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Ising model on a quasiperiodic chain

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The thermodynamic properties of a spin system with two different nearest-neighbor interactions which are ordered in a quasiperiodic pattern along a one-dimensional chain are studied. An exact renormalization technique is used which mimics the deflation rule for a quasiperiodic lattice. The system has a phase transition at zero temperature with the usual scaling form of the thermodynamic functions. These functions have corrections to scaling which do not appear in an ordinary system, and there is a spatial dependence in the correlation function.

I. INTRODUCTION

Quasiperiodic lattices are defined[1] by their lack of periodicity under translation, the incommensurate length scales of their elementary units, their self-similarity under certain inflation and deflation rules, and their long-range orientational order (in two dimensions and higher). The physical properties of such systems whose elementary units represent spins, atoms, potentials, etc. are objects of active investigation.[2] Recent experiments[3] have demonstrated the existence of materials whose symmetries fall outside classical crystallographic theory but which are predicted by quasicrystal models.

A commonly studied one-dimensional quasicrystal is the Fibonacci tiling. This sequence is generated from two elementary incommensurate units, for example, L and S, which we will regard as bonds on a lattice as shown in Fig. 1. The tiling can be inflated or grown from a seed (e.g., an L) by the following inflation or production rule: Each existing S is replaced by L, while each existing L is replaced by LS. This substitution is iterated until the sequence has the desired length. For instance, a growth sequence over five steps would be

$$L \rightarrow LS \rightarrow LSL \rightarrow LSLLS \rightarrow LSLLSLSL \ .$$

This lattice after n iterations has $N_L^{(n)}$ L bonds and $N_S^{(n)}$ S bonds. The inflation rule implies

$$N_L^{(n)} = N_L^{(n-1)} + N_S^{(n-1)} \ , \quad N_S^{(n)} = N_L^{(n-1)} \ . \tag{1.1}$$

Note that the ratio of the number of L bonds to S bonds is the nth rational approximant to the golden mean, $p_n \equiv F_n/F_{n-1}$ ($p_0 \equiv 0$), and F_n are the Fibonacci numbers ($F_{(0)}=0$, $F_1=1$; and $F_{(n)}=F_{(n-2)}+F_{(n-1)}$ for $n \geq 2$). p_n obeys the relation

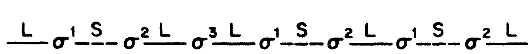

FIG. 1. The quasiperiodic Ising chain. The L interactions are denoted by a solid line. The S interactions are denoted by a broken line. The section in the figure corresponds to the fifth iteration of the inflation rule.

$$p_n = N_L^{(n)}/N_S^{(n)} = 1 + p_{(n-1)}^{-1} \ . \tag{1.2}$$

When $n \rightarrow \infty$, $p_n \rightarrow \tau$, where τ is the golden mean, $\tau = (1+\sqrt{5})/2$. The approach of p_n to τ is, however, oscillatory with

$$\delta p_n = -\frac{1}{\tau^2}\delta p_{n-1} \ , \quad \delta p_n \equiv p_n - \tau \ . \tag{1.3}$$

From (1.1) it follows that $\tau N_L^{(n)} + N_S^{(n)} \sim \tau^n$. We will see in what follows that these properties of the Fibonacci sequence are reflected in the physical properties of the quasiperiodic Ising chain.

The linear quasiperiodic Ising model is a chain of spins, σ_i^k. The spins take the values $\sigma_i^k = \pm 1$. The index $i=1,\ldots,N$ describes the location of a spin along the chain, and the index $k=1,2,3$ describes its local symmetry, i.e., the way it interacts with its neighbors. The reduced Hamiltonian (in units of minus the thermal energy) which describes the system is

$$\overline{H} = \sum_i (K_i \sigma_i \sigma_{i+1} + h^{(k)} \sigma_i^k) \ . \tag{1.4}$$

The Fibonacci tiling generates the types of spins and their interactions. There are two kinds of nearest-neighbor interactions, or bonds, between the spins, $K_i = L$ or $K_i = S$. The ordering of interactions along the chain is determined by the previous chain and the Fibonacci inflation rule: At each iteration stage n, any interaction L is replaced by the couple of interactions, LS (with a spin σ_i^1 between them), and each interaction S is replaced by an L interaction.

Three types of spins can be distinguished according to their two nearest-neighbor couplings (Fig. 1). We have defined σ^1 which interacts to the right by an S interaction. A σ^2 interacts by S and L interactions, but now the S interaction is on the left. A σ^3 interacts via two L interactions. The $h^{(k)}$ is the (reduced) field which couples to spin of type k, σ^k.

Successive application of the inflation rule results in a chain of L and S bonds in which the following is true.

(a) The S bond is always surrounded by two L's.

(b) There are no more than two successive L bonds.

Under the renormalization-group (RG) transformation, a system with N bonds at the nth stage of inflation is transformed into a similar system with N' bonds at the

$(n-1)$th stage of inflation. In the limit $n\to\infty$, $N/N'=\tau$. The RG transformation is a decimation transformation.[4] Each σ_i^1 is traced out of the normalized probability distribution, $P(\{\sigma\})=(1/Z)\exp(\overline{H})$, leaving the other spins untouched. This decimation transformation mimics deflation: L goes to S' and LS goes to L'. The transformation can be formally written as

$$P(\{\mu\})=\mathrm{tr}_\sigma[T(\sigma,\mu)P(\{\sigma\})] . \quad (1.5)$$

The RG operator T is

$$T(\sigma,\mu)=\prod_{i,j,l}\delta_{\mathrm{kr}}(\sigma_i^3,\mu_i^2)\delta_{\mathrm{kr}}(\sigma_j^{2(3)},\mu_j^1)\delta_{\mathrm{kr}}(\sigma_l^{2(1)},\mu_l^3) , \quad (1.6)$$

and δ_{kr} is the Kronecker δ function. The symbol $\sigma^{2(k)}$ is used to characterize σ^2 by its neighbor to the right, σ^k.

II. RG TRANSFORMATION AT ZERO MAGNETIC FIELD

At zero magnetic field the system can be described by a product of transfer matrices with factors

$$\exp(K_i\sigma_i\sigma_{i+1})=\cosh K_i[1+\sigma_i\sigma_{i+1}\tanh K_i] . \quad (2.1)$$

The decimation of the σ^1 spin between the L and the S transfer matrices leads to the renormalized L' transfer matrix,

$$\cosh L'[1+\mu_j\mu_{j+1}\tanh L'] . \quad (2.2)$$

L' is given by the recursion relation

$$Y_L'=Y_LY_S , \quad (2.3a)$$

where $Y_{K_i}\equiv\tanh K_i$. The second recursion relation is obtained by the substitution of an S' for the rest of the L bonds,

$$Y_S'=Y_L . \quad (2.3b)$$

Taking logarithms of (2.3) leads to recursion relations identical to those satisfied by N_{K_i} [Eq.(1.1)]. The solution of these recursion relations, after n renormalizations, is

$$Y_L^{(n)}=Y_L^{p_1p_2\cdots p_n}Y_S^{p_1p_2\cdots p_{n-1}} ,\qquad Y_S^{(n)}=Y_L^{(n-1)} . \quad (2.4)$$

The recursion relations (2.3) have two fixed points, the high-temperature fixed point,

$$Y_L^*=Y_S^*=0 \quad (2.5)$$

and the critical zero-temperature ($T_c=0$) fixed point,

$$Y_L^*=Y_S^*=1 . \quad (2.6)$$

The linear recursion relations, around the zero temperature fixed point, are

$$\begin{bmatrix}(\epsilon_L)'\\(\epsilon_S)'\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}\epsilon_L\\\epsilon_S\end{bmatrix} , \quad (2.7)$$

where $\epsilon_{K_i}\equiv1-Y_{K_i}$. The linearized RG transformation has the following eigenvalues and eigenvectors:

$$\tau , \begin{bmatrix}\tau\\1\end{bmatrix} \text{ and } -\frac{1}{\tau} , \begin{bmatrix}1\\-\tau\end{bmatrix} . \quad (2.8)$$

The system displays several interesting features. Only the first eigenvector is physically accessible to the initial choice of parameters, since the ϵ_{K_i} must be non-negative. Although the second eigenvector describes stable flow in parameter space toward the fixed point, initial conditions in its direction require a negative value of the ϵ_{K_i}.

The small eigenvalue is negative. This is not a common feature of periodic lattices. The negative sign is a consequence of the second recursion relation, (2.3b), which identifies the new value S' with the old value L. It causes oscillations in the ratio L/S around τ which decay rapidly as the parameters leave the unstable fixed point. We note that the recursion relation (2.7) and the inflation rule (1.1) are identical. The oscillations in the direction of the second eigenvector are the same as the oscillations of p_n around the limit τ as $n\to\infty$, (1.3). The value $-1/\tau^2$ in (1.3) is the ratio between the two eigenvalues in (2.8). The largest eigenvalue of (2.8) describes the growth of the Fibonacci numbers under the inflation rule (1.1).

In the $N\to\infty$ system the number of spins in the system is rescaled by a factor of $b=\tau$. This fact is combined with the usual relation [4,5] between the one positive eigenvalue and the rescaling factor to calculate the critical exponent ν which characterizes the dependence of the correlation length on the temperature.[6] From the linearized recursion relations we find

$$\frac{1}{\nu}=\frac{\ln\tau}{\ln b}=1 , \quad (2.9)$$

where we have identified $\tau\exp(-2L)+\exp(-2S)$ as the temperaturelike parameter[4] since the system exhibits an essential singularity at $T=0$ similar to that of the periodic Ising model. Another critical exponent, η, can be obtained immediately. Under the decimation transformation the spin's scale factor is 1. Hence, as in all other models with this scale factor,

$$D-2+\eta=0 , \quad (2.10)$$

where $D=1$ is the dimensionality of the system. This relation leads to the known one-dimensional result, namely, $\eta=1$.

The effect of quasiperiodicity on the thermodynamic properties of the system can be observed by studying the correlation function, $G(R,K_i)=\langle\sigma_x\sigma_{x+R}\rangle$. Without a magnetic field, the transfer matrices (2.1) commute. By grouping all the S transfer matrices and all the L ones, and by using the expression for the ordinary periodic chain, $\langle\sigma_i\sigma_{i+R}\rangle=\tanh^RK$, we obtain the exact expression for the quasiperiodic chain,

$$\langle\sigma_x\sigma_{x+R}\rangle=\tanh^{N_S}S\tanh^{N_L}L , \quad (2.11)$$

where N_S and N_L ($N_S+N_L=R$) are the number of S and L bonds in the segment between x and $x+R$, respectively. The asymptotic limits for N_S and N_L are

$$N_S\to\frac{1}{1+\tau}R , \; N_L\to\frac{\tau}{1+\tau}R \text{ as } R\to\infty . \quad (2.12)$$

Thus, close to T_c ($R\epsilon_{K_i} \ll 1$),

$$G(R\to\infty,S,L)=1-2\frac{R}{1+\tau}(\tau e^{-2L}+e^{-2S}) . \quad (2.13)$$

Identifying the correlation length, $\xi^{-1}\approx\tau\exp(-2L)+\exp(-2S)$ (the component of the temperaturelike field in the direction of the relevant eigenvector of the linearized recursion relations), we recover the scaling form of $G(R,L,S)=G[R/\xi(T)]$.

By taking the limit (2.12) we only examine the behavior of the system in the direction of the relevant eigenvector. Returning to the exact expression (2.11) and using (2.12), we can write it as

$$(Y_S^{1/\tau^2}Y_L^{1/\tau})^R\equiv\exp(R\xi_\infty^{-1}) , \quad \xi_\infty^{-1}=(1/\tau^2)\ln(Y_S Y_L^{\tau}) . \quad (2.14)$$

If the limit (2.12) is not taken, and if $R=F_{n+1}$, ξ becomes

$$\xi^{-1}=\ln(Y_S^{(p_n p_{n+1})^{-1}}Y_L^{(p_{n+1})^{-1}})$$
$$\sim \xi_\infty^{-1}+\delta p_{n+1}\ln(Y_S^{\tau^{-2}}/Y_L^{\tau^{-1}}) , \quad (2.15)$$

where δp_n oscillates according to (1.3).

The above argument is applicable if N_S and N_L are two successive numbers in the Fibonacci sequence, $F_{\tilde{n}-1}$ and $F_{\tilde{n}}$, respectively. However, the segment between 1 and R may not have been obtained by the inflation rule from a single ancestral S. In this case the ratio N_S/N_L is not defined by R alone, but depends also on the location of the segment along the chain. One can expect fluctuations in these numbers, of order $1/R$, which are large at short distances. Thus, even in the infinite system at the thermodynamic limit in the linear regime near T_c the lack of translational invariance is manifest.

This result can be compared with the one obtained by RG arguments. The value $\eta=1$ [Eq.(2.10)] implies

$$G(R^{(n)},S^{(n)},L^{(n)})=G(R,S,L) . \quad (2.16)$$

In the infinite chain the total number of bonds scales under the RG transformation as $N^{(n)}=N/\tau^n$. Inserting this scaling into (2.16) leads to (2.13). However, if R becomes less than infinity, the scaling becomes nonlinear and spatially dependent. If R is the $\tilde{n}$th number in the Fibonacci series, the scale factor is $p_{\tilde{n}}$, as opposed to τ. The correlation function calculated in the renormalized system is

$$G(R^{(n)},S^{(n)},L^{(n)})=\tanh^{N_S^{(n)}}S^{(n)}\tanh^{N_L^{(n)}}L^{(n)} . \quad (2.17)$$

Equation (2.17), using the above scaling of R and (2.4), is consistent with (2.16) and reveals the same oscillatory behavior which has been discussed previously. The space scale factor becomes a nonlinear function of R, and repeating the RG transformation n times does not lead to a power of the scale factor as in translationally invariant systems.

III. TRANSFORMATION OF THE MAGNETIC FIELD

The RG transformation creates different magnetic fields on different sites. Hence, although we start with a homogeneous magnetic field, three kind of fields should be studied. We denote these fields by $h_i^{(k)}$, $k=1,2,3$, where k is the index of the site type k in σ^k.

The contributions to the two untraced neighbor spins, $\tilde{h}_1$ and $\tilde{h}_3$, from a field $h^{(1)}$ on a spin $\sigma^{(1)}$ which is traced out in the RG transformation are obtained by the relation,

$$\mathrm{Tr}_{\sigma_2^1}[\exp(\sigma_1\sigma_2^1 L)\exp(\sigma_2^1 h^{(1)})\exp(\sigma_2^1\sigma_3 S)]$$
$$\equiv A\exp(\sigma_1\tilde{h}_1)\exp(\sigma_1\sigma_3 L')\exp(\sigma_3\tilde{h}_3) . \quad (3.1)$$

These contributions are added to the original fields,

$$(h^{(1)})'=\tilde{h}_3+h^{(2)} ,$$
$$(h^{(2)})'=\tilde{h}_1+h^{(3)} , \quad (3.2)$$
$$(h^{(3)})'=\tilde{h}_1+\tilde{h}_3+h^{(2)} ,$$

leading to exact recursion relations in the entire parameter space,

$$(x_1')^2=x_2^2\frac{y_S+x_1y_L}{x_1y_S+y_L}\frac{x_1+y_Sy_L}{1+x_1y_Sy_L} , \quad (3.3a)$$

$$(x_2')^2=x_3^2\frac{x_1y_S+y_L}{y_S+x_1y_L}\frac{x_1+y_Sy_L}{1+x_1y_Sy_L} , \quad (3.3b)$$

$$(x_3')=x_2\frac{x_1+y_Sy_L}{1+x_1y_Sy_L} , \quad (3.3c)$$

$$(y_L')^2=\frac{y_S+x_1y_L}{1+x_1y_Sy_L}\frac{x_1y_S+y_L}{x_1+y_Sy_L} , \quad (y_S)'=y_L \quad (3.3d)$$

where $\exp(-2K_i)=y_{K_i}$ and $x_i=\exp(-2h^{(i)})$. The recursion relation for the constant term Nc of the free energy[4] in terms of the variable $w\equiv\exp(-4c)$ is

$$w'=w^{p_{\tilde{n}+1}}\left[\frac{x_1y_Sy_L}{(1+x_1y_Sy_L)(x_1+y_Sy_L)(y_S+x_1y_L)(x_1y_S+y_L)}\right]^{1/p_{\tilde{n}}} . \quad (3.3e)$$

$\tilde{n}$ is defined by the number of bonds, N, before and after the RG transformation, $N/\tilde{N}=p_{\tilde{n}+1}$ which has the limit $p_{\tilde{n}+1}\to\tau$, $\tilde{n}\to\infty$.

The fixed points of the full nonlinear recursion relations consist of the points (2.5) and (2.6) and their generalization in the entire parameter space,

$$y_L=y_S=x_i=0 , \quad (3.4a)$$

$$y_L=y_S=0 , \quad x_i=1 , \quad (3.4b)$$

$$y_L=y_S=1 , \quad x_1=x_2=x_3 . \quad (3.4c)$$

These recursion relations can be linearized in the fields using

$$\tilde{h}_1=\tfrac{1}{2}h^{(1)}(Y_{L+S}+Y_{L-S}) ,$$
$$\tilde{h}_2=\tfrac{1}{2}h^{(1)}(Y_{L+S}-Y_{L-S}) , \tag{3.5}$$

and (3.2). Equation (3.5) can be further linearized in ϵ_i near the ferromagnetic fixed point (3.4a). By substituting back into (3.2), the following transformation is obtained:

$$\begin{bmatrix}(h^{(1)})'\\(h^{(2)})'\\(h^{(3)})'\end{bmatrix}=\begin{bmatrix}1-m & 1 & 0\\ m & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}h^{(1)}\\h^{(2)}\\h^{(3)}\end{bmatrix}, \tag{3.6}$$

where $m=1/(1+\epsilon_L/\epsilon_S)$. This recursion relation has the following eigenvalue and eigenvectors (left and right):

$$(\tau,\tau,1)\ \tau\ \begin{bmatrix}1\\ \tau-1+m\\ \dfrac{\tau+m}{\tau}\end{bmatrix},$$

$$(1-\tau^{-2},\tau^{-1},-1)\ -\frac{1}{\tau}\ \begin{bmatrix}-1\\ 1-m+\dfrac{1}{\tau}\\ \tau m-1\end{bmatrix}, \tag{3.7}$$

$$(m^2-1,-m,1)\ -m\ \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

Near the zero-temperature fixed point, all the spins align in the same direction. Hence the recursion relation (3.6) is just the transformation of points of different local symmetry under the Fibonacci inflation rule, which will now be presented. We denote the number of points with local symmetry as that of σ^k by $N_{(k)}$. Under the inflation rule (1.1), $N_{(k)}\to N'_{(k)}$. One can associate with each old point a new one, $(1)\to(2)'$, $(2)\to(3)'$, $(3)\to(2)'$. Although

$$\sum_l (N_{(l)})'=p_{n+2}\sum_k N_{(k)} , \tag{3.8}$$

where n is the iteration stage of the inflation rule, there is no one-to-one correspondence between $(\sigma^1)'$ and the spins at the previous stage. A part of the new points $(1)'$ can be associated with the old point (3). The other part of them can be associated with either (1) or (2). We can introduce a weight function c and describe the transformation $N\to N'$ as

$$\begin{bmatrix}(N_{(1)})'\\(N_{(2)})'\\(N_{(3)})'\end{bmatrix}=\begin{bmatrix}1-m & m & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}N_{(1)}\\N_{(2)}\\N_{(3)}\end{bmatrix}. \tag{3.9}$$

This transformation is the same as (3.6) in which the left and the right eigenvectors have been interchanged. This identification of (3.6) as the inflation rule for the points allows us to verify the known properties of the inflation rule: There is one eigenvalue τ which describes the growth of F_n. There is another eigenvalue, smaller by $-1/\tau^2$, which describes the fluctuations of p_n around τ. There is one more eigenvalue associated with the extra degree of freedom, m, which in the case of the quasiperiodic Ising chain is determined by the initial values of the coupling before the RG transformation.

There is a similarity between the flow trajectories in the quasiperiodic system and the periodic one.[4] In zero magnetic field the fixed points are $S = L$ (0 or ∞). Thus, all the lattice sites become indistinguishable, as in the periodic case. At the fixed line the values of the fields are those of a homogeneous system, again as in the periodic system. The fixed points (3.4a) and (3.4b) describe the ferromagnetic $h^{(i)}=0$, and the "frozen" $h^{(i)}=\infty$ points, respectively. The third fixed point (3.4c) is the paramagnetic critical line. In the quasiperiodic system the parameter space is composed of five components, a fact which makes the flow trajectories more complicated than in the periodic case. The lack of translational invariance means that the unstable direction of flow in the vicinity of the ferromagnetic point is not given by a homogeneous field, $h^{(1)}=h^{(2)}=h^{(3)}$. The initial homogeneous field flows toward the unstable direction accompanied by rapidly decaying fluctuations around this direction. This oscillation is caused by two irrelevant "staggered fields." The oscillations have the same origin and nature as those found in the flow away from the ferromagnetic point in the $S-L$ plane which were discussed in the previous section. The exact compositions of the critical and the "staggered" fields depend on the ratio ϵ_L/ϵ_S [through the factor m in (3.6)]. This ratio depends on the initial values of L and S, and it changes with the iterations of the RG transformation and causes a nonlinear effect even in the "linear" regime. However, this is a fast transient, and the ratio converges rapidly to F_n and then to τ. The set of eigenvalues and eigenvectors which correspond to $\epsilon_L/\epsilon_S=\tau$ are

$$(\tau,\tau,1)\ \tau\ \begin{bmatrix}1\\1\\2\tau^{-1}\end{bmatrix},$$

$$(\tau^{-1},\tau^{-1},-1)\ -\tau^{-1}\ \begin{bmatrix}1\\-2\tau^{-1}\\ \tau^{-2}\end{bmatrix}, \tag{3.10}$$

$$(\tau^{-4}-1,-\tau^{-2},1)\ -\tau^{-2}\ \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

At the ferromagnetic fixed point $m=1/2$. Hence there are two irrelevant "staggered" fields, $(1,-\sqrt{5}/2,1-\tau/2)$ and $(1,-1,0)$. Leaving the linear regime, the trajectories flow to the stable paramagnetic line where the magnetic field is uniform.

IV. CONCLUSION

In the absence of a magnetic field, the quasiperiodic one-dimensional Ising model is exactly solvable. The transfer matrices with different K_i interactions commute. The system can be mapped into two different periodic Ising chains whose properties are well known. The immediate conclusions are that the system has a zero-temperature critical point with discontinuous spontaneous magnetization which scales with the volume of the system.[7] This behavior, which is characterized by the critical exponents $\nu=\eta=1$, has also been found using an exact RG decimation transformation in the absence, as well as in the presence, of a magnetic field.

Any physical property which depends on a limited distance R, such as the two-point spin-spin correlation function, depends on the number of S and L bonds in this interval. The quasiperiodicity of the chain causes these numbers to vary. However, the average of these quantities over the infinite chain is unique. This fact, together with the knowledge of the full RG recursion relation, can be used to study the thermodynamics of the system. For instance, the magnetization has corrections to scaling with exponents equal to -1 and -2 resulting from the larger parameter space, which is a manifestation of the quasiperiodic nature of the system. The corrections to scaling have oscillating amplitudes. This oscillatory behavior is found to be an essential property of the quasicrystal inflation rule.

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