Critical and tricritical microscopic exponents of a Fermi-Bose mixture

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Critical and tricritical "microscopic" exponents (η,ν,z) of a weakly interacting Fermi-Bose mixture are calculated in an approximation scheme. The scheme is based on an approximate form of the effective, low-momentum boson Hamiltonian derived in a previous paper and the Landau-theory equation of state. The approximate form is quadratic in boson operators with nonzero momenta. The scheme predicts its own range of validity. In the critical region of the condensed phase, the range is given by a condition reminiscent of the well-known Ginzburg-Levanyuk criterion. In the tricritical region, however, the scheme is exact. An attempt has been made to extract some information concerning the underlying dynamical structure which leads to the static critical and tricritical behavior observed. It is found that the spatial Fourier transform of the dynamic correlation function of order-parameter fluctuations is periodic in time with time period large compared to the inverse thermal frequency in the critical and tricritical regions. The Fourier transform in space and time conforms to the prediction of Halperin and Hohenberg. A dynamic-scaling assumption at a tricritical point is also verified. The value of the dynamic scaling exponent z, calculated with the aid of this assumption, agrees with that expected on the basis of conventional (Van Hove) prediction.

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I), a system of weakly interacting spinless bosons and spin-$\frac{1}{2}$ fermions was introduced as a model to examine the tricritical behavior of a ^{3}He-^{4}He mixture. The fermion amplitudes and the large-momentum boson amplitudes were eliminated from the problem, using diagrammatic perturbation theory, to obtain an effective, low-momentum boson Hamiltonian. It was shown in another paper[2] (hereafter referred to as II) that if in the effective Hamiltonian, following Bogoliubov,[3] the order parameter (boson operators in the zero-momentum state) is replaced by a c number M and order-parameter fluctuations (boson operators with nonzero momenta) are completely ignored, a Landau-type expansion[4] for an appropriate potential $\Omega'(M)=\Omega+hM$ is possible, where Ω denotes the thermodynamic potential per unit volume and h the field conjugate to the order parameter. The main aim of the present work is to calculate values of the critical and tricritical "microscopic" exponents, viz., η, ν, and z, for the system considered in I and II, in an approximation scheme which, in the effective Hamiltonian, takes into account the order-parameter fluctuations to lowest order. These exponents are microscopic in the sense that they belong to a class different from the class comprising of the exponents α, β, γ, etc. used for discussing macroscopic thermodynamics at criticality.

A move was initiated in I to understand how various scaling assumptions at a tricritical point[5,6] may emerge from a statistical-mechanical theory of tricritical behavior which starts with a microscopic, quantum-mechanical basis. The effort led to a self-consistent Hartree-Fock (HF) theory in II which starts with the quantum-mechanical description in I. It was found that the scaling form of thermodynamic potential per unit volume in HF theory conforms to the corresponding scaling assumption. This work aims at accomplishing the unfinished task, viz., to verify the scaling assumptions for static and dynamic correlation functions of order-parameter fluctuations, starting with the same quantum-mechanical description. It must be added that the main aim of the work, however, does not require a quantum-mechanical basis. Another reason for the choice of the quantum-mechanical description in I for this work is explained below.

The Landau expansion obtained in II could not lead to a Ginzburg-Levanyuk (GL) criterion[7] for its validity in the critical region, for it was obtained by ignoring order-parameter fluctuations completely. This author intends to derive a GL-type criterion here taking the fluctuations into account. As far as the author is concerned, the natural choice for a basis, to achieve this goal, is that in I.

In Sec. III of this paper, the GL-type criterion for the critical region is derived. In the tricritical region, however, the criterion reduces to a trivial one (see Sec. III). The latter explains why values of the tricritical exponents, for $d=3$, in the Riedel-Wegner theory,[8] are the same as the corresponding Landau-theory values. It may be mentioned that these authors,[9] too, had noted in one of their publications that the GL criterion for the molecular-field behavior holds, at all temperatures, only for higher-dimensional $(d>3)$ tricritical systems. But since an examination in sufficient detail was not carried out, probably because these authors were unable to exhibit a tricritical point as the terminus of a first order coexistence line,[10] the need is evident to justify this remark.

The fields (T,μ_3,μ_4) appear as the natural variables in the discussion to follow; μ_3 and μ_4, as in I and II, are the partial chemicals conjugate to the mean number densities of fermions and bosons, respectively, in Ω. The variables (T,μ_3) will be regarded as the independent ones, with μ_4 playing the role of a parameter.

It is interesting to examine underlying dynamical structure which leads to static critical and tricritical behavior of the system. In Sec. IV it is shown that the spatial Fourier transform $\mathscr{R}(q,t)$ of the dynamic correlation function of the order-parameter fluctuations is periodic in time. The time period is large compared to the inverse thermal frequency in the critical and tricritical regions. This indicates that, for dynamic critical phenomena, fields which depend on time small compared to the inverse thermal frequency are relatively unimportant (see Sec. V also). The Fourier transform in time of $\mathscr{R}(q,t)$ conforms to the prediction of Halperin and Hohenberg.[11]

The dynamic scaling exponent z has been calculated in Sec. IV with the help of a scaling form found to hold for the function $\mathscr{R}(q,t)$ at criticality. The value of z agrees with that expected on the basis of conventional prediction.[12] The value is also very close to that estimated from the ϵ expansion of previous workers.[13,14]

An outline of the content of this paper is as follows: The effective boson Hamiltonian is approximated by a quadratic form in Sec. I. An equation of state (i.e., relationship between h and order parameter M) is derived from this Hamiltonian. It is found that the first approximation to this equation, which is nothing but the equation of state in the Landau's theory, is adequate for the goals to be achieved in this paper. In Sec. III an expression for static correlation function of order-parameter fluctuations is derived using the quadratic Hamiltonian and the Landau-type equation of state. The exponents (η,ν) are calculated using this expression. The scaling assumption for the static correlation function is verified in Sec. III. The GL-type criteria for critical and tricritical regions are also derived. In Sec. IV the dynamic correlation function is calculated and a dynamic scaling assumption at a tricritical point (T_t,μ_{3t}) is verified. The dynamic scaling exponent z is also calculated. Section V contains a discussion related to this work. It is pointed out that a fluctuation-dissipation (FD) relation cannot be satisfied under the approximation of replacing boson operators in the zero-momentum state by a c number. In Appendix A, dependence of quantities, used in the derivation of the critical exponents, on the elementary fields $T-T_t$ and $\mu_3-\mu_{3t}$, is discussed. Appendix B contains steps leading to the FD relation alluded to above.

II. LANDAU'S THEORY AND ORDER-PARAMETER FLUCTUATIONS

The effective boson Hamiltonian (in units such that $\hbar=1$) of I is

$$H_e=c_0+\sum_q\left[\frac{q^2}{m_4}-\mu_4'\right]b_q^\dagger b_q+\frac{u_4'}{V}\sum_{q_1,\ldots,q_4}b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3}b_{q_4}\delta_{q_1+q_2,q_3+q_4}$$

$$+\frac{u_6}{V^2}\sum_{\substack{q_1,q_2,q_3\\ q_1',q_2',q_3'}}b_{q_1}^\dagger b_{q_1-q_1'}b_{q_2}^\dagger b_{q_2-q_2'}b_{q_3}^\dagger b_{q_3-q_3'}\delta_{q_1'+q_2',-q_3'}\,, \tag{1}$$

$$c_0=V[\tfrac{1}{2}u_3(n_3^F)^2+u_{34}n_3^Fn_4'+2u_4(n_4')^2+O(u_{34}^2)]\,, \tag{2}$$

$$\mu_4'=\mu_4-u_{34}n_3^F+O(u_{34}^2)\,, \tag{3}$$

$$u_4'=u_4-\tfrac{1}{2}u_{34}^2\frac{\partial n_3^F}{\partial\mu_3}\,, \tag{4}$$

$$u_6=\frac{u_{34}^3}{6}\frac{\partial^2 n_3^F}{\partial\mu_3^2}\,, \tag{5}$$

$$n_3^F=2V^{-1}\sum_k\left\{\exp\left[\beta\left[\frac{k^2}{m_3}-\mu_3\right]\right]+1\right\}^{-1}, \tag{6}$$

$$n_4'=V^{-1}\sum_{|p|>p_c}\left\{\exp\left[\beta\left[\frac{p^2}{m_4}-\mu_4\right]\right]-1\right\}^{-1}. \tag{7}$$

Here, u_3, u_4, and u_{34}, respectively, denote the fermion-fermion interaction strength, the boson-boson interaction strength, and the boson-fermion interaction strength. It has been assumed in I that $u_4=O(u_{34}^2(\partial n_3^F/\partial u_3))$. All q's in (1) are such that $0\le|q|<p_c$, where p_c is small compared to the boson thermal momentum

$$\lambda_B^{-1}(T)=(m_4/4\pi\beta)^{1/2}\,. \tag{8}$$

The expression for c_0, second-order contributions to μ_4', and third-order contributions to u_4' will not be needed for calculations to follow.

The thermodynamic potential per unit volume of the system can be written as

$$\Omega(T,\mu_3,\mu_4,h)=\Omega_1(T,\mu_3,\mu_4)+\Omega_2(T,\mu_3,\mu_4,h)\ , \tag{9}$$

where

$$\Omega_1=-(\beta V)^{-1}\ln\Xi_F\ , \tag{10}$$

$$\Omega_2=-(\beta V)^{-1}\ln \mathrm{Tr}\exp[-\beta(H_e+H_s)]\ , \tag{11}$$

$$\Xi_F=\prod_k\left\{1+\exp\left[-\beta\left[\frac{k^2}{m_3}-\mu_3\right]\right]\right\}\times\prod_{|p|>p_c}\left\{1-\exp\left[-\beta\left[\frac{p^2}{m_4}-\mu_4\right]\right]\right\}^{-1}\ , \tag{12}$$

$$H_s=-\frac{hV}{2}\left[\frac{b_0}{\sqrt{V}}+\frac{b_0^\dagger}{\sqrt{V}}\right]. \tag{13}$$

In (13), h denotes the field conjugate to the real part of the order parameter $b_0/\sqrt{V}$. Upon replacing $b_0/\sqrt{V}$ by a c number M, one gets

$$H_e+H_s=c_0'+h_1+h_2\ , \tag{14}$$

$$c_0'=c_0+V[(-\mu_4'+u_6f^2)M^2+(u_4'+3u_6f)M^4+u_6M^6-hM]\ , \tag{15}$$

$$h_1=\tfrac{1}{2}\sum_q{}' E(q)(b_q^\dagger b_q+b_{-q}^\dagger b_{-q})+\tfrac{1}{2}UM^2\sum_q{}'(b_q^\dagger b_{-q}^\dagger+b_{-q}b_q)\ , \tag{16}$$

$$\begin{aligned}h_2=&\frac{(2u_4'+9u_6M^2)M}{V^{1/2}}\sum_{q_1,q_2}{}'(b_{q_1+q_2}^\dagger b_{q_1}b_{q_2}+b_{q_1}^\dagger b_{q_2}^\dagger b_{q_1+q_2})+\frac{(u_4'+9u_6M^2)}{V}\sum_{q_1,\dots,q_4}{}' b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3}b_{q_4}\delta_{q_1+q_2,q_3+q_4}\\&+\frac{3u_6M^2}{V}\sum_{q_1,q_2,q_3,q_1'}{}'(b_{q_1}^\dagger b_{q_1-q_1'}b_{q_2}^\dagger b_{q_3}^\dagger+b_{q_2}b_{q_3}b_{q_1-q_1'}^\dagger b_{q_1})\delta_{q_1'+q_2,-q_3}\\&+\frac{3u_6M}{V^{3/2}}\sum_{q_1,q_2,q_3}{}'(b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3}^\dagger b_{q_1+q_2}b_{q_3}+b_{q_1+q_2}^\dagger b_{q_3}^\dagger b_{q_1}b_{q_2}b_{q_3})\\&+\frac{u_6}{V^2}\sum_{\substack{q_1,q_2,q_3\\q_1',q_2',q_3'}}{}' b_{q_1}^\dagger b_{q_1-q_1'}b_{q_2}^\dagger b_{q_2-q_2'}b_{q_3}^\dagger b_{q_3-q_3'}\delta_{q_1'+q_2',-q_3'}\ ,\end{aligned} \tag{17}$$

$$E(q)=\frac{q^2}{m_4}-\mu_4'+4(u_4'+3u_6f)M^2+9u_6M^4\ , \tag{18}$$

$$U=2(u_4'+3u_6M^2)\ , \tag{19}$$

$$f=V^{-1}\sum_{|q|<p_c}1\ . \tag{20}$$

The primed q summations above exclude the point $q=0$. In c_0' as well as $E(q)$, $3u_6f$ appears as a correction of order u_{34}^3 to u_4', and will be omitted. Similarly, u_6f^2 appears as a small renormalization of μ_4 and will be omitted. The unknown M will be determined by the requirement that the thermodynamic potential per unit volume be minimum with respect to variations in M.

In a microscopic calculation of the exponents (η,ν,z), one needs explicit expressions for correlation functions of order-parameter fluctuations. For the present system, M is the order parameter and the boson operators b_q with $q\neq0$ are fluctuations in M. The simplest way to obtain expressions for correlation functions of the fluctuations corresponds to starting with a truncated Bogoliubov-type Hamiltonian where the fluctuations are retained up to second order. From (14) one finds that, for the system under consideration, the Bogoliubov-type Hamiltonian is

$$H_{\mathrm{B}}=c_0'+h_1\ . \tag{21}$$

In the following sections, various ensemble averages will be calculated with this quadratic Hamiltonian. The equation for M which will be used in calculating these averages will be derived now by demanding stationarity of Ω_{B} with respect to M, where

$$\Omega_{\mathrm{B}}=\Omega_1-(\beta V)^{-1}\ln\mathrm{Tr}\exp(-\beta H_{\mathrm{B}})\ . \tag{22}$$

The Hamiltonian H_{B} can be diagonalized by defining a new set of creation and destruction operators $(c_q,c_q^\dagger)$ with the help of the Bogoliubov transformation. Upon diagonalizing, one gets

$$H_{\mathrm{B}}=c_0'+\tfrac{1}{2}\sum_q{}'\{[E^2(q)-U^2M^4]^{1/2}-E(q)\}+\tfrac{1}{2}\sum_q{}'\{[E^2(q)-U^2M^4]^{1/2}\}(c_q^\dagger c_q+c_q^\dagger c_q)\ . \tag{23}$$

Consequently, the thermodynamic potential per unit volume

$$\Omega_B=\Omega_1+V^{-1}c_0'+(2V)^{-1}\sum_q{}'\{[E^2(q)-U^2M^4]^{1/2}-E(q)\}+(\beta V)^{-1}\sum_q{}'\ln(1-\exp\{-\beta[E^2(q)-U^2M^4]^{1/2}\})\ . \tag{24}$$

From (24) one gets the following equation for M:

$$h=2A_2M+4A_4M^3+6u_6M^5\ , \tag{25}$$

$$A_2=-\mu_4'+4u_4'g_1-2u_4'g_2\ , \tag{26}$$

$$A_4=u_4'+9u_6g_1-6u_6g_2\ , \tag{27}$$

$$g_1=(2V)^{-1}\sum_q{}'\left[\frac{E(q)}{[E^2(q)-U^2M^4]^{1/2}}\frac{\exp\{\beta[E^2(q)-U^2M^4]^{1/2}\}+1}{\exp\{\beta[E^2(q)-U^2M^4]^{1/2}\}-1}-1\right]\ , \tag{28}$$

$$g_2=\frac{UM^2}{2V}\sum_q{}'\left[\frac{1}{[E^2(q)-U^2M^4]^{1/2}}\frac{\exp\{\beta[E^2(q)-U^2M^4]^{1/2}\}+1}{\exp\{\beta[E^2(q)-U^2M^4]^{1/2}\}-1}\right]\ . \tag{29}$$

Equation (25) is not the same as that in Landau's theory, inasmuch as the coefficients A_2 and A_4 are functions of M. As explained below, these are, however, independent of M to the leading orders in the interaction strength u_{34}.

In A_4, the terms $9u_6g_1$ and $6u_6g_2$ are small corrections to u_4', the reason being u_4' is of order u_{34}^2, while these terms are of higher order. In A_2, $4u_4'g_1$ and $2u_4'g_2$ appear as small corrections to the first-order term of μ_4', viz., $u_{34}n_3^F$ [cf. Eq. (3)]. Thus Eq. (25) can be written as

$$h\simeq-2\mu_4'M+4u_4'M^3+6u_6M^5\ , \tag{30}$$

where μ_4' and u_4', respectively, are correct to $O(u_{34})$ and $O(u_{34}^2)$ only. Equation (30) is the Landau-type equation of state. The expression, for the thermodynamic potential per unit volume, consistent with (30), is

$$\Omega_L=\Omega_1+V^{-1}c_0+(-\mu_4')M^2+u_4'M^4+u_6M^6-hM\ . \tag{31}$$

This is of the form of the Landau expansion appropriate for discussing tricritical behavior. From (31) one finds that only those solutions of (30) are admissible which satisfy the condition

$$(-\mu_4'+6u_4'M^2+15u_6M^4)>0\ . \tag{32}$$

The conclusions drawn from the analysis of (30)–(32), for $h=0$, are stated below. These will be used for calculations in the remaining sections and Appendix A.

In the region $u_4'\geq 0$ in the T-μ_3 plane, those points correspond to the normal phase ($h=0$, $M=0$, $h/M\neq 0$) where $\mu_4'<0$; those where $\mu_4'>0$ correspond to the condensed phase ($h=0$, $M\neq 0$). The curve $\mu_4'=0$ is the critical line. The intersection (T_t,μ_{3t}) of the curves $\mu_4'=0$ and $u_4'=0$ is also a critical point. If the fermions are degenerate, the intersection exists, provided

$$\mu_4>\frac{64\pi^4}{3}\frac{u_4^3}{m_3^3u_{34}^5}\ . \tag{33}$$

In the limit of a nondegenerate Fermi gas as well, the intersection exists in the region $\mu_3<0$, provided

$$\mu_4>\frac{128\pi^3u_4^3}{m_3^3u_{34}^5}\ . \tag{34}$$

In both the cases, $u_6>0$ (cf. Sec. IV of I).

In the region $u_4'<0$, those points correspond to the condensed phase where $(u_4')^2+4\mu_4'\mu_6>0$; those where $(u_4')^2+4\mu_4'u_6<0$ correspond to the normal phase. The equation $(u_4')^2+4\mu_4'u_6=0$ corresponds to the coexistence curve (CXC), in the T-μ_3 plane, along which the normal phase ($M=0$) and the condensed phase ($M^2=-u_4'/2u_6$) coexist. In the region $u_4'<0$ there is no critical line. The coexisting phases become identical at the terminus of the CXC. The terminus, usually referred to as the tricritical point (TCP), is nothing but the intersection (T_t,μ_{3t}). Therefore, it is also the terminus of the critical line $\mu_4'=0$.

A possible form of the curves $\mu_4'=0$ and $u_4'=0$ in the T-μ_3 plane, for the case of degenerate fermions, is depicted in Fig. 1, and that for the case of nondegenerate fermions is depicted in Fig. 2. The positive and negative sides of the curves are indicated in each of these figures. The dashed lines correspond to the CXC. In these figures, the hatched regions correspond to the normal phase.

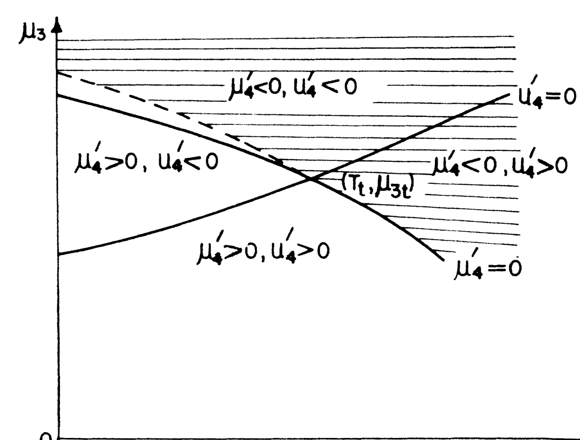

FIG. 1. Qualitative phase diagram in the limit of a degenerate Fermi gas. Normal phase is possible in the hatched region only. The dashed line corresponds to $(u_4')^2+4\mu_4'u_6=0$.

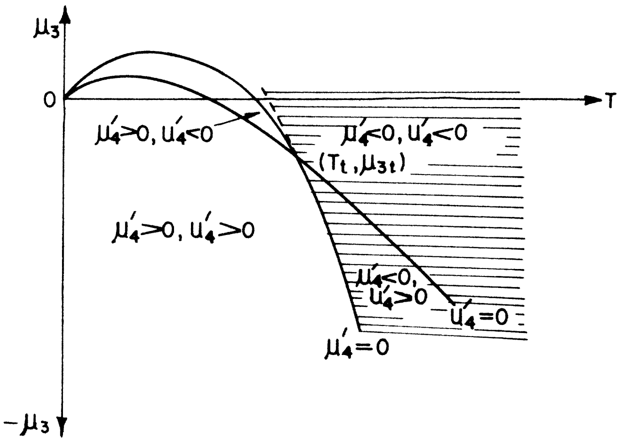

FIG. 2. Qualitative phase diagram in the limit of a nondegenerate Fermi gas. Hatched region corresponds to normal phase. The dashed line represents the coexistence curve.

Although these phase diagrams are different in several aspects, such as the segment $T<T_t$ of the line $\mu_3=\mu_{3t}$ lying in the region $\mu_4'>0, u_4'<0$ for the degenerate case, whereas this segment lies in the region $\mu_4'>0, u_4'>0$ for the nondegenerate case, values of critical and tricritical exponents are same for both cases. This conclusion upholds the universality hypothesis regarding tricritical exponents. As pointed out by Fisher and Sarbach,[15] tricritical behavior has certain nonuniversal aspects too.

III. STATIC CORRELATION OF ORDER-PARAMETER FLUCTUATIONS

In this section the exponents (η,ν) will be calculated. The scaling assumption[5,6] for a correlation function at a tricritical point will be verified. A GL-type criterion for the critical region will also be derived toward the end. The definitions of the tricritical exponents (η,ν) can be found in Griffith's paper.[6] The calculations require expression for the spatial Fourier transform $S(q)$ of the static correlation function of the order-parameter fluctuations. For a Fermi-Bose mixture these functions can be written in the following manner:

$$S(q;T,\mu_3,\mu_4,h)=V^{-1}\int d^3r\,d^3r'\,e^{iq\cdot(r-r')}\times\Gamma(r-r';T,\mu_3,\mu_4,h)\,, \tag{35}$$

where

$$\Gamma(r-r';T,\mu_3,\mu_4,h)=\langle \tfrac{1}{2}[\psi(r)+\psi^\dagger(r)]\tfrac{1}{2}[\psi(r')+\psi^\dagger(r')]\rangle - \langle \tfrac{1}{2}[\psi(r)+\psi^\dagger(r)]\rangle \times \langle \tfrac{1}{2}[\psi(r')+\psi^\dagger(r')]\rangle\,, \tag{36}$$

$$\psi(r)=V^{-1/2}\sum_q b_q e^{iq\cdot r}\,, \tag{37}$$

and the angular brackets, $\langle\cdots\rangle$, denote an ensemble calculated with H_e+H_s. In what follows, the average, however, will be calculated with $H_{\rm B}$.

From (35)–(37), following Bogoliubov,[3] one gets

$$S(q\neq 0;T,\mu_3,\mu_4,h)=\tfrac{1}{4}\langle b_q^\dagger b_q+b_{-q}b_{-q}^\dagger + b_{-q}b_q+b_q^\dagger b_{-q}^\dagger\rangle\,. \tag{38}$$

Introducing the following elementary Green's functions defined with $H_{\rm B}$, viz.,

$$\begin{aligned}
\mathcal{G}_1(q\tau,q'\tau')&=-\langle T_\tau[b_q(\tau)b_{q'}^\dagger(\tau')]\rangle\,,\\
\mathcal{G}_2(q\tau,q'\tau')&=-\langle T_\tau[b_q^\dagger(\tau)b_{q'}(\tau')]\rangle\,,\\
\mathcal{G}_3(q\tau,q'\tau')&=-\langle T_\tau[b_q(\tau)b_{q'}(\tau')]\rangle\,,\\
\mathcal{G}_4(q\tau,q'\tau')&=-\langle T_\tau[b_q^\dagger(\tau)b_{q'}^\dagger(\tau')]\rangle\,,
\end{aligned} \tag{39}$$

where

$$b_q(\tau)=\exp(H_{\rm B}\tau)b_q\exp(-H_{\rm B}\tau)\,, \tag{40}$$

one can write

$$S(q\neq 0)=-\tfrac{1}{4}[\mathcal{G}_1(q\tau,q\tau^+)+\mathcal{G}_2(-q\tau,-q\tau^+) + \mathcal{G}_3(q\tau,-q\tau^+)+\mathcal{G}_4(-q\tau,q\tau^+)]\,. \tag{41}$$

Here, $0<|q|<p_c$ $[p_c \ll \lambda_B^{-1}(T)]$. The symbol T_τ orders the operators according to their τ values with the smallest to the right. τ^+ denotes the limiting value $\tau+\eta$ as η approaches zero from positive values. These Green's functions can be calculated by solving equations of motion for them. The equations of motion for the operators $b_q(\tau)$, $b_{-q}^\dagger(\tau)$, etc. are required for this purpose. From (21) and (40) we get these equations. For example, the equation for $b_{-q}^\dagger(\tau)$ is

$$\frac{\partial}{\partial\tau}b_{-q}^\dagger(\tau)=E(q)b_{-q}^\dagger(\tau)+UM^2b_q(\tau)\,. \tag{42}$$

With the help of (42) and similar equations, one finds

$$\begin{aligned}
-\frac{\partial}{\partial\tau}\mathcal{G}_1(q\tau,q\tau^+)&=E(q)\mathcal{G}_1(q\tau,q\tau^+)+UM^2\mathcal{G}_4(-q\tau,q\tau^+)+\delta(-\eta)\,,\\
\frac{\partial}{\partial\tau}\mathcal{G}_2(-q\tau,-q\tau^+)&=E(q)\mathcal{G}_2(-q\tau,-q\tau^+)+UM^2\mathcal{G}_3(q\tau,-q\tau^+)+\delta(-\eta)\,,\\
-\frac{\partial}{\partial\tau}\mathcal{G}_3(q\tau,-q\tau^+)&=E(q)\mathcal{G}_3(q\tau,-q\tau^+)+UM^2\mathcal{G}_2(-q\tau,-q\tau^+)\,,\\
\frac{\partial}{\partial\tau}\mathcal{G}_4(-q\tau,q\tau^+)&=E(q)\mathcal{G}_4(-q\tau,q\tau^+)+UM^2\mathcal{G}_1(q\tau,q\tau^+)\,.
\end{aligned} \tag{43}$$

The Green's functions above are expressed in the usual Fourier representation. In units such that $\hbar=1$, the representation is

$$\mathcal{G}(q\tau,q'\tau')=\beta^{-1}\sum_n e^{-i\omega_n(\tau-\tau')}\mathcal{G}(q,q',\omega_n)\,, \tag{44}$$

where $\omega_n = 2n\pi/\beta$. Upon using (44) in (43), one gets the following equations for the Fourier coefficients $\mathcal{G}(q,\omega_n)$:

$$\begin{aligned}
&[i\omega_n - E(q)]\mathcal{G}_1(q,\omega_n) - UM^2\mathcal{G}_4(q,\omega_n) = 1 \ ,\\
&[i\omega_n + E(q)]\mathcal{G}_2(q,\omega_n) + UM^2\mathcal{G}_3(q,\omega_n) = -1 \ ,\\
&[i\omega_n - E(q)]\mathcal{G}_3(q,\omega_n) - UM^2\mathcal{G}_2(q,\omega_n) = 0 \ ,\\
&[i\omega_n + E(q)]\mathcal{G}_4(q,\omega_n) + UM^2\mathcal{G}_1(q,\omega_n) = 0 \ .
\end{aligned} \tag{45}$$

In the normal phase, one finds

$$\begin{aligned}
&\mathcal{G}_1(q,\omega_n) = [i\omega_n - E_N(q)]^{-1} \ ,\\
&\mathcal{G}_2(q,\omega_n) = -[i\omega_n + E_N(q)]^{-1} \ ,\\
&\mathcal{G}_3(q,\omega_n) = 0 = \mathcal{G}_4(q,\omega_n) \ ,\\
&E_N(q) = (q^2/m_4 - \mu_4') \ .
\end{aligned} \tag{46}$$

On using the well-known result for bosons, viz.,

$$\lim_{\eta\to 0} \beta^{-1} \sum_{n \text{ even}} e^{i\omega_n\eta}/(i\omega_n - z) = (1 - e^{\beta z})^{-1} \ , \tag{47}$$

from (38), (44), and (46) one gets

$$S(q\neq 0) = \frac{1}{4}\left[\frac{\exp[\beta E_N(q)]+1}{\exp[\beta E_N(q)]-1}\right] . \tag{48}$$

In view of the fact that $|q| \ll \lambda_B^{-1}(T)$ (see I), one finds from (48) that at any point on the critical line ($\mu_4'=0$) including the TCP ($\mu_4'=0=u_4'$), $S(q\neq 0)\sim q^{-2}[1+R(q)]$, where $R(q)\to 0$ as $q\to 0$. This shows that the exponents η and η_t both equal zero. In the neighborhood of a critical point including the TCP, one gets

$$S(q\neq 0) \simeq [2\beta E_N(q)]^{-1} \ . \tag{49}$$

Therefore, in the normal phase close to a critical point,

$$\Gamma(r-r') \simeq \frac{m_4 k_B T}{8\pi} \frac{\exp(-|r-r'|/\xi)}{|r-r'|} \ , \tag{50}$$

where $\xi = (-m_4\mu_4')^{-1/2}$. The quantity ξ is the correlation length in this phase. When μ_3 is held fixed and $T - T_\lambda$ is small, in view of (A1) (see Appendix A), one finds $\xi \sim (T-T_\lambda)^{-\nu}$, $\nu = \frac{1}{2}$. Similarly, for $\mu_3 = \mu_{3t}$ and $T > T_t$, it follows from (A9) that $\xi \sim (T-T_t)^{-\nu_t}$, $\nu_t = \frac{1}{2}$.

In the preceding section the domains of the condensed phase have been obtained on the basis of the equation

$$-\mu_4' + 2u_4'M^2 + 3u_6M^4 \simeq 0 \ , \tag{51}$$

which follows from (30) for $h=0, M\neq 0$. This equation will now be used to determine the Fourier coefficients $\mathcal{G}(q,\omega_n)$ in the condensed phase. From (45) and (51) one finds that

$$\begin{aligned}
&\mathcal{G}_1(q,\omega_n) = u_q^2[i\omega_n - E_C(q)]^{-1} - v_q^2[i\omega_n + E_C(q)]^{-1} \ ,\\
&\mathcal{G}_2(q,\omega_n) = -u_q^2[i\omega_n + E_C(q)]^{-1} + v_q^2[i\omega_n - E_C(q)]^{-1} \ ,\\
&\mathcal{G}_3(q,\omega_n) = \mathcal{G}_4(q,\omega_n)\\
&\qquad = u_q v_q\{[i\omega_n + E_C(q)]^{-1} - [i\omega_n - E_C(q)]^{-1}\} \ ,
\end{aligned} \tag{52}$$

where

$$E_C(q) = \frac{q^2}{m_4}[1+2(q\xi)^{-2}]^{1/2} \ , \tag{53}$$

$$u_q^2 = \frac{1}{2}\left[\frac{1+(q\xi)^{-2}}{[1+2(q\xi)^{-2}]^{1/2}} + 1\right] , \tag{54}$$

$$v_q^2 = \frac{1}{2}\left[\frac{1+(q\xi)^{-2}}{[1+2(q\xi)^{-2}]^{1/2}} - 1\right] , \tag{55}$$

$$\xi = (UM^2 m_4)^{-1/2} \ . \tag{56}$$

From (56) one sees that in the region $u_4' < 0$ in the T-μ_3 plane, for ξ to exist, $M^2 \geq |u_4'|/3u_6$. Thus, once again one finds that in this region a critical line does not exist.

In view of (38), (47), and (52), $S(q\neq 0)$ for the condensed phase can be easily written. One finds

$$S(q\neq 0) = \tfrac{1}{4}(u_q - v_q)^2\left[\frac{\exp[\beta E_C(q)]+1}{\exp[\beta E_C(q)]-1}\right] . \tag{57}$$

Upon using the fact that $|q| \ll \lambda_B^{-1}(T)$, in the neighborhood of a critical point including the TCP, one can write

$$S(q\neq 0) \simeq \frac{m_4}{2\beta}[q^2 + (\xi')^{-2}]^{-1}, \quad \xi' = \xi/\sqrt{2} \ . \tag{58}$$

Therefore, the correlation function

$$\Gamma(r-r') \simeq \frac{m_4 k_B T}{8\pi} \frac{\exp(-|r-r'|/\xi')}{|r-r'|} \ . \tag{59}$$

The length ξ' can now be identified as the correlation length in the condensed phase. For fixed μ_3 and small $T_\lambda - T$, in view of (A6), one gets

$$\xi' \simeq (2\bar{C}_2 m_4)^{-1/2}(T_\lambda - T)^{-\nu'} \ , \tag{60}$$

where $\nu' = \frac{1}{2}$. From (A10) and (A17), for $\mu_3 = \mu_{3t}$ and $T < T_t$, the corresponding result is

$$\xi' \simeq (4C_2 m_4)^{-1/2}(T_t - T)^{-\nu_t'} \ , \tag{61}$$

where $\nu_t' = \frac{1}{2}$. It may be noted that the values of (η_t, ν_t, ν_t') here are the same as the corresponding renormalization-group- (RG-) theory[8,9] values for $d=3$.

In the neighborhood of the TCP, u_6 can be approximated by u_6^0 given by (A16). Therefore, while $E_N(q)$ and $\beta S(q)$, respectively, in (46) and (49), can be regarded as functions of (q,a_2), $E_C(q)$ and $\beta S(q)$, respectively, in (53) and (58), can be regarded as functions of (q,a_2,a_4). Equation (51) gives M as a function of (a_2,a_4). Here, $a_2 = -\mu_4'$ and $a_4 = u_4'$. It is easy to see that $E_I(q)$ $(I=N,C)$ and $S'(q) = \beta S(q)$, respectively, scale as

$$E_I(l^{\nu_t}q, la_2, l^{\phi_t}a_4) = l^{2\nu_t}E_I(q,a_2,a_4) \tag{62}$$

and

$$S'(l^{\nu_t}q, la_2, l^{\phi_t}a_4) = l^{-2\nu_t}S'(q,a_2,a_4) \ , \tag{63}$$

with $\nu_t = \phi_t = \frac{1}{2}$, provided M scales as

$$M(la_2, l^{\phi_t}a_4) = l^{\beta_t}M(a_2,a_4) \ , \tag{64}$$

with $\beta_t = \frac{1}{4}$. Here, l is a positive number. Equation (51) shows that (64) holds. The scaling hypothesis for the

TCP requires the scaling variables to be linear combinations of the elementary physical fields $T-T_t$ and $\mu_3-\mu_{3t}$. Since a_2 and a_4 fulfill this requirement [cf. Eqs. (A9) and (A10)], they have been chosen as the scaling variables above. Equations (62) and (63) can be written as

$$E_I(q,a_2,a_4)=|q|^{2-\eta_t}E_I\left[\pm 1, \frac{a_2}{|q|^{1/\nu_t}}, \frac{a_4}{|q|^{\phi_t/\nu_t}}\right] \tag{65}$$

and

$$S'(q,a_2,a_4)=|q|^{-(2-\eta_t)}S'\left[\pm 1, \frac{a_2}{|q|^{1/\nu_t}}, \frac{a_4}{|q|^{\phi_t/\nu_t}}\right], \tag{66}$$

with $\eta_t=0$. The scaling form of the order-parameter correlation function in (66) is essentially the same as the corresponding outcome of the parameter scaling assumption of Riedel and Wegner.[5,8,9]

In the neighborhood of a critical point far away from the TCP, choosing a_2 $(=-\mu_4')$ as the scaling variable (u_4' is a constant), similarly, one finds from (46), (49), (51), (53), and (58), that

$$E_I(l^\nu q,la_2)=l^{2\nu}E_I(q,a_2) \tag{67}$$

and

$$S'(l_q^\nu,la_2)=l^{-2\nu}S'(q,a_2)\ , \tag{68}$$

with $\nu=\frac{1}{2}$. The scaling forms (62), (63), (67), and (68) will be useful for calculations in Sec. IV.

Now (59) will be used to predict the range of validity of the approximation made to calculate the exponents. The approximation is valid provided the fluctuations in M over distances of order ξ' are relatively small, or roughly $\Gamma(r-r')$ in (59), for $|r-r'|\sim\xi'$, is small in comparison with M^2. This means

$$\left[\frac{m_4k_BT}{8\pi e\xi'}\right] \ll M^2\ . \tag{69}$$

In view of (A6) and (A8), where the latter gives discontinuity ΔC in the specific heat C_{μ_3,μ_4} at $T=T_\lambda$, (69) can be expressed as

$$|\bar{\tau}|^{1/2} \gg \frac{(m_4T_\lambda\bar{C}_2)^{3/2}}{4\sqrt{2}\pi e}\left[\frac{k_B}{\Delta C}\right]. \tag{70}$$

Here, $\bar{\tau}=(T-T_\lambda)/T_\lambda$. Upon substituting for $\bar{C}_2$ from (A4) into (70) and taking $m_4/m_3=\frac{4}{3}$, one gets

$$|\bar{\tau}| \gg 0.044\epsilon\left[\frac{k_B}{\lambda_B^3(T_\lambda)\Delta C}\right]^2, \tag{71}$$

where

$$\epsilon=[m_4u_{34}k_F^{-1}/\lambda_B^2(T_\lambda)]^3\ . \tag{72}$$

The inequality (71) is the GL-type criterion corresponding to the critical region. Its form is reminiscent of the GL form.[7] Since the calculations are performed in the high-density fermion limit, the dimensionless quantity $\lambda_B(T_\lambda)k_F \gg 1$. In view of the weak-interaction assumption in I and this inequality, it follows that $\epsilon \ll 1$.

In the neighborhood of the TCP (T_t,μ_{3t}), the condition (69) reduces to a trivial one. To clarify, one may consider a point on the segment $T<T_t$ of the line $\mu_3=\mu_{3t}$. At this point, ξ' and M^2 are given by (61) and (A17), respectively. Substituting these equations in (69), one gets

$$u_6^0 \ll \frac{16\pi^2e^2}{3m_4^3k_B^2T_t^2}\ . \tag{73}$$

In view of (A16), (73) is valid anyway. One may also consider a point, in the tricritical region corresponding to the condensed phase, close to the CXC in the T-μ_3 plane. Here, $M^2\sim|u_4'|/u_6^0$. Consequently, $\xi'\sim[(u_4')^2m_4/u_6^0]^{-1/2}$. In view of these results, from (69), one again gets a condition effectively the same as (73). The conclusion is that GL criterion spares the tricritical region in a Fermi-Bose mixture.

IV. DYNAMIC CORRELATION OF ORDER-PARAMETER FLUCTUATIONS

A dynamic scaling assumption for a TCP (Ref. 16) can be stated as follows: In the neighborhood of the TCP, the dynamic correlation function $\Gamma(r,a_2,a_4,t)$ of order-parameter fluctuations satisfies the generalized homogeneity relation

$$\Gamma(l^{b_r}r,l^{b_2}a_2,l^{b_4}a_4,l^{b_t}t)=l\Gamma(r,a_2,a_4,t)\ . \tag{74}$$

Here, l denotes an arbitrary positive number while the exponents b_r, b_2, etc. are unknown quantities, the inputs of the theory. a_2,a_4 are the scaling variables. In writing (74) it has been assumed that field conjugate to the order parameter is zero. It follows that the spatial Fourier transform $\mathscr{R}(q,a_2,a_4,t)$ is also a generalized homogeneous function:

$$\mathscr{R}(l^{b_q}q,l^{b_2}a_2,l^{b_4}a_4,l^{b_t}t)=l\mathscr{R}(q,a_2,a_4,t)\ , \tag{75}$$

where $b_q=-b_r$. One of the aims in this section is to show that (75) holds for the present system. The other is to calculate the dynamic scaling exponent z.

To achieve these goals, an expression for $\mathscr{R}(q,t)$ will be needed. In what follows, we will derive first an expression for Fourier transform $\mathscr{R}(q,\omega)$ in space and time of $\Gamma(r,t)$. This will lead to one for $\mathscr{R}(q,t)$. Frequencies in $\mathscr{R}(q,\omega)$ may range from $-\infty$ to $+\infty$, but momenta will lie in the range $0<|q|<p_c$ as in Sec. III.

In the present approximation scheme, $\mathscr{R}(q,\omega)$ is given by

$$\mathscr{R}(q,\omega)=\tfrac{1}{4}\int dt\, e^{i\omega t}[\langle b_q^\dagger(t)b_q(0)\rangle+\langle b_{-q}(t)b_{-q}^\dagger(0)\rangle + \langle b_{-q}(t)b_q(0)\rangle+\langle b_q^\dagger(t)b_{-q}^\dagger(0)\rangle]\ , \tag{76}$$

where

$$b_q(t)=\exp(iH_Bt)b_q\exp(-iH_Bt)\ , \tag{77}$$

and the angular brackets denote an ensemble average calculated with H_B. An attempt will now be made to express $\mathscr{R}(q,\omega)$ in terms of the spectral weight function

$$A_1(q,\omega)=i[\mathscr{G}_1(q,\omega_n)|_{i\omega_n=\omega+i\eta} - \mathscr{G}_1(q,\omega_n)|_{i\omega_n=\omega-i\eta}] , \quad (78)$$

where the Fourier coefficient $\mathscr{G}_1(q,\omega_n)$ is given by (45). In view of (46) and (52), one finds that while in the normal phase

$$A_1(q,\omega)=2\pi\delta(\omega-E_N(q)) , \quad (79)$$

in the condensed phase

$$A_1(q,\omega)=2\pi[u_q^2\delta(\omega-E_C(q))-v_q^2\delta(\omega+E_C(q))] . \quad (80)$$

Thus, in the present approximation, $A_1(q,\omega)$ is an even function of q in both the phases.

It is useful to introduce the functions

$$\begin{aligned} f_1(q,\omega)&=-i\int dt\, e^{i\omega t}\langle b_q^\dagger(t)b_q(0)\rangle\Theta(t) , \\ f_2(q,\omega)&=-i\int dt\, e^{i\omega t}\langle b_{-q}(t)b_{-q}^\dagger(0)\rangle\Theta(t) , \\ f_3(q,\omega)&=-i\int dt\, e^{i\omega t}\langle b_{-q}(t)b_q(0)\rangle\Theta(t) , \\ f_4(q,\omega)&=-i\int dt\, e^{i\omega t}\langle b_q^\dagger(t)b_{-q}^\dagger(0)\rangle\Theta(t) , \end{aligned} \quad (81)$$

where $\Theta(t)$ is the step function, for the purpose of expressing $\mathscr{R}(q,\omega)$ in terms of $A_1(q,\omega)$. Using the integral representation of $\Theta(t)$, viz.,

$$\Theta(t)=i\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega+i\eta} , \quad (82)$$

one gets the following Lehmann representation for $f_1(q,\omega)$:

$$f_1(q,\omega)=e^{\beta V\Omega_B}\sum_{m,n}e^{-\beta H_{B,m}}\langle m\,|b_q^\dagger|\,n\rangle\langle n\,|b_q|\,m\rangle \times(\omega+H_{B,m}-H_{B,n}+i\eta)^{-1} , \quad (83)$$

where Ω_B is given by (22), $|\,m\rangle$ is an exact eigenstate of H_B, and

$$H_B|\,m\rangle=H_{B,m}|\,m\rangle . \quad (84)$$

From (78) and (83) one finds that

$$\mathrm{Re} f_1(q,\omega)=-\mathrm{P}\int_{-\infty}^{+\infty}\frac{d\omega'}{\pi}\frac{\mathrm{Im} f_1(q,\omega')}{\omega-\omega'} , \quad (85)$$

$$\mathrm{Im} f_1(q,\omega)=-\tfrac{1}{2}A_1(q,-\omega)[\exp(-\beta\omega)-1]^{-1} , \quad (86)$$

where P denotes a Cauchy principal value. In writing (86), in particular, the Lehmann representation of $\mathscr{G}_1(q,\omega_n)$, viz.,

$$\mathscr{G}_1(q,\omega_n)=e^{\beta V\Omega_B}\sum_{m,n}e^{-\beta H_{B,m}}\langle m\,|b_q|\,n\rangle\langle n\,|b_q^\dagger|\,m\rangle \times\left[\frac{1-\exp[-\beta(H_{B,n}-H_{B,m})]}{i\omega_n+H_{B,m}-H_{B,n}}\right] , \quad (87)$$

has also been used. The representation is easy to obtain using the result that the Fourier coefficient

$$\mathscr{G}_1(q,\omega_n)=\int_0^\beta d\tau\, e^{i\omega_n\tau}\mathscr{G}_1(q,\tau) . \quad (88)$$

From (82), (85), and (86) one gets

$$f_1(q,\omega)=-i\int dt\, e^{i\omega t}\Theta(t) \times\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}A_1(q,-\omega')(e^{-\beta\omega'}-1)^{-1} \times e^{-i\omega' t} . \quad (89)$$

Comparison of (81) and (89) yields an expression for the average $\langle b_q^\dagger(t)b_q(0)\rangle$. It follows that

$$\int dt\, e^{i\omega t}\langle b_q^\dagger(t)b_q(0)\rangle =-\frac{A_1(q,-\omega)}{2\pi}[1-\exp(-\beta\omega)]^{-1} . \quad (90)$$

Upon considering the functions f_2, f_3, and f_4 and proceeding as above, one gets the similar results

$$\begin{aligned} \int dt\, e^{i\omega t}\langle b_{-q}(t)b_{-q}^\dagger(0)\rangle&=\frac{A_1(-q,\omega)}{2\pi}[1-\exp(-\beta\omega)]^{-1}, \\ \int dt\, e^{i\omega t}\langle b_{-q}(t)b_q(0)\rangle&=\int dt\, e^{i\omega t}\langle b_q^\dagger(t)b_{-q}^\dagger(0)\rangle \\ &=\frac{A_2(q,\omega)}{2\pi}[1-\exp(-\beta\omega)]^{-1} . \end{aligned} \quad (91)$$

Here,

$$A_2(q,\omega)=i[\mathscr{G}_j(q,\omega_n)|_{i\omega_n=\omega+i\eta} - \mathscr{G}_j(q,\omega_n)|_{i\omega_n=\omega-i\eta}],\ j=3,4 \quad (92)$$

$$\mathscr{G}_j(q,\omega_n)=\int_0^\beta d\tau\, e^{i\omega_n\tau}\mathscr{G}_j(q,\tau) . \quad (93)$$

In view of (46) and (52), one finds that while in the normal phase $A_2(q,\omega)=0$, in the condensed phase

$$A_2(q,\omega)=2\pi u_q v_q[\delta(\omega+E_C(q))-\delta(\omega-E_C(q))] . \quad (94)$$

The results above show that whereas for the normal phase

$$\mathscr{R}(q,\omega)=\frac{1}{8\pi}(1-e^{-\beta\omega})^{-1}[A_1(q,\omega)-A_1(q,-\omega)] , \quad (95)$$

for the condensed phase

$$\mathscr{R}(q,\omega)=\frac{1}{8\pi}(1-e^{-\beta\omega})^{-1} \times[A_1(q,\omega)-A_1(q,-\omega)+2A_2(q,\omega)] . \quad (96)$$

These are the expressions sought for. It can be seen from (95) and (96), respectively, that $\mathscr{R}(q,\omega)$ satisfies the identity

$$(1-e^{-\beta\omega})\mathscr{R}(q,\omega)+(1-e^{\beta\omega})\mathscr{R}(q,-\omega)=0 \quad (97)$$

in the normal phase, and the identity

$$[(1-e^{-\beta\omega})\mathscr{R}(q,\omega)+(1-e^{\beta\omega})\mathscr{R}(q,-\omega)] =\frac{1}{4\pi}[A_2(q,\omega)+A_2(q,-\omega)] \quad (98)$$

in the condensed phase. With the help of (79), (80), (94), (95), and (96) one finds that, for the former,

$$\mathscr{R}(q,\omega)=\tfrac{1}{4}(1-e^{-\beta\omega})^{-1}[\delta(\omega-E_N(q)) -\delta(\omega+E_N(q))] , \quad (99)$$

and, for the latter,

$$\mathscr{R}(q,\omega)=\tfrac{1}{4}(1-e^{-\beta\omega})^{-1}(u_q-v_q)^2[\delta(\omega-E_C(q)) -\delta(\omega+E_C(q))] . \quad (100)$$

In the critical and tricritical regions, (99) and (100) can be expressed as

$$\mathscr{R}(q,\omega)\simeq\omega_0^{-1}S_I(q)g(\omega/\omega_0), \quad I=N,C \quad (101)$$

where the static functions $S_I(q)$ are given by (49) and (58) and

$$g\left[\frac{\omega}{\omega_0}\right]=\frac{1}{2}\left[\delta\left[\frac{\omega}{\omega_0}+\frac{E_I(q)}{\omega_0}\right]+\delta\left[\frac{\omega}{\omega_0}-\frac{E_I(q)}{\omega_0}\right]\right] . \quad (102)$$

The quantity ω_0 is called the characteristic frequency. It is defined through the equation

$$\int_{-\omega_0}^{\omega_0}\mathscr{R}(q,\omega)d\omega=\tfrac{1}{2}\int_{-\infty}^{+\infty}\mathscr{R}(q,\omega)d\omega . \quad (103)$$

The result in (101) complies with the prediction of Halperin and Hohenberg.[11]

From (99) and (100) one finds that the Fourier transform $\mathscr{R}(q,t)$ for the normal and condensed phases, respectively, are given by

$$\mathscr{R}(q,t)=\tfrac{1}{4}[(1-e^{-\beta E_N(q)})^{-1}e^{-iE_N(q)t} -(1-e^{\beta E_N(q)})^{-1}e^{iE_N(q)t}] \quad (104)$$

and

$$\mathscr{R}(q,t)=\tfrac{1}{4}(u_q-v_q)^2[(1-e^{-\beta E_C(q)})^{-1}e^{-iE_C(q)t} -(1-e^{\beta E_C(q)})^{-1}e^{iE_C(q)t}] . \quad (105)$$

As expected, setting $t=0$ above, one gets back $S_I(q)$ in (48) and (57). In the critical and tricritical regions (104) and (105) are given by

$$\mathscr{R}(q,t)\simeq S_I(q)\cos[E_I(q)t] , \quad (106)$$

where $S_I(q)$ are given by (49) and (58). One sees that $\mathscr{R}(q,t)$ is periodic in time with time periods $2\pi E_I^{-1}(q)$. From Eqs. (8), (46), (53), and the fact that $|q| \ll \lambda_B^{-1}(T)$, it can be checked that these periods are large compared to $(k_BT)^{-1}$ in the critical and tricritical regions.

Upon choosing $-\mu_4'$ and u_4', respectively, as the scaling variables a_2 and a_4, with the help of (62) and (63) one finds that (106) satisfies the relation (75) with

$$b_q=-\tfrac{1}{2}, \; b_2=-1, \; b_4=-\tfrac{1}{2}, \; b_t=1 . \quad (107)$$

It is thus established that the dynamic scaling assumption (74) holds for the present system.

For the critical region, relation similar to (75) is

$$\mathscr{R}(l^{c_q}q,l^{c_2}a_2,l^{c_t}t)=l\mathscr{R}(q,a_2,t) , \quad (108)$$

where (c_q,c_2,c_t) are unknown exponents and a_2 is the scaling variable. Upon choosing $-\mu_4'$ as the variable a_2, with the help of (67) and (68) it can be easily shown that (106) satisfies the relation (108) with

$$c_q=-\tfrac{1}{2}, \; c_2=-1, \; c_t=1 . \quad (109)$$

The critical dynamic scaling exponent z is given by the relation[17] $z=-c_t/c_q$. We thus find that in the present approximation $z=2$. Upon defining a tricritical dynamic scaling exponent z_t by a similar relation, we find from (107) that this, too, has the same value.

In the model chosen for this work, order parameter is not conserved, the reason being the order-parameter field $\psi(r)$ does not commute with the Hamiltonian. Moreover, it has been found in Sec. III that $\eta=0$. Therefore, the value of z obtained above agree with that expected on the basis of conventional prediction[12] $(z=2-\eta)$ for the order-parameter nonconservation case.

Many years ago, Halperin *et al.*[13] calculated the exponent z to order ϵ^2 for the two cases, viz., the order-parameter nonconservation and conservation cases of a time-dependent Ginzburg-Landau (TDGL) model. For the two-component $(n=2)$, nonconserved order parameter of this model, from the expansion to $O(\epsilon^2)$, the estimated value of z is 2.01, for $d=3$. The subsequent work of Dominicis *et al.*[14] provides a slightly better estimate. The present system is a member of the universality class alluded to above, viz., the class with $n=2$, $d=3$, and nonconserved order parameter. One can therefore conclude that the result obtained above is quite good. In fact, this is the best possible outcome that can be expected from the present theory, which is valid only when one stays away from a critical point specified by the GL-type criterion (71). Since for tricritical region there is no such criterion, the value $z_t=2$ is exact, like those for β_t, γ_t, δ_t, etc.

V. DISCUSSION

The work reported in this paper is based on the findings in I. An attempt has been made here to extract some information regarding the statics and dynamics of a statistical-mechanical theory for a Fermi-Bose mixture. The theory goes beyond Landau's theory to incorporate some measure of the fluctuations in the order parameter. The method of temperature and real-time Green's function was used in the analysis. The approximation $H_e+H_s\simeq H_B$ enabled us to write exact equations of motion for the temperature Green's function used.

The additional facts related to the contents of the preceding sections are as follows:

(1) In the neighborhood of a critical point the correlation length ξ' is very large. Therefore, for a finite q, the region $q\xi'\gg 1$ in the T-μ_3 plane corresponds to the critical and tricritical regions of the condensed phase. The region far away from this, usually referred to as the hydrodynamic region, corresponds to $q\xi'\ll 1$. In the hydro-

dynamic region, from (53) one finds $E_C(q)\simeq vq$, where $v=(2M^2Um_4^{-1})^{1/2}$. The sound velocity v exists, in the region $u_4'<0$, provided $M^2\geq |u_4'|/3u_6$. Thus, the condensed Bose component of the mixture shows a phononlike dispersion relation in the hydrodynamic region.

(2) In Sec. II, $b_0/\sqrt{V}$ was approximated by a c number M. This approximation does not allow one to derive the thermodynamics of the mixture through the static correlation function of the order-parameter fluctuations. The reason is that a fluctuation-dissipation (FD) relation (see Appendix B for derivation), with the help of which this can be achieved, cannot be satisfied under this approximation. The FD relation in question is

$$\left[\frac{\partial m}{\partial h}\right]_{T,\mu_3,\mu_4}\simeq\beta S(q=0;T,\mu_3,\mu_4,h)\ , \tag{110}$$

where

$$m=\left\langle\frac{1}{2}\left[\frac{b_0}{\sqrt{V}}+\frac{b_0^\dagger}{\sqrt{V}}\right]\right\rangle . \tag{111}$$

Here, $\langle\cdots\rangle$ denotes the ensemble average calculated with H_e+H_s. The function $S(q=0)$ in (110) is given by (35).

The problem of calculating the function $S(q)$, in a manner such that (110) is satisfied, is being studied presently by the author of this paper. The problem appears to be a challenging one.

(3) It has been noticed that the scale set by the thermal momentum is a special one. One of the reasons for this, being exponentiation of the connected graphs in I belonging to the first and second orders, can be proved to all orders provided momenta associated with external legs are small in comparison with the thermal momentum. The results in Sec. III of this paper, e.g., $S[\,|q|\ll\lambda_B^{-1}(T)]\sim q^{-2}$ at any point on the critical line including the TCP, highlight the fact that the scale is a special one for statics. The result obtained in Sec. IV, viz., the periodic time of $\mathscr{R}(q,t)$ is large compared to $(k_BT)^{-1}$ in the critical and tricritical regions, indicates that there is an analogous scale for dynamics, viz., the scale set by the inverse thermal frequency $\hbar/k_BT$. It appears that for dynamic critical phenomena, fields which depend on time, small compared to $\hbar/k_BT$ or of order $\hbar/k_BT$, are relatively unimportant. Therefore, it is a task for a future theory of dynamic critical behavior to find a scheme to integrate out these fields. It may be noted that this remark concerning dynamics is meaningful only for a model with a quantum-mechanical basis. For classical models, such as the TDGL model,[13,14] the remark is not meaningful, since the inverse thermal frequency tends toward zero in the classical case.

APPENDIX A

It has been noted in Sec. II that, for $h=0$ and $u_4'\geq 0$, the curve $\mu_4'=0$, in the T-μ_3 plane, separates the regions corresponding to the normal ($\mu_4'<0$) and condensed ($\mu_4'>0$) phases. This fact is also depicted in Figs. 1 and 2. Therefore, this curve is the critical line, in the region $u_4'\geq 0$, when $h=0$.

We consider the transition at fixed values of the fields other than temperature. Let $T_\lambda(\mu_3,\mu_4)$ be the solution of $\mu_4'=0$. If μ_4' is assumed to be analytic at the transition temperature T_λ, we may write, for T close to T_λ,

$$-\mu_4'\simeq\bar{C}_2(T-T_\lambda)+O(T-T_\lambda)^2\ , \tag{A1}$$

$$\bar{C}_2=-(\partial\mu_4'/\partial T)_{\substack{\mu_3,\mu_4\\ T=T_\lambda}}\ . \tag{A2}$$

To justify this assumption, one has to show that $\bar{C}_2$ exists.

In the high-density fermion limit ($\mu_3\gg k_BT$),

$$n_3^F=\frac{(m_3\mu_3)^{3/2}}{3\pi^2}\left[1+\frac{\pi^2}{8}\left[\frac{k_BT}{\mu_3}\right]^2+\cdots\right] . \tag{A3}$$

Upon using (A3) in (3), one finds that

$$\bar{C}_2=\frac{u_{34}m_3^2k_B^2T_\lambda}{12k_F}\ , \tag{A4}$$

$$k_F=(m_3\mu_3)^{1/2}\ . \tag{A5}$$

Thus $\bar{C}_2$ exists, in the limit of a degenerate Fermi gas, provided T_λ exists in the same limit. Equations (3) and (A3) indicate that, for μ_4 slightly greater than $u_{34}(m_3\mu_3)^{3/2}/3\pi^2$, T_λ does exist. Similarly, it can be shown that in the limit of a nondegenerate Fermi gas $[n_3^F\lambda_B^3(T)\sim e^{\beta\mu_3}\ll 1]$ also, $\bar{C}_2$ is finite.

From (51) and (A1) one finds that, far away from the TCP, for fixed μ_3 and μ_4,

$$M^2\simeq\left[\frac{\bar{C}_2}{2u_4'}\right](T_\lambda-T)\ . \tag{A6}$$

Upon using this result in (31), for the specific heat C_{μ_3,μ_4} one finds

$$\begin{aligned}C_{\mu_3,\mu_4}&=-T\frac{\partial^2}{\partial T^2}(\Omega_1+V^{-1}c_0),\quad T>T_\lambda\\ &=-T\frac{\partial^2}{\partial T^2}(\Omega_1+V^{-1}c_0)+T\frac{\bar{C}_2^2}{2u_4'},\quad T<T_\lambda\ .\end{aligned} \tag{A7}$$

The magnitude of the discontinuity in C_{μ_3,μ_4} at $T=T_\lambda$ is thus

$$\Delta C=T_\lambda\frac{\bar{C}_2^2}{2u_4'}\ . \tag{A8}$$

The equations to determine the TCP (T_t,μ_{3t}) are $\mu_4'=0=u_4'$. In the neighborhood of the TCP, one can write

$$-\mu_4'\simeq C_2(T-T_t)+K_2(\mu_3-\mu_{3t})\ , \tag{A9}$$

$$u_4'\simeq C_4(T-T_t)+K_4(\mu_3-\mu_{3t})\ , \tag{A10}$$

$$u_6\simeq u_6^0+O(T-T_t)+O(\mu_3-\mu_{3t})\ . \tag{A11}$$

In the high-density fermion limit, one finds that

$$\begin{aligned}C_2&=-\left[\frac{\partial\mu_4'}{\partial T}\right]_{\substack{\mu_3,\mu_4\\ T=T_t,\,\mu_3=\mu_{3t}}}\\ &\simeq\frac{u_{34}m_3^3k_B^2T_t}{12k_F(\mu_3=\mu_{3t})}+O(u_{34}^2)\ ,\end{aligned} \tag{A12}$$

$$K_2=-\left[\frac{\partial\mu_4'}{\partial\mu_3}\right]_{T=T_t,\,\mu_3=\mu_{3t}}^{T,\mu_4}$$

$$\simeq\frac{u_{34}m_3k_F(\mu_3=\mu_{3t})}{2\pi^2}+O(u_{34}^2)\ , \qquad \text{(A13)}$$

$$C_4=\left[\frac{\partial u_4'}{\partial T}\right]_{T=T_t,\,\mu_3=\mu_{3t}}^{\mu_3,\mu_4}$$

$$\simeq\frac{u_{34}^2m_3^3k_B^2T_t}{48k_F^3(\mu_3=\mu_{3t})}+O(u_{34}^3)\ , \qquad \text{(A14)}$$

$$K_4=\left[\frac{\partial u_4'}{\partial\mu_3}\right]_{T=T_t,\,\mu_3=\mu_{3t}}^{T,\mu_4}$$

$$\simeq-\frac{u_{34}^2m_3^2}{8\pi^2k_F(\mu_3=\mu_{3t})}+O(u_{34}^3)\ , \qquad \text{(A15)}$$

$$u_6^0=\frac{u_{34}^3m_3^2}{24\pi^2k_F(\mu_3=\mu_{3t})}\ . \qquad \text{(A16)}$$

From (51), (A9), (A10), and (A11), one finds that along the segment $T<T_t$ of the line $\mu_3=\mu_{3t}$ in the T-μ_3 plane,

$$M^2\simeq\left[\frac{C_2}{3u_6^0}\right]^{1/2}(T_t-T)^{1/2}\ . \qquad \text{(A17)}$$

APPENDIX B

Let the order parameter (m) of the system be defined by

$$m=\frac{1}{2}\left\langle\frac{b_0}{\sqrt{V}}+\frac{b_0^\dagger}{\sqrt{V}}\right\rangle\ , \qquad \text{(B1)}$$

where $\langle\cdots\rangle$ denotes the ensemble average calculated with $H=H_e+H_s$. From (35) and (37) one then finds that

$$S(q=0;T,\mu_3,\mu_4,h)=\tfrac{1}{4}\langle(b_0+b_0^\dagger)(b_0+b_0^\dagger)\rangle-Vm^2\ . \qquad \text{(B2)}$$

Upon giving an increment δh in h, for fixed (T,μ_3,μ_4), $H\to H+\delta H_s$ and $m\to m+\delta m$, where

$$\delta H_s=-\frac{\delta h}{2}V\left[\frac{b_0}{\sqrt{V}}+\frac{b_0^\dagger}{\sqrt{V}}\right] \qquad \text{(B3)}$$

and

$$m+\delta m=\frac{\frac{1}{2}\mathrm{Tr}[e^{-\beta(H+\delta H_s)}(b_0/\sqrt{V}+b_0^\dagger/\sqrt{V})]}{\mathrm{Tr}\,e^{-\beta(H+\delta H_s)}}\ . \qquad \text{(B4)}$$

Expansion for $\exp[-\beta(H+\delta H_s)]$ in powers of δH_s can be obtained with the help of an operator $U(\tau,\tau')$ defined by

$$U(\tau,\tau')=\exp(H\tau)\exp[-(H+\delta H_s)(\tau-\tau')]\exp(-H\tau')\ . \qquad \text{(B5)}$$

By setting up differential equation for U and solving it, one gets

$$U(\tau,\tau')=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{\tau'}^{\tau}d\tau_1\cdots\int_{\tau'}^{\tau}d\tau_n T_t[\delta H_s(\tau_1)\cdots\times\delta H_s(\tau_n)]\ , \qquad \text{(B6)}$$

where

$$\delta H_s(\tau)=\exp(H\tau)\delta H_s\exp(-H\tau)\ . \qquad \text{(B7)}$$

Hence, the required expansion is

$$e^{-\beta(H+\delta H_s)}=e^{-\beta H}U(\beta,0)=e^{-\beta H}\left[1-\int_0^\beta d\tau\,\delta H_s(\tau)+\cdots\right]\ . \qquad \text{(B8)}$$

With the help of (B8), from (B1) and (B4) one gets

$$\delta m=\delta h V\int_0^\beta d\tau[\langle B_0(\tau)B_0(0)\rangle-\langle B_0(\tau)\rangle\langle B_0(0)\rangle]+\cdots\ , \qquad \text{(B9)}$$

where

$$B_0=\frac{1}{2}\left[\frac{b_0}{\sqrt{V}}+\frac{b_0^\dagger}{\sqrt{V}}\right]\ . \qquad \text{(B10)}$$

Upon using the cyclic property of the trace and the fact that any two functions of the same operator commute, one finds that

$$\langle B_0(\tau)\rangle=\langle B_0(0)\rangle\ . \qquad \text{(B11)}$$

It follows that

$$\frac{\delta m}{\delta h}=\left[\frac{\partial m}{\partial h}\right]_{T,\mu_3,\mu_4}\simeq\beta S(q=0;T,\mu_3,\mu_4,h)\ . \qquad \text{(B12)}$$

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