag{2}$$

This equation of motion describes the motion of the spin both quantum mechanically and classically.

We allow $S_0(\mathbf{R})$ to vary over short length scales, but since we are only interested in low-energy excitations, we assume that the angles describing the deviations of the magnetization from the z axis are small and slowly varying. Within the harmonic approximation, $\psi(\mathbf{R})/[2S_0(\mathbf{R})]^{1/2}$ is a field operator obeying Bose commutation relations, as can be verified using the Holstein-Primakoff transformation. Thus we write $S^{\dagger}(\mathbf{R})=S_0(\mathbf{R})\psi(\mathbf{R})$, where $\psi(\mathbf{R})$ is a slowly varying function of $\mathbf{R}$. We find a Schrödinger-type equation for ψ:

$$-i\hbar\frac{\partial\psi}{\partial t}=-\sum_{i,j=1}^{3}\left[\frac{\hbar^2}{2m}\right]_{ij}\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\psi - \mathbf{A}(\mathbf{r})\cdot\nabla\psi(\mathbf{r})+U(\mathbf{r})\psi(\mathbf{r})\ , \tag{3}$$

where $U(\mathbf{r})=U_0+\delta U(\mathbf{r})$ and $U_0=g\mu_0H+3D_0S_{00}/2$. The quantities $(\hbar^2/2m)_{ij}$, $\mathbf{A}$, and δU are continuum limits of

$$\left[\frac{\hbar^2}{2m}\right]_{ij}=\tfrac{1}{2}\sum_{\boldsymbol{\delta}}J(\mathbf{R},\mathbf{R}+\boldsymbol{\delta})S_0(\mathbf{R}+\boldsymbol{\delta})\delta_i\delta_j\ , \tag{4a}$$

$$\mathbf{A}(\mathbf{R})=\sum_{\boldsymbol{\delta}}J(\mathbf{R},\mathbf{R}+\boldsymbol{\delta})S_0(\mathbf{R}+\boldsymbol{\delta})\boldsymbol{\delta}\ , \tag{4b}$$

$$\delta U(\mathbf{R})=g\delta\mu(\mathbf{R})H+\tfrac{3}{2}\delta D(\mathbf{R})S_0(\mathbf{R})\ . \tag{4c}$$

Equation (3) is the Schrödinger equation for a particle with mass of order $(\hbar^2/S_0Ja_0^2z)$ moving in a random vector potential and a random scalar potential, where a_0 is the lattice constant and z the number of nearest neighbors, i.e., the Anderson model for electron localization.[6] The U_0 term ensures that all the excitations have positive energy so that the uniform state is stable. This correspondence holds only for the Heisenberg ferromagnet, which has a quadratic dispersion: $\omega\propto k^2$. Spin waves of antiferromagnets obey linear dispersion laws, so they cannot be mapped simply onto the Anderson Hamiltonian. The random scalar potential term consists of two parts. The first is proportional to the product of the magnetic field and the random moment, and the second is proportional to the magnitude of the random anisotropy. This result can be understood in terms of broken rotational symmetry of the system. If we only include the first term in Eq. (1) (e.g., a diluted magnet) the system still possesses this symmetry, so it has an extended mode at zero frequency. The anisotropy and magnetic field break rotational symmetry (so Goldstone's theorem no longer applies), and since the moments precess at different frequencies, there is no longer a simply constructed uniform spin-wave mode.

It has been shown for the Anderson model that the lowest-energy eigenstates are localized for energies less than a critical value E_0, the mobility edge. This result has implications for the response functions of the Anderson model, which we now translate into magnetic language to compute the neutron-scattering cross section.

We must compute the dynamic structure factor $S_{ii}(q,\omega)$, defined as

$$S_{ii}(\mathbf{q},\omega)=\int\frac{d^3r}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\langle\langle M_i(\mathbf{r},t)M_i(0,0)\rangle\rangle_{\text{imp}}\ , \tag{5}$$

where ii stands for zz (longitudinal) or xx (transverse), $\langle\ \rangle$ implies a thermal average, and $\langle\ \rangle_{\text{imp}}$ denotes a quenched average. Using a definition of ψ and the harmonic approximation for S_z, one finds

$$S_{xx}(\mathbf{q},\omega)=\int\frac{d^3r}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}g^2\langle\mu(\mathbf{r})\mu(0)S_0(\mathbf{r})S_0(0)\langle\psi^{\dagger}(\mathbf{r},t)\psi(0,0)\rangle\rangle_{\text{imp}}+\text{c.c.}\ , \tag{6a}$$

$$S_{zz}(\mathbf{q},\omega)=\delta(\omega)\int\frac{d^3r}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{r}}\langle\langle M_z(\mathbf{r})\rangle\langle M_z(0)\rangle\rangle_{\text{imp}} + \tfrac{1}{4}\int\frac{d^3r}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)}g^2\langle\mu(\mathbf{r})\mu(0)S_0(\mathbf{r})S_0(0)\langle|\psi(\mathbf{r},t)|^2|\psi(0,0)|^2\rangle_c\rangle_{\text{imp}}\ . \tag{6b}$$

The subscript c in (6b) stands for cumulant. The transverse structure factor corresponds to a *single-particle* correlation function. The longitudinal structure factor is comprised of two terms. The first term is the elastic Bragg reflection, and the second inelastic term S_{zz}^{I} is a measure of how the probability density $|\psi|^2$ of a particle obeying the Schrödinger equation spreads in time. It corresponds to the quenched-averaged density-density correlation function, a *two-particle* Green's function. It is analogous to the imaginary part of the dielectric function for electrons. In the harmonic approximation

$$S_{zz}^{I}(\mathbf{q},\omega)=\int_0^{\infty}dE\,N(E)n(E)[1+n(E)]S_E(\mathbf{q},\omega)\ , \tag{7}$$

with $N(E)$ the magnon density of states, $n(E)$ the Bose occupation factor, and S_E is

$$S_E(\mathbf{q},\omega)=\tfrac{1}{4}g^2\mu_0^2S_{00}^2N(E)\left\langle\left|\int d^3r\,\phi_n^*(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\phi_m(\mathbf{r})\right|^2\right\rangle_{\text{imp}} , \tag{8}$$

where ϕ_m and ϕ_n are eigenstates of Eq. (3) with eigenvalues $E_m=E$ and $E_n=E-\hbar\omega$, respectively. S_E can be interpreted as the dynamical structure factor[7] of particles whose wave function obeys Eq. (3) and whose energy is of order E (see Ref. 8).

The most relevant random contribution to Eq. (3) is the random scalar potential. It was shown by Wegner[9] for that case that the averaged *single-particle* Green's function is analytic in E and can be computed in mean-field theory, but that the averaged *density-density correlation function* is nonanalytic in E and mean-field theory breaks down.

Starting with S_{xx}, we find to lowest order in perturbation theory,

$$S_{xx}(\mathbf{q},\omega)=\tfrac{1}{2}g^2\mu_0^2S_{00}\left[\frac{n_q\tau^{-1}(\omega)}{(\omega+\omega_q)^2+\tau^{-2}(\omega)}+\frac{(1+n_q)\tau^{-1}(\omega)}{(\omega-\omega_q)^2+\tau^{-2}(\omega)}\right] , \tag{9}$$

where $\hbar^2/2m$ is $JS_{00}a_0^2z/2$, ω_q is the magnon dispersion of a pure anisotropic ferromagnet, and $\tau^{-1}(\omega)$ is the elastic scattering time:

$$\hbar\tau^{-1}(E/\hbar)\cong\pi N(E)\langle\delta U^2\rangle_{\text{imp}}=\frac{1}{\pi\sqrt{2}}\left[\frac{1}{J_0S_{00}a_0^2z}\right]^{3/2}(E-g\mu_0H-\tfrac{3}{2}D_0S_{00})^{1/2}(g^2H^2\langle\delta\mu^2\rangle_{\text{imp}}+\tfrac{9}{4}S_0^2\langle\delta D^2\rangle_{\text{imp}}) , \tag{10}$$

which is proportional to k. The contributions to τ^{-1} from the randomness in J and S_0 are of higher order in k. Notice that τ diverges as the wave vector goes to zero. The randomness has not changed the dispersion of the low-frequency modes, only added a damping term. Turning next to the longitudinal structure factor, we find, again within lowest-order perturbation theory, a diffusive central peak[10]

$$S_E(\mathbf{q},\omega)\cong\frac{1}{\pi}\frac{D_E(\omega)q^2}{\omega^2+D_E^2(\omega)q^4} , \tag{11}$$

with $D_E(\omega)$ a diffusion constant[11]

$$D_E(\omega)=v^2(E)\tau(E)\times\left[1+A\frac{\hbar}{mv^2(E)\tau(E)}\left[\frac{\omega\lambda_E^2}{v^2(E)\tau(E)}\right]^{1/2}+\cdots\right] , \tag{12}$$

with A a constant and with $v(E)=[2(E-g\mu_0H-3D_0S_{00}/2)/m]^{1/2}$, the velocity of a particle with wavelength $\lambda_E=h/mv(E)$. The diffusion constant depends both on the average energy E of the wave packet and the absorbed energy $\hbar\omega$.

The first term in Eq. (12) is the classical Einstein relation. The correction term is the first term in a perturbation series in $\hbar/[mv^2(E)\tau(E)]$. As E approaches $g\mu_0H+3D_{00}S_0/2$, the series starts to diverge and perturbation theory fails. In the analogous electron problem this signals the onset of localization: There is a critical energy E_0 below which the diffusion constant vanishes. Since to lowest order in perturbation theory neither the randomness in the mass nor the random vector potential in Eq. (3) enters for small q, we will ignore these terms so we can use the results of electron localization theory to include the effect of the remaining random scalar potential. The mobility threshold E_0 can then be determined using the Ioffe-Regel rule $\tau(E_0)v(E_0)\cong\lambda(E_0)$ (equivalent to $\hbar/mv^2\tau\cong1$) or

$$E_0-g\mu_0H-\tfrac{3}{2}D_0S_{00}\sim\frac{2}{(zJ_0S_{00}a_0^2)^3}\langle|\delta U|^2\rangle_{\text{imp}}^2 . \tag{13}$$

The corrections to τ^{-1} from randomness in J and S_0 do not give rise to localization of low-energy spin waves in three dimensions. As E approaches E_0, the diffusion constant vanishes as $D_E\sim(E-E_0)^\nu$ with ν the correlation-length exponent for Anderson localization. As D_E vanishes, the correlation length ξ_E diverges as $\xi_E\sim1/(E-E_0)^\nu$. For either $q>>\xi_E^{-1}$ or $\omega>>D_E\xi_E^{-2}$, Eq. (11) no longer applies, and the structure factor assumes a form characteristic of the *critical* point $E=E_0$ (Ref. 8)

$$S_E^{\text{crit}}(\mathbf{q},\omega)\approx\frac{A_1q^2}{A_2q^5+\omega^{5/3}} , \tag{14}$$

with the A_i constants. For $E<E_0$ Eq. (14) remains valid as long as $q>>\xi_E^{-1}$, but for smaller E one leaves the critical regime and enters the *localized* regime ($q<<\xi_E^{-1}$) where[8]

$$S_E^{\text{loc}}(\mathbf{q},\omega)=A_3\delta(\omega)\exp(-q^2\xi_E^2)+\frac{A_4q^2}{A_5\xi_E^{-5}+\omega^{5/3}} . \tag{15}$$

The δ function reflects that, in the localized regime, a wave packet with energy E less than E_0 remains within a distance ξ_E of its starting position. The longitudinal structure factor is an integral [Eq. (7)] over these three regimes.

We only discuss the most singular behavior of S_{zz} for small q and ω. The critical, localized, and extended regimes, respectively, contribute

$$S_{zz}^{\text{crit}}(\mathbf{q},\omega)\sim N(E_0)n(E_0)[1+n(E_0)]\times\begin{cases}q^2\omega^{-(5-1/\nu)/3}, & q^3<<\omega/D_0a_0,\\ q^{-(3-1/\nu)}, & q^3>>\omega/D_0a_0,\end{cases} \tag{16a}$$

$$S_{zz}^{\text{loc}}(\mathbf{q},\omega)\sim N(E_0)n(E_0)[1+n(E_0)]\delta(\omega), \tag{16b}$$

$$S_{zz}^{\text{ext}}(\mathbf{q},\omega)\sim N(E_0)n(E_0)[1+n(E_0)]\times\begin{cases}q^2/\omega^2, & q^2<<\omega/D_0,\\ q^{-(3-1/\nu)}, & q^3>>\omega/D_0a_0,\\ q^{-2/\nu}\omega^{-(1-1/\nu)}, & \left[\frac{\omega}{D_0}\right]^{3/2}<<q^3<<\frac{\omega}{D_0a_0},\end{cases} \tag{16c}$$

with D_0 and a_0 the microscopic diffusion and lattice constants. For $\omega=0$, $S_{zz}(\mathbf{q},0)$ has acquired a power-law divergence for small q with an exponent $3-1/\nu$, while for $q=0$, $S_{zz}(0,\omega)$ is dominated for small ω by the delta-function peak. Equations (16) are limiting forms: At $\omega=0$ the extended regime still contributes the usual $1/q^2$ divergence which is less singular only if $\nu>1$.

The singular behavior of S_{zz} will be broadened by inelastic magnon-magnon collisions, which induce a finite temperature-dependent lifetime $\tau_{\rm in}$. The resulting line shape is complex but the main features are as follows: For $q=0$, the delta-function singularity is broadened to a Lorenzian $\tau_{\rm in}^{-1}/(\omega^2+\tau_{\rm in}^{-2})$, while for $\omega=0$ the power-law divergence becomes weaker $(1/q^{2/\nu})$ for $(D_0\tau_{\rm in})^{-1/2}<q<(D_0\tau_{\rm in}a_0)^{-1/3}$ and then saturates for $q<(D_0\tau_{\rm in})^{-1/2}$. As for electron localization, we estimate $\tau_{\rm in}$ using its value for the nonrandom system. The lifetime of $q=0$ magnons in pure ferromagnets[12] is $\tau_{\rm in}\sim T^{-2}\exp(\epsilon_0/k_BT)$ for $k_BT\ll\epsilon_0$ and $\tau_{\rm in}\sim T^{-2}$ for $k_BT\gg\epsilon_0$, where ϵ_0 is the gap in the spin-wave dispersion. Measuring the temperature dependence of the peak height and linewidth of the localized central peak as $T\to 0$ may be the most promising test of the theory, just as the temperature dependence of the dc conductivity is used to test electron localization theory.

We now comment on limitations and complications involving the application of our results to experiment. First, the longitudinal structure factor of a *pure* Heisenberg ferromagnet is expected to display a power-law divergence,[13] though it has proven very difficult to observe experimentally (perhaps because a small anisotropy energy is present).[14] This divergence in S_{zz} is not expected to occur in the presence of a field or anisotropy, unlike the divergences expected from localization effects. Second, we have only considered on-site random anisotropy, whereas the experimentally relevant case involves anisotropic exchange. This complicates the theory because the total magnetization no longer is a conserved variable and probability is not conserved in the corresponding Schrödinger equation. However, we estimate the rate at which a particle decays to be proportional to the cube of its wave vector k, so that it is much slower than the elastic scattering rate in the region of interest. Finally, localization effects are pronounced only if there are sufficiently many thermally excited localized modes and if $\tau_{\rm in}$ is long. We estimate from Eq. (13) that random moments in a field lead to a mobility edge only of order 1 μK, but the mobility edge for random-anisotropy magnets ($\sim\langle\delta D^2\rangle/J_0^3$) could be of order T_c satisfying the first condition. However, in that case ϵ_0 is also of that order, so to achieve a long $\tau_{\rm in}$ one must have $k_BT\ll\epsilon_0$. Since S_{zz} depends on T as $\exp(-\epsilon_0/k_BT)$ [see Eq. (7)], achieving sufficient signal may be the major experimental problem.

In summary, we have found that hydrodynamics breaks down in ferromagnets with random anisotropy (or with random moments in a uniform field) because of the appearance of Anderson localization. If one compares spin-wave localization with electron localization, then one expects the effects of interactions to be less severe for the former. For electrons, Coulomb interactions cause a gap in the density of states at the Fermi surface (below the mobility edge) which is believed to change the critical properties of the metal-insulator transition. Therefore, the dynamics of random anisotropy ferromagnets may offer a better opportunity to measure ν.

We would like to thank G. Aeppli, B. Halperin, P. Hohenberg, J. Igirashi, and A. Pruisken for useful conversations. One of us (S.C.) acknowledges the support of the U. S. Department of Energy Grant No. DE-AC02-76CH00016.

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