Hyperscaling, dimensional reduction, and the random-field Ising model

Ho-Fai Cheung
Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801
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We show that the hyperscaling law is modified whenever the relevant fixed point of the renormalization-group transformation is at zero temperature (infinite coupling). Instead of $2-\alpha=d\nu$, the hyperscaling law now reads $2-\alpha=d'\nu$ where d' is the anomalous dimension which is equal to $d-|y|$ and y is the exponent for the leading irrelevant operator that flows to zero temperature. We discuss the random-field Ising model in detail. We also perform a numerical domain-wall renormalization-group analysis for the $d=3$ model. For a Gaussian random-field distribution, our results are consistent with a second-order transition at low temperatures, and $|y|$ is found to be 1.3—1.7. For a $\pm h$ distribution, the transition appears to be first order at $T=0$.

The concept of dimensional reduction originates from the study of the random-field spin model. Written explicitly, the model of interest is a Ginzburg-Landau model with a random field,[1]

$$\mathscr{H}=\int\left[\frac{1}{2}(\nabla\phi)^2+\frac{r}{2}\phi^2+\frac{u}{4!}\phi^4+h(x)\phi(x)\right]dx \ . \qquad (1)$$

In a paper by Aharony, Imry, and Ma,[1] it was shown that, to all orders in perturbation theory, the exponents of the d-dimensional random-field model are equivalent to those of the $d-2$ dimensional pure model (no random field). In particular, the hyperscaling relation is modified from $2-\alpha=d\nu$ to $2-\alpha=(d-2)\nu$. In other words, the effective dimension is $d'=d-2$. Later, Parisi and Sourlas[2] showed that this reduction is due to a hidden supersymmetry. However, later studies suggest other variations. Schwartz[3] showed that $d'=d-2+\eta(d')$, which was originally suggested by Aharony *et al.* Shapir[4] suggested, for the short-time behavior,

$$d'=\begin{cases} d-1/\nu' & \text{for } \alpha'\geq 0 \ , \\ (d+1/\nu')/2 & \text{for } \alpha'\leq 0 \ , \end{cases} \qquad (2)$$

where $\nu'=\nu(d')$, $\alpha'=\alpha(d')$. His long-time results agree with those of Schwartz. Some other studies of the model can be found in Refs. 5—13 and experimental studies are in Refs. 14—17.

In the studies of the random-field model, the term "dimensional reduction" means that the critical exponents of the random-field model are equal to those of the pure model at a lower spatial dimension. Suppose that there is a dimensional reduction of y; then the d-dimensional random-field model has the same critical exponents as the $(d-y)$-dimensional pure model. We know that, for this pure model, the hyperscaling law is $2-\alpha=(d-y)\nu$. Therefore, the same relation must hold for the random-field model. This means that the hyperscaling law for the random-field model is modified. Dimensional reduction implies modification of the hyperscaling law. However, modification of the hyperscaling law does not necessarily imply dimensional reduction. In this paper, we discuss modification of the hyperscaling law in terms of the renormalization-group (RG) formalism. We found that the hyperscaling law is modified if the RG fixed point that controls the critical behavior is at zero temperature (infinite coupling). There is an anomalous dimension $d'=d-|y|$, where y is the exponent for the leading irrelevant operator that flows to zero temperature. The modified hyperscaling law now reads $2-\alpha=d'\nu$. We believe that this happens in a random-field model; however, our theory says nothing about dimensional reduction in this model. We use the random-field Ising model (RFIM) as an example for our discussion throughout. Then, we present a numerical RG calculation for the three-dimensional RFIM.

It was Fisher[10] who first suggested that modification of the hyperscaling law in RFIM is related to a zero-temperature fixed point. He traced the idea back to Grinstein.[5] However, the fixed point that Fisher referred to is from ϵ expansion of the continuous model [Eq. (1)]. This fixed point is the same one found by Aharony *et al.*[1] The calculation started with a double expansion in u and h_r. Feynman-diagram techniques were used. Thermal fluctuations is not important because the most divergent terms are represented by tree diagrams. In this sense, we are looking at zero-temperature properties. Aharony *et al.*[1] found a fixed point at $u=0$, $uh_r^2=0(\epsilon)$, and $r=0$. u is an irrelevant operator, with exponent $y=-2$ to all order in ϵ. The calculation showed a dimensional reduction by the amount $|y|$. The anomalous dimension is $d'=d-|y|$. Our approach differs from the above. It is not known whether the two approaches contain the same physics.

We start with the RFIM model on a lattice. The Hamiltonian for this model is

$$\mathscr{H}=\sum_{\langle i,j\rangle}JS_iS_j+\sum_i(H+H_i)S_i \ , \qquad (3)$$

where J is the nearest-neighbor coupling, H is a uniform magnetic field, and H_i is a random magnetic field distributed according to a Gaussian having zero mean and standard deviation H_r. We write

$$P(H_i)=\frac{1}{(2\pi)^{1/2}H_r}\exp(-H_i^2/2H_r^2)\ . \quad (4)$$

The coupling space of this model is (J,H,H_r). Now, consider some sort of RG that restricts the coupling space to the above three couplings. Examples of such an RG are the phenomenological RG of Nightingale,[18] or the domain-wall RG of McMillan.[19] The actual physics does not depend on the particular RG, but using this type of RG makes our discussion easier. Let T be the temperature. By a zero-temperature fixed point, we mean a fixed point at $T/J=0$, $T/H_r=0$, and $T/H=0$. We also need to specify the relative ratios of J, H_r, and H, since the properties of the system at zero temperature depend only on these ratios. The fixed point of interest is at $H_r/J=(H_r/J)_c$ and $H/J=0$. The relevant scaling parameters are

$$\hbar_R=H_r/J-(H_r/J)_c\ , \quad (5)$$

$$\hbar_H=H/J\ . \quad (6)$$

Here, $\hbar_R$ is the temperaturelike scaling parameter, while $\hbar_H$ is the fieldlike scaling parameter, because the critical region is $H/J\ll 1$ instead of $H\ll 1$. Linearizing around the fixed point, we define the following exponents,

$$\hbar'_R=b^{y_t}\hbar_R\ , \quad (7)$$

$$\hbar'_H=b^{y_h}\hbar_H\ , \quad (8)$$

$$J'=b^{y_j}J\ . \quad (9)$$

The prime on $\hbar_R$, $\hbar_H$, and J denotes the corresponding renormalized parameter and b is the scale factor. y_t and y_h are like the usual temperature and magnetic field exponents. y_j is the exponent for J. We could have used y_r, the exponent for H_r, but y_r is equal to y_j, since H_r/J is a constant at the fixed point. $y_j<0$ means that the flow is towards the weak-coupling regime, so that the zero-temperature fixed point controls the critical properties at zero temperature only. $y_j>0$ means that the flow is towards the strong-coupling regime, so that the zero-temperature fixed point controls the critical properties at sufficiently low temperatures.

Next, we extract the critical behavior at this zero-temperature fixed point. At zero magnetic field H, the flow away from the fixed point is controlled by y_t. Thus we can identify the behavior of the correlation length ξ,

$$\xi\sim(\hbar_R)^{-\nu}\ , \quad (10)$$

where $\nu=1/y_t$. Next, we find the singular part of the free energy. In the context[20] of the RG formalism,

$$f(J,\hbar_R,\hbar_H)=b^{-d}f(J_1,\hbar_{R_1},\hbar_{H_1})+R(J,\hbar_R,\hbar_H)\ , \quad (11)$$

where $R(J,\hbar_R,\hbar_H)$ is the additive contribution to the free energy which arises in the process of renormalization. J_n is the coupling renormalized n times. We can keep on renormalizing, so

$$f(J,\hbar_R,\hbar_H)=\sum_{n=0}^{\infty}b^{-nd}R(J_n,\hbar_{R_n},\hbar_{H_n})\ . \quad (12)$$

J_n is not a relevant scaling parameter; however, it is the only energy scale in the system, so it must be taken out in front for dimensional consistency. Thus,

$$f(J,\hbar_R,\hbar_H)=\sum_{n=0}^{\infty}b^{-nd}J_nR'(\hbar_{R_n},\hbar_{H_n}) \quad (13)$$

$$=J\sum_{n=0}^{\infty}b^{-nd}b^{ny_j}R'(b^{ny_t}\hbar_R,b^{ny_h}\hbar_H)\ . \quad (14)$$

Using the standard way[20] of extracting the singular part of the free energy, we find

$$f_s(\hbar_R,\hbar_H)=\hbar_R^{(d-y_j)\nu}F(\hbar_H/\hbar_R^{\nu y_h})\ . \quad (15)$$

We consider only $y_j>0$, so that this fixed point controls at least the low-temperature critical behavior (see Fig. 1). Suppose we keep the coupling J and the random field H_r fixed. We vary the temperature T and the magnetic field H. The scaling parameters would then be

$$t=(T-T_c)/T_c\ , \quad (16)$$

$$h=H/T_c\ , \quad (17)$$

where T_c is the critical temperature. If the critical behavior is controlled by the zero-temperature fixed point, then, because of the smoothness of the RG flow near the critical point, t and h will be mapped linearly into $\hbar_R$ and $\hbar_H$, respectively. Thus the singular part of the free energy at nonzero temperature is

$$f_s(t,h)=t^{(d-y_j)\nu}F(h/t^{\nu y_h})\ . \quad (18)$$

Comparing this with the standard form, we find the hyperscaling law,

$$2-\alpha=(d-y_j)\nu\ . \quad (19)$$

For the RFIM, $y_j=2$ corresponds to the result of Aharony *et al.*[1] and $y_j=2-\eta$ corresponds to the result of Schwartz.[3] In order for the $d=3$ RFIM to have $d=2$

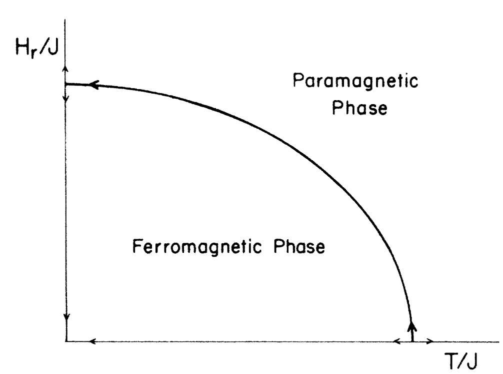

FIG. 1. Phase diagram of the $d=3$ random-field Ising model. H_r is the strength of the random field. J is the nearest-neighbor coupling. T is the temperature. Arrows show the renormalization flow as found by using the domain-wall renormalization group.

pure Ising exponents, as suggested by some experiments,[14] it would be necessary to have $y_j=1$, $y_t=1$, and $y_h=1.875$. In our analysis, y_j is an independent variable.

Our above discussion on modification of the hyperscaling law concentrates on the RFIM. The same argument holds for any other fixed point at zero temperature. Examples are the random-bond spin model[19,21] (a spin-glass model). The fact that the zero-temperature fixed point controls the critical behavior at finite temperature comes about because randomness is more effective than thermal fluctuations in destroying the long-range order.

In the following, we present numerical work on the RFIM. We have done a domain-wall renormalization-group (DWRG) calculation for this model. The DWRG developed by McMillan has been a very useful method in the study of spin-glass models. When applied to the RFIM described by Eq. (3), the DWRG transformation equations at zero temperature are

$$\overline{W}_L(J,H_r)=\overline{W}_{L'}(J',H_r'),\quad H=0 \tag{20}$$

$$W_{rL}(J,H_r)=W_{rL'}(J',H_r'),\quad H=0 \tag{21}$$

$$d\overline{F}_L(J,H_r,H)=d\overline{F}'_L(J',H_r',H'),\quad H/J\ll 1 \tag{22}$$

where $\overline{W}_L$ and W_{rL} are the mean and standard deviation of the domain-wall free energy of the RFIM on a lattice of size L. At $T=0$, the domain-wall free energy is the ground-state energy with antiperiodic boundary conditions minus the ground-state energy with periodic boundary conditions. $d\overline{F}_L$ in Eq. (22) is the mean of the absolute change in free energy when a small constant magnetic field is switched on. At $T=0$, dF_L is calculated from the ground-state magnetization M by

$$dF_L(J,H_r,H)=L^dM(J,H_r,0)H\ . \tag{23}$$

The RG transformation equations (20)—(22) determine the renormalized couplings J', H_r', and H' implicitly. This renormalization procedure is equivalent to a finite-size scaling analysis of the domain-wall free-energy distribution and the magnetization.

In order to carry out the RG transformation, we need the ground-state energies of the model with both periodic and antiperiodic boundary conditions, and also the ground-state magnetization. The phase transition occurs at $H=0$, so the fixed point is at $H/J=0$. We first used Eqs. (20) and (21) to find the fixed-point value of H_r/J and the exponents y_j and y_t; then, we used Eq. (22) to find the magnetic exponent y_h.

We employ the so-called Monte Carlo quench method to find the ground state. This method was developed by McMillan.[22] Basically, we perform a Monte Carlo simulation at finite temperature to generate a set of spin configurations. Starting from each configuration, we perform a zero-temperature quench. That is, we check to see if we can lower the energy by flipping one spin at a time. If the energy can be lowered, we flip that spin. This is repeated until we can no longer lower the energy. We call this a metastable state. We get one metastable state per quench. Different initial spin configurations may produce the same metastable state. We approximate the ground state as the lowest-energy metastable state. The method works if the number of quenches is large enough. This can be checked by looking at the mean of the lowest energy (averaged over many random-field configurations) as a function of the number of quenches. Using this method, we find the ground state of the model with different boundary conditions.

In the actual calculation, we used a simple-cubic lattice of size $L=2$, 3, 4, and 5. The numbers of configurations are 40 000, 10 000, 8000, and 3000, respectively. Figure 2 shows the plot of $W_r/\overline{W}$ against H_r/J with $H=0$. According to the RG equations (20) and (21), the intersection point is the fixed point. The error bars shown are one standard deviation. In analyzing the data, we must remember that there are errors due to the small lattices used. The greatest weight should be given to the result from the largest lattice, which is a $5\times5\times5$ lattice. However, we do not have as good statistics for the large lattice as for the small ones. We deduce from the graph that the fixed point is at $(H_r/J)_c=2.2$—2.5. Next, we determine the various exponents. In doing so, the most significant error comes from the uncertainty of the fixed point. We find $y_j=1.3$—1.7, $y_t=0.75$—1.05, and $y_j+y_h=2.90$—2.99. If we assume $(H_r/J)_c=2.2$, the best-fitted values are $y_j=1.7$, $y_t=0.80$, and $y_h=1.29$. These give $\alpha=0.38$, $\beta=0.01$, $\gamma=1.60$, $\eta=0.72$, and $\nu=1.25$. If we assume $(H_r/J)_c=2.5$, the best-fitted values are $y_j=1.4$, $y_t=0.95$, and $y_h=1.55$. These give $\alpha=0.32$, $\beta=0.05$, $\gamma=1.58$, $\eta=0.50$, and $\nu=1.05$. These values are somewhat different from those seen in experiments.[14-16] Experiments give $\alpha=0.0$, $\nu=1.0$, and no consistent value for the magnetic exponent. Note that the magnetic field ex-

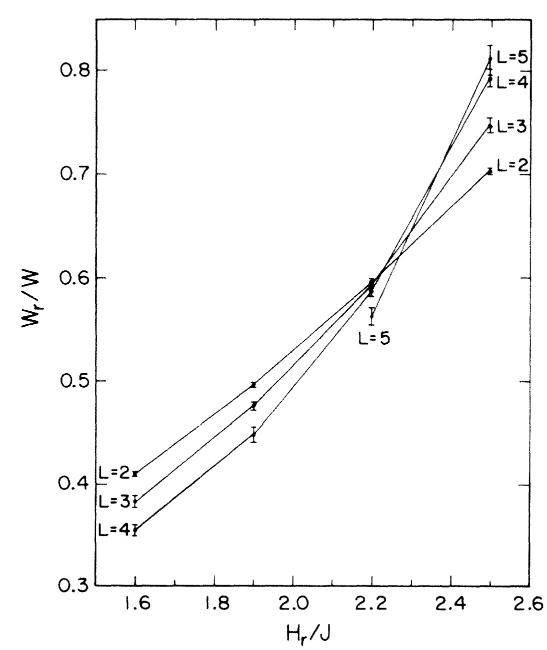

FIG. 2. Plot of $W_r/\overline{W}$ against H_r/J to find the critical point at $T=0$. $\overline{W}$ and W_r are the mean and standard deviation of the domain-wall free energy. Error bars represent one standard deviation.

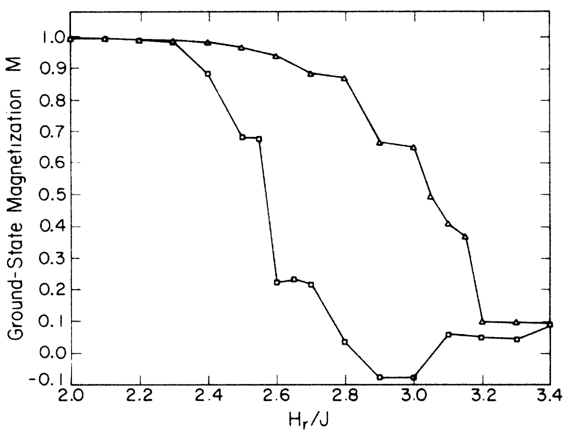

FIG. 3. Ground-state magnetization of the RFIM with Gaussian distribution against the random-field strength. Squares represent $L=20$. Triangles represent $L=10$. Each graph is for one particular random-field distribution. The lines are drawn to guide the eye.

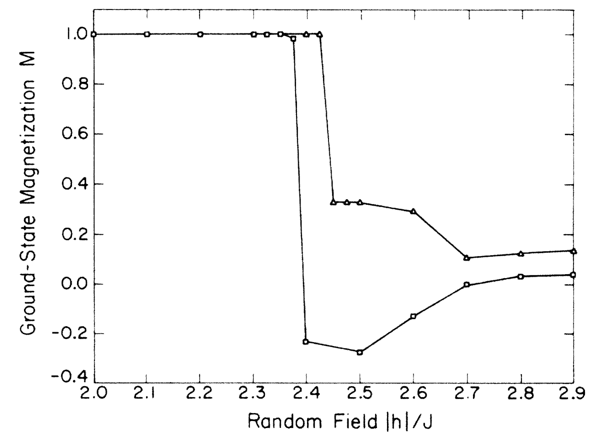

FIG. 4. Ground-state magnetization of the RFIM with $\pm h$ distribution against the random-field strength. Squares represent $L=20$. Triangles represent $L=10$. Each graph is for one particular random-field distribution. The lines are drawn to guide the eye.

ponent, which is equal to y_j+y_h in this case, is quite close to the spatial dimension, which is three. According to standard RG arguments,[23] this signals a discontinuity in the order parameter, i.e., a first-order transition. Other indicators come from η and β. According to an exact inequality of Schwartz and Soffer,[24] at a second-order transition η must be no less than 0.5. Our result is close to this limit. Also, β must be positive for a second-order transition. Our β is quite small. Thus our results certainly do not rule out the possibility of a first-order transition; however, if it is first order, it is probably weakly so.

For the Gaussian RFIM, there have been suggestions[12,13] that the transition is actually first order. However, we know that both mean-field theory[6] and the $d=2$ model[9] have a second-order transition. (In $d=2$, there is no stable ferromagnetic phase at nonzero random field, but the correlation length diverges at zero random field.) The simplest possibility would be that the transition is second order for all $d>2$. If it is a first-order transition at $d=3$, this should show up as a discontinuity in the slope of the ground-state energy and a discontinuity in the ground-state magnetization. We have checked the ground-state magnetization by using the maximum-flow[25] method to find the ground state. Using this method, the largest size we can look at is $20\times20\times20$. This method finds the ground state exactly for periodic boundary conditions, but does not work for the antiperiodic case. Results for the typical ground-state magnetization are shown in Fig. 3. As the random field increases, the ground-state magnetization drops to zero in a series of small jumps. It seems likely that there is no sharp discontinuity in the ground-state magnetization for an infinite system. (The erratic jumps will be averaged out to give a continuous magnetization curve.) Our graph also shows that the magnetization curves differ greatly between $L=20$ and $L=10$, indicating a large finite-size effect, which prevents a definitive conclusion. Our guess is that there is a second-order transition at zero temperature, although the result is also consistent with a very weak first-order transition. Because the RG flows towards this fixed point, the second-order transition should persist at low temperatures. The same calculation on the $\pm h$ model (the random field can only take the values $+h$ or $-h$) is shown in Fig. 4. Here, we do see a large jump in the ground-state magnetization as we increase the random field, suggesting a first-order transition at zero temperature. Incidentally, mean-field theory[6] for the $\pm h$ model also shows a first-order transition. The above results suggest that models with different random-field distributions may have transitions of different order.

In conclusion, we have deduced that the hyperscaling law is modified when the fixed point of the RG transformation is at infinite coupling. The flow of the parameters that set the energy scale is relevant to the critical behavior. Such modification can occur in systems with randomness, and is related to the fact that randomness is more important than thermal fluctuation in destroying the long-range order. For the $d=3$ random-field Ising model, our DWRG analysis suggests that the transition is second order.

After this study was completed, we received a communication from Bray and Moore[26] reporting a very similar theory of the modification of the hyperscaling law for the random-field Ising model.

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[1]A. Aharony, Y. Imry, and S.-k. Ma, Phys. Rev. Lett. **37**, 1364 (1976); A. P. Young, J. Phys. C **10**, L257 (1976).
[2]G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[3]M. Schwartz, J. Phys. C **18**, 135 (1985).
[4]Y. Shapir, J. Phys. C **17**, L809 (1985).
[5]G. Grinstein, Phys. Rev. Lett. **37**, 944 (1976).
[6]A. Aharony, Phys. Rev. B **18**, 3318 (1978).
[7]G. Grinstein and S.-k. Ma, Phys. Rev. B **28**, 2588 (1983).
[8]J. T. Chalker, J. Phys. C **16**, 6615 (1983).
[9]J. Fernandez and E. Pytte, Phys. Rev. B **31**, 2886 (1985).
[10]D. S. Fisher, Phys. Rev. B **31**, 7233 (1985).
[11]J. Villain (unpublished).
[12]A. P. Young and M. Nauenberg, Phys. Rev. Lett. **54**, 2429 (1985).
[13]A. Houghton, A. Khurana, and F. J. Seco, Phys. Rev. Lett. **55**, 856 (1985).
[14]R. J. Birgeneau, R. A. Cowley, G. Shirane, and H. Yoshizawa, J. Stat. Phys. **34**, 817 (1984).
[15]P.-Z. Wong, J. W. Cable, and P. Dimon, J. Appl. Phys. **55**, 2377 (1984).
[16]D. P. Belanger, A. R. King, and V. Jaccarino, J. Appl. Phys. **55**, 2383 (1984).
[17]S. M. Rezende, A. R. King, and V. Jaccarino, J. Appl. Phys. **55**, 2413 (1984).
[18]M. P. Nightingale, Phys. Lett. **59A**, 486 (1977); J. Appl. Phys. **53**, 7927 (1982).
[19]W. L. McMillan, Phys. Rev. B **29**, 4026 (1984).
[20]Th. Niemeijer and J. M. J. Van Leeuwen, in *Phase Transition and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 6.
[21]B. W. Southern and A. P. Young, J. Phys. C **10**, 2179 (1977); A. Benyoussef and N. Boccara, *ibid.* **16**, 1901 (1983).
[22]W. L. McMillan, Phys. Rev. B **30**, 476 (1984).
[23]B. Nienhuis and M. Nauenbery, Phys. Rev. Lett. **35**, 477 (1975).
[24]M. Schwartz and A. Soffer, Phys. Rev. Lett. **55**, 2499 (1985).
[25]J. C. Angles d'Auriac, M. Preissmann, and R. Rammal, J. Phys. (Paris) Lett. **46**, L173 (1985); F. Barahona, J. Phys. A **18**, L673 (1985).
[26]A. J. Bray and M. A. Moore, J. Phys. C **18**, L927 (1985).