Response functions and collective modes of superfluid ^{3}He-B in strong magnetic fields

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(Received 28 October 1985)

The field dependence of the magnetic susceptibility and collective mode frequencies of superfluid ^{3}He-B are calculated perturbatively in $(\gamma H/\Delta)$ using the quasiclassical theory. With the aid of this perturbation theory, we interpret the gap distortion, nonlinear susceptibility, and collective excitations of the order parameter in terms of additional correlations between Cooper pairs that are induced by a magnetic field. We also calculate the dispersion splittings of the real squashing modes in a magnetic field. In addition to the nonlinear Zeeman shifts of these modes, which occur in strong magnetic fields $(\gamma H \sim 0.10\Delta)$, there are low-field $(\gamma H \sim q^2 v_f^2/\Delta)$ nonlinearities of the collective mode frequencies resulting from the field dependence of the quantization axis. Our results for all of these response properties of ^{3}He-B depend upon a small number of material parameters; thus measurements of these properties can provide detailed information on the quasiparticle interactions.

I. INTRODUCTION

The behavior of superfluid ^{3}He-B in a strong magnetic field $[\gamma H \lesssim \Delta(T)]$ has received increased study in the past few years. Qualitatively new phenomena are observed in the properties of ^{3}He-B when external magnetic fields are sufficiently large to significantly distort the B-phase order parameter.[1-3] This gap distortion, which is partly due to depairing of the $S_z = 0$ Cooper pairs, affects both the equilibrium and nonequilibrium properties of the B-phase. NMR measurements of the nonlinear field dependence of the magnetic susceptibility by Hoyt *et al.*[3] clearly exhibit the magnetic field distortion of the order parameter. The nonlinear Zeeman splittings of the real squashing modes observed by Shivaram *et al.*[1] result from field-induced mixing of the nonequilibrium order parameter with states of different total angular momentum, in addition to distortion of the equilibrium order parameter. Both sets of measurements provide a probe of the underlying quasiparticle interactions in ^{3}He.

In Sec. II we present our calculation of the B-phase order parameter in a strong magnetic field. An interesting result is that the B phase is no longer described by a pure p-wave order parameter; an f-wave component of the equilibrium order parameter enters at quadratic order in the magnetic field. These f-wave pairing correlations also affect the nonlinear magnetic susceptibility, which we calculate through order $(\gamma H/\Delta)^2$. In Sec. III we derive equations for the time-dependent order-parameter response in strong magnetic fields. These equations are solved to obtain the quadratic nonlinear Zeeman splittings, as well as the field-dependent dispersion splittings, of the real squashing (RSQ) modes. Our results are exact in weak-coupling BCS theory through order $[\gamma H/\Delta(T)]^2$, and include all quasiparticle molecular-field effects. Strong-coupling corrections are not included, so our theoretical results are only applicable to an interpretation of the low-pressure response properties of ^{3}He-B, where the specific-heat jump (a direct measure of strong-coupling effects) is close to the BCS prediction. The remainder of the introduction briefly reviews the current status of collective mode spectroscopy in superfluid ^{3}He-B.

The order parameter collective modes of superfluid ^{3}He, which have been studied theoretically and experimentally by several authors,[4] are unique signatures of the intricate structure of the ^{3}He order parameter. In the B phase, several order parameter collective modes have been observed as resonances in the ultrasound attenuation for frequencies $\omega < 2\Delta(T)$. In this frequency range, zero sound phonons can excite Cooper pairs out of the ground state (or equilibrium state at $T \neq 0$) into excited Cooper-pair states that lie below the pair-breaking threshold of $2\Delta(T)$. The sharp peaks in the ultrasound attenuation correspond to the resonant excitation of these Cooper pairs. The width of the excited Cooper-pair states is determined by quasiparticle damping; since $\tau_{QP}^{-1} \ll \Delta$ at low temperature, the resonant structures in the sound attenuation are very sharp. In consequence, ultrasound has developed as a powerful spectroscopic tool to study ordered states of superfluid ^{3}He-B.

It is well known that the equilibrium B phase order parameter and the collective modes are eigenfunctions of the square of the total twisted angular momentum,[5]

$$\mathbf{J} = \mathbf{L} + \underline{R}^{\,tr} \cdot \mathbf{S} \,, \tag{1}$$

where $\mathbf{L}$ ($\mathbf{S}$) is the orbital (spin) angular momentum operator and $\underline{R}$ is the rotation matrix that defines the equilibrium B-phase order parameter in zero field (the Balian-Werthamer, or BW state),

$$\Delta(\hat{\mathbf{p}}) = \Delta(T)(i\sigma\sigma^2) \cdot \underline{R} \cdot (\hat{\mathbf{p}}) \,. \tag{2}$$

The equilibrium state is $J = 0$, in addition to $l = 1$ and $S = 1$. The sharp features that appear in the ultrasound attenuation correspond to the $J = 2$ order-parameter modes; there are ten such modes associated with the real ($2+$ mode) and imaginary ($2-$ mode) parts of the $J = 2$

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order parameter. The 2+ and 2− modes, also referred to as the real and imaginary squashing modes, are fivefold degenerate in zero field for zero wave vector. Unlike the equilibrium state in zero field, which is a pure p-wave ($l=1$) state, the excited states are not eigenfunctions of $\mathbf{L}^2$; the $J=2$ states are superpositions of $l=1$ and $l=3$ states.[6]

The order-parameter collective modes are also indexed by their signature under the particle-hole transformation.[7] The 2+ (2−) modes have signature + (−) under this operation. The particle-hole transformation is not an exact symmetry of the quasiparticle effective Hamiltonian, but the deviation from exact particle-hole symmetry is small. This asymmetry is nevertheless important. Zero sound and the 2− modes have the same signature under the particle-hole transformation, which is opposite to that of the 2+ modes. In the limit of exact particle-hole symmetry a selection rule would prohibit the coupling between the 2+ modes and zero sound. Thus, the observed weak coupling of the 2+ mode to zero sound is a direct measure of particle-hole asymmetry in the underlying Hamiltonian. The identification by Koch and Wölfle[7] of the 2+ mode with the "weak" attenuation feature observed by Giannetta *et al.*[8] and Mast *et al.*[9] was confirmed by Avenel *et al.*[10] who observed the fivefold Zeeman splitting of 2+ modes in weak magnetic fields ($H \lesssim 0.3$ kG). The five magnetic substates are labeled by the eigenvalues of $\mathbf{J}\cdot\hat{\mathbf{z}}=J_z$, where $\hat{\mathbf{z}}$ is the direction of the magnetic field. Schopohl and Tewordt[11,12] first predicted a Zeeman splitting of the 2± modes. Sauls and Serene[13] calculated the g factors for the 2± modes including, as they did for the zero-field collective-mode frequencies, Fermi liquid, and pairing interaction effects. The detailed dependence of the collective-mode frequencies on the quasiparticle interactions is discussed in Sec. III. The main results are equations for the nonlinear Zeeman splittings and field-dependent dispersion splittings of the RSQ modes which can be solved numerically to analyze the extensive data on these collective modes. It is worth emphasizing that spectroscopic determinations of the collective-mode frequencies can provide detailed and precise information about the underlying interactions between ^{3}He quasiparticles.

Depending on the strength and sign of these interactions, subtle dynamical correlations between quasiparticles and Cooper pairs may be observable at very low temperatures.[6] Reliable determinations of the higher angular momentum Fermi liquid and pairing interaction parameters also place stringent constraints on any microscopic theory designed to predict the effective interactions in liquid ^{3}He.[14] The information that can be obtained on the quasiparticle interactions from NMR and collective-mode spectroscopy is considerably richer than any information on Fermi liquid parameters obtainable from measurements in the normal state. One reason is that the higher angular momentum pairing interactions, which parametrize the quasiparticle interaction in the particle-particle channel, enter observable properties of the superfluid phases on equal footing with the Landau parameters, which parametrize the quasiparticle interaction in the particle-hole channel, and which are the only relevant interactions in the normal Fermi liquid.

The collection of our results for the gap distortion, susceptibility, and nonlinear Zeeman splitting and those of Refs. 6 and 13 for the zero-field and linear Zeeman shifts provide stringent constraints on the ultrasound and NMR data bases of Refs. 1 and 3, and should ultimately provide precise values for the $l=2$ Fermi liquid parameters ($F_2^{s,a}$) and the $l=3$ transition temperature (T_{c3}), at least at low pressures where strong-coupling effects are believed to be small. Since theoretical results for the response functions of ^{3}He-*B* overdetermine the interaction parameters ($F_2^{s,a}, F_0^a, T_{c3}$), significant discrepancies between the parameters obtained from different data bases may indicate either (i) discrepancies in temperature scales or (ii) important strong-coupling corrections. We discuss in detail the comparison between our theoretical results and the data obtained from NMR and ultrasound by several research groups in a separate publication.[15]

II. QUASICLASSICAL THEORY OF ^{3}He-*B* IN A MAGNETIC FIELD

The quasiclassical theory of superconductivity formulated by Eilenberger,[16] by Larkin and Ovchinnikov,[17] and by Eliashberg[18] has been applied by several authors[19,20] to superfluid ^{3}He. Our notation follows that of the recent review article of Serene and Rainer.[21] We use this theory to calculate the equilibrium and nonequilibrium properties of superfluid ^{3}He-*B* in strong magnetic fields. The quasiclassical theory is formulated in terms of quasiclassical propagators, 4×4 matrix Green's functions in particle-hole and spin space, that are integrated over the magnitude of the quasiparticle momentum, or equivalently the normal-state quasiparticle energy $\xi_{\mathbf{p}}=v_f(|\mathbf{p}|-p_f)$ near the Fermi surface (v_f and p_f are the quasiparticle speed and momentum on the Fermi surface). The thermodynamic and static response functions may be calculated from the Matsubara propagator

$$\hat{g}^{\,m}(\hat{\mathbf{p}},\mathbf{R};\epsilon_n) \propto \int_{-\epsilon_c}^{\epsilon_c} d\xi_{\mathbf{p}}\hat{\tau}_3\hat{G}(\mathbf{p},\mathbf{R};\epsilon_n) , \tag{3}$$

where $\hat{G}$ is the one-particle Matsubara Green's function and $\hat{\tau}_3=(\tau_3)_{p\text{-}h}\times \mathbb{1}_{\text{spin}}$; we use τ (σ) for the Pauli matrices in particle-hole (spin) space. For a complete derivation of the quasiclassical equations and a full explanation of the notation see Serene and Rainer[22] or the original references.[16–18] The structure of $\hat{g}^{\,m}$ in particle-hole space is

$$\hat{g}^{\,m}=\begin{bmatrix} g^m & f^m \\ \bar{f}^{\,m} & \bar{g}^{\,m}\end{bmatrix} , \tag{4}$$

where $g^m(\hat{\mathbf{p}},\mathbf{R};\epsilon_n)$ is the conventional one-particle Green's function integrated over $\xi_{\mathbf{p}}$ and $f^m(\hat{\mathbf{p}},\mathbf{R};\epsilon_n)$ is the corresponding anomalous propagator. The Matsubara propagator depends on $\hat{\mathbf{p}}$, the quasiparticle position on the Fermi surface, $\mathbf{R}$, the center-of-mass coordinate, and $\epsilon_n=(2n+1)\pi T$, the Fermion Matsubara frequencies. The time-reversed propagators are $\bar{g}^{\,m}=g^m(-\hat{\mathbf{p}},\mathbf{R};\epsilon_n)^*$ and $\bar{f}^{\,m}=f^m(-\hat{\mathbf{p}},\mathbf{R};\epsilon_n)^*$. The particle-hole degree of freedom, which is essential for the description of superfluidity, combined with spin must be given a fully quantum-mechanical treatment. The quasiclassical propagator satisfies a Boltzmann-like transport equation,

$$[i\epsilon_n\hat{\tau}_3-\hat{\Delta}-\hat{\sigma},\hat{g}^m]+iv_f\hat{\mathbf{p}}\cdot\nabla_{\mathbf{R}}\hat{g}^m=0 . \quad (5)$$

The other central equations of the quasiclassical theory are the normalization equation for $\hat{g}^m$,

$$\hat{g}^m(\hat{\mathbf{p}},\mathbf{R};\epsilon_n)^2=-\pi^2\hat{1} , \quad (6)$$

and the self-energy equations for the diagonal ($\hat{\sigma}$) and off-diagonal ($\hat{\Delta}$) self-energies, which are defined diagrammatically in terms of the propagator $\hat{g}^m$ and the quasiparticle interactions that are taken as inputs to the theory. The diagrammatic contributions to $\hat{\Delta}$ and $\hat{\sigma}$ are classified by a small parameter $(T_c/T_f)\ll 1$.[19] The leading order contributions to the self energies

$$\hat{\sigma}=\begin{bmatrix}\sigma & 0\\ 0 & \bar{\sigma}\end{bmatrix},\ \hat{\Delta}=\begin{bmatrix}0 & \Delta\\ \bar{\Delta} & 0\end{bmatrix}, \quad (7)$$

are the quasiparticle mean fields, which are independent of energy $i\epsilon_n$; $\bar{\sigma}$ and $\bar{\Delta}$ are related to σ and Δ by the same symmetries as the corresponding propagators,

$$\sigma(\hat{\mathbf{p}},\mathbf{R})=v(\hat{\mathbf{p}},\mathbf{R})+T\sum_n\int\frac{d\Omega'}{4\pi}[A^s(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')g(\hat{\mathbf{p}}',\mathbf{R};\epsilon_n)\mathbb{1}+A^a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\mathbf{g}(\hat{\mathbf{p}}',\mathbf{R};\epsilon_n)\cdot\boldsymbol{\sigma}] , \quad (8)$$

$$\Delta(\hat{\mathbf{p}},\mathbf{R})=T\sum_n\int\frac{d\Omega'}{4\pi}[V^0(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')f_0(\hat{\mathbf{p}}',\mathbf{R};\epsilon_n)i\sigma^2+V^1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\mathbf{f}(\hat{\mathbf{p}}',\mathbf{R};\epsilon_n)\cdot i\boldsymbol{\sigma}\sigma^2] , \quad (9)$$

where v represents an external field that couples to the quasiparticles. The decomposition of the propagators in spin space is given in Eq. (48). The Landau interaction functions $A^{s,a}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')$ are related to the conventional Fermi liquid parameters $F_l^{s,a}$ by

$$A^{s,a}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')=\sum_{l\geq 0}A_l^{s,a}P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') ,$$
$$A_l^{s,a}=F_l^{s,a}/(1+F_l^{s,a}/2l+1) , \quad (10)$$

and the pairing interactions V^0 (V^1) for the singlet (triplet) channel is parametrized as

$$V^{0(1)}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')=\sum_{l}^{\substack{(\mathrm{odd})\\ \mathrm{even}}}(2l+1)V_lP_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') . \quad (11)$$

The physical transition temperatures for pairing in the lth partial wave,

$$T_{cl}=1.13\epsilon_c e^{-1/V_l} , \quad (12)$$

can always be used to eliminate the ill-defined cutoff parameter ϵ_c and the pairing interaction parameters $\{V_l\}$ that enter the "gap equation" [Eq. (9)]. Equations (5)–(9) define the propagator and molecular field in weak-coupling BCS theory. The static response functions are calculated from the propagator g^m; in particular, the magnetization is given by

$$\mathbf{M}=\chi_N\mathbf{H}+\chi_N(\gamma/2)^{-1}\int\frac{d\Omega}{4\pi}T\sum_n\mathbf{g}(\hat{\mathbf{p}},\mathbf{R};\epsilon_n) , \quad (13)$$

where $\chi_N=2N(0)(\gamma/2)^2/(1+F_0^a)$ is the normal-state susceptibility, $2N(0)$ is the total normal-state quasiparticle density of states at the Fermi surface, and γ is the gyromagnetic ratio that determines the Larmor frequency γH in terms of the external field H.

The effects of a magnetic field on the B phase are calculated perturbatively in the parameter

$$(\gamma H/\Delta)\cong 0.16(H/\mathrm{kG})/(\Delta/\mathrm{mK}) , \quad (14)$$

which is typically small even for fields as large as 1 kG, except for temperatures very close to T_c. To calculate the magnetic field distortion of the order parameter we formally expand $\hat{g}^m$, $\hat{\sigma}$, and $\hat{\Delta}$ in powers of the field:

$$\hat{g}^m=\hat{g}_0^m+\hat{g}_1^m+\hat{g}_2^m+\cdots ,$$
$$\hat{\sigma}=\hat{\sigma}_1+\hat{\sigma}_2+\cdots , \quad (15)$$
$$\hat{\Delta}=\hat{\Delta}_0+\hat{\Delta}_1+\hat{\Delta}_2+\cdots ,$$

where the $|\hat{g}_i^m|\sim(\gamma H/\Delta)|\hat{g}_{i-1}^m|$, etc. Starting from the zero field propagator

$$\hat{g}_0^m=N(\hat{\Delta}_0-i\epsilon_n\hat{\tau}_3) ,$$
$$N=\pi/(\epsilon_n^2+\Delta_0^2)^{1/2} , \quad (16)$$

and zero-field gap matrix (order parameter) for the BW state given in Eq. (2), the finite field corrections to the propagator are obtained by first solving the transport equation and normalization condition, order by order in $(\gamma H/\Delta)$, in terms of the mean field and order parameter. It is convenient to parametrize the diagonal mean field in terms of the total effective field

$$\mathbf{h}(\hat{\mathbf{p}})=\tfrac{1}{2}\,\mathrm{tr}[\boldsymbol{\sigma}\sigma(\hat{\mathbf{p}})]=-(\gamma/2)(1+F_0^a)^{-1}\mathbf{H}+T\sum_n\int\frac{d\Omega'}{4\pi}A^a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\mathbf{g}(\hat{\mathbf{p}}',\epsilon_n) . \quad (17)$$

Similarly, the gap matrix is represented by the vector order parameter,

$$\boldsymbol{\Delta}(\hat{\mathbf{p}})=-\tfrac{1}{2}\,\mathrm{tr}[i\sigma^2\boldsymbol{\sigma}\Delta(\hat{\mathbf{p}})] . \quad (18)$$

It is straightforward to show that σ and Δ are represented only by $\mathbf{h}$ and $\boldsymbol{\Delta}$ provided $\mathbf{H}$ is uniform in space. It is also simple to show that $\mathbf{h}(\boldsymbol{\Delta})$ contains only terms of odd (even) order in $(\gamma H/\Delta_0)$. The resulting perturbative solution for the propagator through third order in $(\gamma H/\Delta_0)$ becomes

$$g_1=N^3/\pi^2(\boldsymbol{\Delta}_0\cdot\mathbf{h}_1)(\boldsymbol{\Delta}_0\cdot\boldsymbol{\sigma}) , \quad (19a)$$

$$f_1=N^3/\pi^2(-i\epsilon_n)(\boldsymbol{\Delta}_0\cdot\mathbf{h}_1)(i\sigma^2) , \quad (19b)$$

$$g_2=(i\epsilon_n)[N^3/\pi^2(\boldsymbol{\Delta}_0\cdot\boldsymbol{\Delta}_2)-\tfrac{3}{2}N^5/\pi^4(\boldsymbol{\Delta}_0\cdot\mathbf{h}_1)^2]\sigma^0 , \quad (19c)$$

$$f_2=\{N^3/\pi^2[(\epsilon_n^2+\Delta_0^2)\mathbf{\Delta}_2-(\mathbf{\Delta}_0\cdot\mathbf{\Delta}_2)\mathbf{\Delta}_0-(\mathbf{\Delta}_0\cdot\mathbf{h}_1)\mathbf{h}_1] + \tfrac{3}{2}N^5/\pi^4(\mathbf{\Delta}_0\cdot\mathbf{h}_1)^2\mathbf{\Delta}_0\}\cdot(i\boldsymbol{\sigma}\sigma^2)\ , \tag{19d}$$

$$g_3=\{N^3/\pi^2[(\mathbf{\Delta}_0\cdot\mathbf{h}_3)\mathbf{\Delta}_0+(\mathbf{\Delta}_2\cdot\mathbf{h}_1)\mathbf{\Delta}_0+(\mathbf{\Delta}_0\cdot\mathbf{h}_1)\mathbf{\Delta}_2] - N^5/\pi^4[3(\mathbf{\Delta}_0\cdot\mathbf{\Delta}_2)(\mathbf{\Delta}_0\cdot\mathbf{h}_1)\mathbf{\Delta}_0 + (\mathbf{\Delta}_0\cdot\mathbf{h}_1)h_1^2\mathbf{\Delta}_0+(\mathbf{\Delta}_0\cdot\mathbf{h}_1)^2\mathbf{h}_1] + \tfrac{5}{2}N^7/\pi^6(\mathbf{\Delta}_0\cdot\mathbf{h}_1)^3\mathbf{\Delta}_0\}\cdot\boldsymbol{\sigma}\ , \tag{19e}$$

$$f_3=(-i\epsilon_n)\{N^3/\pi^2[\mathbf{\Delta}_0\cdot\mathbf{h}_3+\mathbf{\Delta}_2\cdot\mathbf{h}_1] - N^5/\pi^4[3(\mathbf{\Delta}_0\cdot\mathbf{\Delta}_2)(\mathbf{\Delta}_0\cdot\mathbf{h}_1)+(\mathbf{\Delta}_0\cdot\mathbf{h}_1)h_1^2] + \tfrac{5}{2}N^7/\pi^6(\mathbf{\Delta}_0\cdot\mathbf{h}_1)^3\}(i\sigma^2)\ . \tag{19f}$$

Inserting g_1 into Eq. (17) we obtain the self-consistency equation for the first-order effective field,

$$\mathbf{h}_1(\hat{\mathbf{p}})=\mathbf{h}_{\rm ext}+\int\frac{d\Omega'}{4\pi}A^a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\times\{Y_{3/2}[\mathbf{\Delta}_0(\hat{\mathbf{p}}')\cdot\mathbf{h}_1(\hat{\mathbf{p}}')]\mathbf{\Delta}_0(\hat{\mathbf{p}}')\}\ , \tag{20}$$

where

$$\mathbf{h}_{\rm ext}=-(\gamma/2)\mathbf{H}/(1+F_0^a)\ ,\qquad Y_{m/2}=\pi T\sum_n(\epsilon_n^2+\Delta_0^2)^{-m/2},\ \ m>1\ . \tag{21}$$

Note that $\mathbf{h}_1(\hat{\mathbf{p}})$ is necessarily even in $\hat{\mathbf{p}}$. For the BW state Δ_0^2 is independent of $\hat{\mathbf{p}}$ and the equation for $\mathbf{h}_1(\hat{\mathbf{p}})$ is easily solved to obtain the standard result,

$$\mathbf{h}_1=\{[1+(\tfrac{1}{3}+\tfrac{2}{3}y)(F_2^a/5)](1+F_0^a)\mathbf{h}_{\rm ext} + (1-y)(F_2^a/5)[(\mathbf{\Delta}_0\cdot\mathbf{h}_{\rm ext})\mathbf{\Delta}_0/\Delta_0^2-\tfrac{1}{3}\mathbf{h}_{\rm ext}]\times(1+F_0^a)\}/D\ ,$$
$$D=1+(\tfrac{2}{3}+\tfrac{1}{3}y)F_0^a+(\tfrac{1}{3}+\tfrac{2}{3}y)(F_2^a/5)+yF_0^aF_2^a/5\ , \tag{22}$$

where $y=1-\Delta_0^2Y_{3/2}$ is the Yoshida function. Having determined the first-order effective field, the linear susceptibility is easily found to be

$$\chi_1=M_1/H=\chi_N[\tfrac{2}{3}+(\tfrac{1}{3}+F_2^a/5)y](1+F_0^a)/D\ , \tag{23}$$

the result first obtained by Serene and Rainer.[22]

Finite field corrections to the order parameter first appear in second order; the first-order correction from f_1 vanishes when summed over frequency. Thus $\mathbf{\Delta}_2$ is obtained as the solution to the inhomogeneous gap equation,

$$\mathbf{\Delta}_2(\hat{\mathbf{p}})=\int\frac{d\Omega'}{4\pi}V^1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\{[K_0(T)+\ln(T/T_{c1})]\mathbf{\Delta}_2 - Y_{3/2}[(\mathbf{\Delta}_0\cdot\mathbf{\Delta}_2)\mathbf{\Delta}_0+(\mathbf{\Delta}_0\cdot\mathbf{h}_1)\mathbf{h}_1] + \tfrac{3}{2}Y_{5/2}(\mathbf{\Delta}_0\cdot\mathbf{h}_1)^2\mathbf{\Delta}_0\}\ , \tag{24}$$

$$K_0(T)=\pi T\sum_{|\epsilon_n|<\epsilon_c}|\epsilon_n|^{-1}=\ln(1.13\epsilon_c/T)\ . \tag{25}$$

This logarithmically divergent sum also enters the linearized gap equation which defines the transition temperatures for l-wave pairing,

$$K_0(T_{cl})=1/V_l\ . \tag{26}$$

Thus, the pairing interactions and cutoff ϵ_c can be eliminated in favor of the physical transition temperatures. We solve Eq. (24) by projecting out the angular momentum components of $\mathbf{\Delta}_2(\hat{\mathbf{p}})$,

$$\mathbf{\Delta}_{2(l)}(\hat{\mathbf{p}})=(2l+1)\int\frac{d\Omega'}{4\pi}P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\mathbf{\Delta}_2(\hat{\mathbf{p}}')\ , \tag{27}$$

and by introducing traceless and symmetric tensors of ranks $l-1$ and $l+1$ to represent the components of $\hat{\mathbf{p}}\cdot\mathbf{\Delta}_{2(l)}$,

$$\hat{\mathbf{p}}\cdot\mathbf{\Delta}_{2(l)}=\Delta_0B_{[\mu]}^{(l-1,l)}p_{\mu_1}\cdots p_{\mu_{l-1}}+\Delta_0B_{[\mu]}^{(l+1,l)}\cdots p_{\mu_1}p_{\mu_{l+1}}\ , \tag{28}$$

where $\{p_\mu,\mu=1,2,3\}$ are the direction cosines of $\hat{\mathbf{p}}$. We use the notation $[\mu]$ to denote the relevant collection of indices. It is sufficient to solve the equation for $\mathbf{\Delta}_2$ using the reference order parameter $\mathbf{\Delta}_0=\Delta_0\hat{\mathbf{p}}$; since spin-orbit corrections to V^1 are negligible, the physical $\mathbf{\Delta}_2$ is obtained by operating on the reference solution with the rotation $\underline{R}$ that defines the physical $\mathbf{\Delta}_0$.[23] The resulting solution for $\mathbf{\Delta}_2(\hat{\mathbf{p}})$ contains three terms labeled by the total angular momentum (J) and orbital angular momentum (l),

$$\Delta_2^i=\Delta_0\{(B_{01}\delta_{ij}+B_{21}Z_{ij})p_j + B_{23}[(Z_{ij}\delta_{kl}+Z_{ik}\delta_{jl}+Z_{il}\delta_{jk}) - \tfrac{5}{2}(\delta_{ij}Z_{kl}+\delta_{ik}Z_{jl}+\delta_{il}Z_{jk})]p_jp_kp_l\}\ , \tag{29a}$$

$$B_{01}=\frac{1}{3}\left[\left[\frac{\frac{3}{2}\Delta_0^4Y_{5/2}}{1-y}\right]-1\right]\left[\frac{\omega_L}{2\Delta_0}\right]^2, \tag{29b}$$

$$B_{21}=\left[\left[\frac{\frac{3}{2}\Delta_0^4Y_{5/2}}{1-y}\right]-\left[\frac{\frac{5}{2}+F_2^a/5+\frac{3}{2}yF_2^a/5}{1+F_2^a/5}\right] + \tfrac{3}{2}x_3^{-1}\left[\frac{(1-y)(1+yF_2^a/5)}{1+F_2^a/5}\right]\right]\left[\frac{\omega_L}{2\Delta_0}\right]^2, \tag{29c}$$

$$B_{23}=\tfrac{1}{3}x_3^{-1}\left[\frac{(1-y)(1+yF_2^a/5)}{1+F_2^a/5}\right]\left[\frac{\omega_L}{2\Delta_0}\right]^2, \tag{29d}$$

where $Z_{ij}=\hat{\mathbf{z}}_i\hat{\mathbf{z}}_j-\frac{1}{3}\delta_{ij}$, $\omega_L\equiv(\gamma H)(1+F_2^a/5)/D$ is the Larmor frequency corresponding to the effective field, and $x_3=V_1^{-1}-V_3^{-1}=\ln(T_{c3}/T_{c1})$ measures the relative importance of the f-wave pairing interaction to the p-wave interaction. Because of mixing between the $l=1$ and $l=3$ channels with $J=2$, x_3^{-1} enters B_{21} as well as the f-wave component of $\mathbf{\Delta}_2$ proportional to B_{23}. However, x_3^{-1} drops out of the angle-dependent energy gap through order $(\omega_L/2\Delta_0)^2$,

$$|\Delta|^2=\Delta_0^2[1+2B_{01}+(B_{21}-\tfrac{9}{2}B_{23})(p_z^2-\tfrac{1}{3})]\ . \tag{30}$$

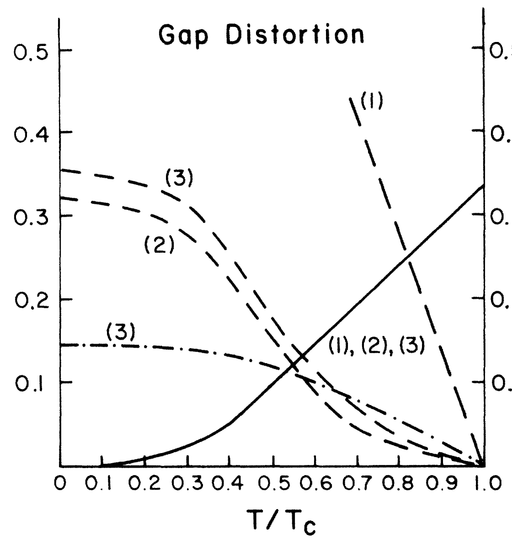

FIG. 1. Gap distortion versus T/T_c. Coefficients of the quadratic gap distortion, Eqs. (29), are plotted versus T/T_c. The solid curve is $[-B_{01}/(\omega_L/2\Delta_0)^2]$, the dashed curves are $[B_{21}/(\omega_L/2\Delta_0)^2+\frac{5}{2}]$, and the dashed-dotted curve is $[-B_{23}/(\omega_L/2\Delta_0)^2]$. The numerical labels refer to the following sets of interactions: (1) $x_3^{-1}=F_2^a=0$, (2) $x_3^{-1}=0$ and $F_2^a=-1.56$, (3) $x_3^{-1}=-0.43$ and $F_2^a=0$. Cases (2) and (3) are different choices of Fermi liquid and f-wave pairing interaction which give the same value for the RSQ mode frequency at $T=0$ and $H=0$; $\omega_0=1.07\Delta_0$.

Furthermore, for the angle average of $|\Delta|^2$,

$$\langle|\Delta|^2\rangle=\Delta_0^2\left[1+\frac{2}{3}\left[\frac{\frac{3}{2}\Delta_0^4Y_{5/2}}{1-y}-1\right]\left[\frac{\omega_L}{2\Delta_0}\right]^2\right], \tag{31}$$

only the Fermi liquid corrections that define ω_L contribute. Even these terms drop out at $T=0$, leaving Δ distorted, but $\langle|\Delta|^2\rangle=\Delta_0^2$ unperturbed to order $(\omega_L/2\Delta_0)^2$. Finally we note that if we neglect quasiparticle interaction effects by setting F_2^a and x_3^{-1} to zero then our result for Δ_2 agrees with Eq. (24) of Tewordt and Schopohl[12] at $T=0$. In Fig. 1 we show the temperature dependence of the gap distortion for realistic values of the quasiparticle interaction parameters. Note that there is a sizeable renormalization of the $l=1$, $J=2$ gap distortion by quasiparticle interactions.

The nonlinear magnetic susceptibility in ^{3}He-B is sensitive to the distortion of the order parameter $\mathbf{\Delta}_2(\hat{\mathbf{p}})$; thus both f-wave pairing effects and higher order Fermi liquid terms contribute to the nonlinear susceptibility. The susceptibility is formally expanded in powers of $(\gamma H/\Delta_0)$, $\chi=\chi_1+\chi_3+\cdots$. The leading nonlinear correction to χ_1,

$$\chi_3=(\gamma/2)^{-1}\chi_N\int\frac{d\Omega}{4\pi}\bar{g}_3^z/H\ , \tag{32}$$

is of order $\chi_3\sim(\gamma H/\Delta_0)^2\chi_1$. It is convenient to define the frequency summed propagator,

$$\bar{g}_3^i=T\sum_n\tfrac{1}{2}\,\mathrm{tr}(\sigma^i g_3)=Y_{3/2}(\mathbf{\Delta}_0\cdot\mathbf{h}_3)\Delta_0^i+R^i(\hat{\mathbf{p}})\ . \tag{33}$$

We explicitly separate out the third-order contribution to the effective field; $R^i(\hat{\mathbf{p}})$ represents the remaining terms in Eq. (19e). The calculation of χ_3 is straightforward except that $\mathbf{h}_3$ is unknown *a priori*, but fortunately it can be eliminated from Eq. (32) without direct calculation. The general form of the third order effective field is

$$h_3^i=G^i+G_{\alpha\beta}^i p_\alpha p_\beta+\cdots\ , \tag{34}$$

where $G^i(G_{\alpha\beta}^i)$ is the $l=0$ ($l=2$) contribution to h_3^i given by

$$G^i=A_0^a\langle\bar{g}_3^i\rangle\ , \tag{35a}$$

$$G_{\alpha\beta}^i=\tfrac{3}{2}A_2^a(\langle p_\alpha p_\beta\bar{g}_3^i\rangle-\tfrac{1}{3}\delta_{\alpha\beta}\langle\bar{g}_3^i\rangle)\ , \tag{35b}$$

and $\langle\cdots\rangle\equiv\int d\Omega/4\pi(\cdots)$. There is an $l=4$ contribution to $h_3^i(\hat{\mathbf{p}})$, but it does not contribute to χ_3. The susceptibility is independent of the rotation $\underline{R}(\hat{n},\theta)$ that defines the BW state, so we calculate χ_3 with the reference order parameter $\Delta_0^i=\Delta_0 p_i$, in which case χ_3 is proportional to

$$\langle\bar{g}_3^z\rangle=(\Delta_0^2Y_{3/2})(\tfrac{1}{3}G^z+\tfrac{2}{15}G_{iz}^i)+\langle R^z\rangle\ . \tag{36}$$

The $l=0$ effective field is eliminated with Eq. (35a); the $l=2$ effective field is eliminated by using the identity,

$$\langle(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})p_i\bar{g}_3^i\rangle=\langle g_3^z\rangle-\langle R^z\rangle+\langle(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})(p_iR^i)\rangle\ , \tag{37}$$

which follows from the *form* of the coupling of $\mathbf{h}_3$ to $\bar{g}_3^i$ in Eq. (33). The resulting susceptibility is determined by moments of the function $R^i(\hat{\mathbf{p}})$,

$$\chi_3=(\gamma/2)^{-1}\chi_N(1+F_0^a)(1+F_2^a/5)D^{-1}\times[\langle R^z\rangle+(\Delta_0^2Y_{3/2})(A_2^a/5)\langle(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})p_iR^i-R^z\rangle]\ . \tag{38}$$

It is now straightforward to show that χ_3 is given by

$$\chi_3=\tfrac{4}{3}\chi_N(1+F_0^a)(1+F_2^a/5)^2D^{-2}\left[\frac{\omega_L}{2\Delta_0}\right]^2\times\left\{(\Delta_0^2Y_{3/2})[1-A+\tfrac{3}{10}A^2-\tfrac{1}{2}x_3^{-1}(\Delta_0^2Y_{3/2})(1-A)^2]+(\Delta_0^4Y_{5/2})\left[-1+\frac{27}{40}\left[\frac{\Delta_0^4Y_{5/2}}{\Delta_0^2Y_{3/2}}\right]+\tfrac{1}{5}A+\tfrac{1}{5}A^2\right]-\tfrac{3}{8}(\Delta_0^6Y_{7/2})\right\}, \tag{39}$$

where $A=(\Delta_0^2Y_{3/2})(A_2^a/5)$. This result for the quadratic field correction to the B-phase susceptibility is exact in weak-coupling BCS theory. At zero temperature Eq. (39) reduces to

$$\chi_3=\tfrac{2}{3}\chi_N(1+F_0^a)D^{-2}\left[\frac{\omega_L}{2\Delta_0}\right]^2(\tfrac{13}{15}-x_3^{-1})\ . \tag{40}$$

Tewordt and Schopohl[12] give an incorrect expression for

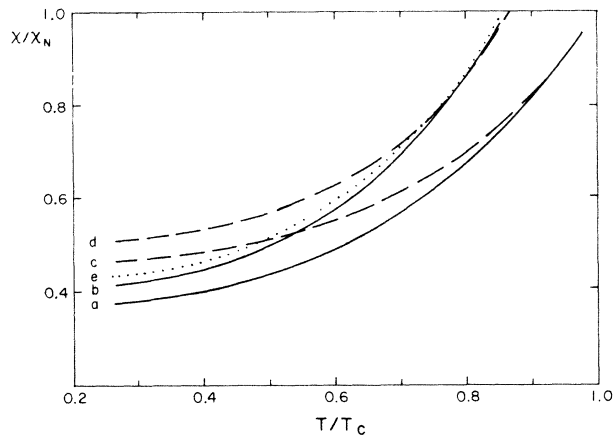

FIG. 2. Susceptibility versus T/T_c. B-phase susceptibility is shown for magnetic fields of $\gamma H/2\pi = 1$ MHz (curves a and c) and $\gamma H/2\pi = 10$ MHz (curves b, d, and e), and $F_0^a = -0.7$. The solid curves were calculated with $F_2^a = x_3^{-1} = 0$, while the two dashed curves correspond to $F_2^a = -1.56$ and $x_3^{-1} = 0$. The dotted curve e was calculated with $F_2^a = 0$ and $x_3^{-1} = -0.43$. Note that the interactions used to calculate d and e give the same value for the RSQ mode frequency, $\omega_0 = 1.07\Delta_0$, at $T = 0$.

χ_3, with $F_2^a = x_3^{-1} = 0$ at $T = 0$.

Equation (39) also simplifies in the high-temperature limit, $T \lesssim T_c$,

$$\chi_3(T \to T_c -) = \tfrac{7}{24}\zeta(3)\chi_N(1+F_0^a)^{-3}(\gamma H/\pi T_c)^2 , \qquad (41)$$

which is independent of x_3^{-1} and F_2^a. However, this result is valid only in the restricted range of temperature and fields satisfying $\gamma H \ll \Delta_0(T) \ll \pi T_c$, in which case the χ_3 correction is exceedingly small. Higher angular momentum pairing correlations and Fermi liquid effects are only exhibited away from the Ginzburg-Landau region of the phase diagram.

In Fig. 2 we show the temperature and field dependence of $\chi(T,H) = \chi_1 + \chi_3$ calculated from Eqs. (23) and (39).

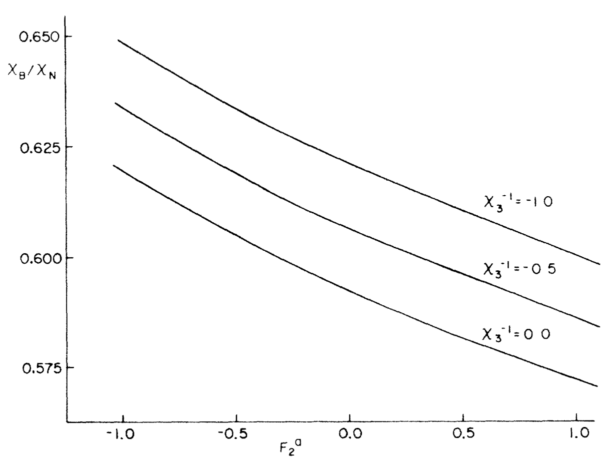

FIG. 3. Interaction dependence of the nonlinear susceptibility. This figure shows $\chi(H,T)/\chi_N$ vs F_2^a and $x_3^{-1}$for $\gamma H/2\pi = 8$ MHz, $T/T_c = 0.65$, with $T_c = 1.04$ mK and $F_0^a = -0.7$ appropriate for low pressures.

For the highest field shown, $\gamma H/2\pi = 10$ MHz, the expansion parameter is still small; $(\gamma H/\Delta_0)^2 = 0.07$ at $T = 0$ and becomes of order 1 only for $T/T_c \gtrsim 0.98$, well above the A-B transition at this field. The general effect of a finite magnetic field is to increase the susceptibility by depairing the $S_z = 0$ Cooper pairs. Attractive f-wave pairing and negative Fermi liquid interactions enhance this effect, as shown in Fig. 3.

Schopohl[24] has recently calculated the nonlinear susceptibility numerically for $x_3^{-1} = 0$. Our perturbative result for $\chi(H,T)$ compares favorably with Schopohl at low fields, but does not show the double-valuedness obtained by Schopohl for very high fields. This double-valued behavior is unphysical since it appears only for temperatures and fields where the B phase is thermodynamically unstable.

III. MAGNETIC FIELD DEPENDENCE AND DISPERSION OF THE COLLECTIVE MODES IN ^{3}He-B

We extend the work of Sauls and Serene[6,13] on the collisionless dynamics of ^{3}He-B by deriving quasiclassical equations for the order-parameter collective modes, valid to order $(\gamma H/\Delta)^2$. We solve these equations for both the nonlinear Zeeman splittings and the field-dependent dispersion splittings of the RSQ modes.

As noted earlier a magnetic field lifts the fivefold degeneracy of the $J = 2+$ modes. For weak fields, $\gamma H \ll \Delta$, the degeneracy splitting is linear in the field. A larger magnetic field distorts the $l = 1$ order parameter and generates an $l = 3$ component of order $(\gamma H/\Delta)^2$. This "gap distortion" is partly responsible for the nonlinear Zeeman splitting of the $2+$ modes.[2] Like the nonlinear susceptibility, the Zeeman splitting of the $2+$ modes depends on the $l = 0$ and $l = 2$ Fermi liquid parameters and the $l = 3$ pairing interaction. Our result for the nonlinear Zeeman effect includes all interaction effects that enter the weak coupling theory. Perturbation theory reveals that to first order in the field, the real squashing (RSQ) modes are no longer purely $J = 2$, but also contain $J = 1$ and $J = 3$ components. The $1+$ and $3+$ amplitudes lead to additional nonlinear field dependence of the mode frequencies; this mode repulsion weakens the effect of the gap distortion. In addition to the interaction parameters that enter the linear field splitting $-F_0^a, F_2^a, x_3^{-1}$—the nonlinear splitting also depends on F_4^a, albeit weakly. If we neglect all Fermi liquid and pairing interactions except F_0^a, our results reduce to those of Schopohl *et al.* only when numerous terms are neglected in the "effective Hamiltonian" of our eigenvalue equations (see below).

In zero field, but finite wave vector, the degeneracy of the RSQ modes is again partially lifted. The low field data of Shivaram *et al.*[1] reveals this dispersion splitting of the RSQ modes; the RSQ modes are excited at finite wave vector $q = \omega/c_0$, where c_0 is the velocity of zero sound. The quantization axis is determined by $\hat{\mathbf{q}}$, the direction of propagation of zero sound, and the modes split into three groups labeled by $|m_q|$. As the magnetic field is increased the quantization axis rotates from the $\hat{\mathbf{q}}$ direction to the $\hat{\mathbf{z}}$ direction; the dispersion splitting of the modes also changes. The dispersion effect depends on the Fermi liquid interactions F_1^a, F_3^a, F_2^a, as well as the f-wave (x_3^{-1}) and d-wave (x_2^{-1}) pairing interactions. The dependence of the RSQ modes on x_2^{-1} results from mode repulsion between the $J=2+$ modes and the $S=0$, $l=2$ modes. Detailed measurements of the mode dispersion can in principle yield information about the presently unknown d-wave pairing interaction and Fermi liquid parameter $F_{1,3}^a$. We find that the mode frequencies approach their zero-field limits without crossing. For zero field, in the absence of superflow, only the $m_q=0$ mode couples to zero sound. Our result also shows that for $\hat{\mathbf{q}}\perp\hat{\mathbf{z}}$ the $m_q=0$ mode evolves into the $m_z=+2$ mode for high magnetic fields. We postpone detailed examination of the mode dispersion and consider first the field dependence of the 2+ modes for $\mathbf{q}=0$.

A. Quasiclassical equations for the RSQ modes in a magnetic field

A theoretical description of collective excitations in superfluid ^{3}He requires a nonequilibrium generalization of the quasiclassical theory. The quasiclassical kinetic equations were originally derived by Larkin and Ovchinnikov.[17] We follow the review by Serene and Rainer.[21] The central equations are the (1) nonequilibrium transport equation,

$$\epsilon\hat{\tau}_3\hat{g}^K(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')-\hat{g}^K(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')\hat{\tau}_3\epsilon'+iv_F\hat{\mathbf{p}}\cdot\nabla_{\mathbf{R}}\hat{g}^K(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')-(\hat{\sigma}^R+\hat{v})\cdot\hat{g}^K(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')+\hat{g}^K\cdot(\hat{\sigma}^A+\hat{v})(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')$$
$$-\hat{\sigma}^K\cdot\hat{g}^A(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')+\hat{g}^R\cdot\hat{\sigma}^K(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')=0\ , \tag{42}$$

(2) the normalization conditions for the Keldysh ($\hat{g}^K$), retarded ($\hat{g}^R$), and advanced ($\hat{g}^A$) propagators,

$$\hat{g}^{R(A)}\cdot\hat{g}^{R(A)}(\hat{\mathbf{p}},\mathbf{R};\epsilon,\epsilon')=-2\pi^3\delta(\epsilon-\epsilon')\hat{1}\ , \tag{43}$$

with

$$A\cdot B(\epsilon,\epsilon')\equiv\int\frac{d\epsilon''}{2\pi}A(\epsilon,\epsilon'')B(\epsilon'',\epsilon')\ ,$$

and (3) the self-energy equations for $\hat{\sigma}^{R,A,K}$. In weak-coupling theory the self-energies are given by the mean field equations: $\hat{\sigma}^R=\hat{\sigma}^A=\hat{\sigma}$ and $\hat{\sigma}^K=0$, with

$$\sigma(\hat{\mathbf{p}},\mathbf{R};\omega)=\int\frac{d\epsilon'}{4\pi i}\int\frac{d\Omega'}{4\pi}[A^s(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')g^K(\hat{\mathbf{p}}',\mathbf{R};\epsilon'+\omega/2,\epsilon'-\omega/2)\mathbb{1}+A^a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\mathbf{g}^K(\hat{\mathbf{p}}',\mathbf{R};\epsilon'+\omega/2,\epsilon'-\omega/2)\cdot\boldsymbol{\sigma}]\ , \tag{44}$$

$$\Delta(\hat{\mathbf{p}},\mathbf{R};\omega)=\int\frac{d\epsilon'}{4\pi i}\int\frac{d\Omega'}{4\pi}[V^0(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')f_0^K(\hat{\mathbf{p}}',\mathbf{R};\epsilon'+\omega/2,\epsilon'-\omega/2)(i\sigma^2)+V^1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\mathbf{f}^K(\hat{\mathbf{p}}',\mathbf{R};\epsilon'+\omega/2,\epsilon'-\omega/2)\cdot(i\boldsymbol{\sigma}\sigma^2)]\ . \tag{45}$$

What is left out of the weak-coupling theory are (i) quasiparticle collision effects and (ii) strong-coupling corrections to the mean field self-energy. These omissions restrict the kinetic equations to high frequencies $\omega\tau_{QP}\gg 1$. Since our interest is the collective mode spectrum in the frequency regime $\omega\sim\Delta$, $\omega\tau_{QP}\gg 1$ is always satisfied except in a very narrow temperature interval close to T_c. Thus, the damping of the collective modes can in principle be calculated perturbatively in $(\omega\tau_{QP})^{-1}$. The neglect of strong-coupling corrections is potentially more serious. At least the omission of strong-coupling corrections restricts the theory to low pressures where all strong-coupling effects are supposed negligible.

The diagonal components of $\hat{g}^K(\hat{\mathbf{p}},\mathbf{R};\epsilon+\omega/2,\epsilon-\omega/2)$, first introduced by Keldysh, are simply related to the distribution function for quasiparticles with energy ϵ traveling with velocity $v_f\hat{\mathbf{p}}$. In equilibrium for $\mathbf{H}=0$, $g_{eq}^K=2\pi\delta(\epsilon-\epsilon')g^{eq}(\hat{\mathbf{p}},\epsilon)$ is simply proportional to the normalized BCS density of states and the Fermi distribution [Eq. (51)]. The off-diagonal Keldysh function f^K has the interpretation of the time-dependent pair amplitude. The equilibrium, zero-field pair amplitude may be expressed directly in terms of the equilibrium gap matrix $\Delta=\boldsymbol{\Delta}_0\cdot i\boldsymbol{\sigma}\sigma^2$. Quite generally the static Keldysh propagator may be obtained by analytic continuation of the Matsubara propagator to the real frequency axis,

$$g^{eq}=[g^m(\hat{\mathbf{p}},\mathbf{R};\epsilon_n=-i\epsilon+\eta)-g^m(\hat{\mathbf{p}},\mathbf{R};\epsilon_n=-i\epsilon-\eta)]_{\eta=0^+}\tanh\left[\frac{\beta\epsilon}{2}\right]\ . \tag{46}$$

This equation is used to construct the equilibrium Keldysh functions in a magnetic field, which are *inputs* to the linearized kinetic equation for the nonequilibrium Keldysh propagator. The linearized version of Eq. (42) for $\hat{g}^K$ has a much simpler structure since the internal energy integrals are trivial. In terms of the equilibrium Keldysh function g^{eq} and mean field σ^{eq}, the linearized kinetic equation becomes (we drop the superscript K from here on)

$$[\epsilon\hat{\tau}_3,\delta\hat{g}]+\left\{\frac{\omega}{2}\hat{\tau}_3,\delta\hat{g}\right\}-[\hat{\sigma}^{\mathrm{eq}},\delta\hat{g}]+iv_f\hat{\mathbf{p}}\cdot\nabla_{\mathbf{R}}\delta\hat{g}$$
$$-(\hat{v}+\delta\hat{\sigma})\hat{g}^{\mathrm{eq}}(\epsilon-\omega/2)+\hat{g}^{\mathrm{eq}}(\epsilon+\omega/2)(\hat{v}+\delta\hat{\sigma})=0\ , \tag{47}$$

where the square brackets and large curly brackets represent the usual matrix commutator and anticommutator, and $\delta\hat{g}(\hat{\mathbf{p}},\mathbf{R};\epsilon,\omega)$ and $\delta\hat{\sigma}(p,\mathbf{R};\omega)$ are functions of the quasiparticle energy ϵ and the frequency ω of the external field $\hat{v}(\hat{\mathbf{p}},\mathbf{R};\omega)$.

Equations (44), (45), and (47), with $\hat{g}^{\mathrm{eq}}$ and $\hat{\sigma}^{\mathrm{eq}}$ supplied as inputs, describe all the collisionless collective modes of superfluid ^{3}He, at least in the weak-coupling approximation. There are numerous collective modes corresponding to oscillations of the spin and particle-hole components of the time-dependent mean field $\delta\hat{\sigma}$. To exhibit these collective excitations we expand the diagonal propagator in scalar and vector components, and the off-diagonal propagator in singlet and triplet components,

$$\delta\hat{g}=\begin{bmatrix}\delta g+\delta\mathbf{g}\cdot\boldsymbol{\sigma} & (\delta f+\delta\mathbf{f}\cdot\boldsymbol{\sigma})i\sigma_2\\ i\sigma_2(\delta\bar{f}+\delta\bar{\mathbf{f}}\cdot\boldsymbol{\sigma}) & \delta\bar{g}+\delta\bar{\mathbf{g}}\cdot\boldsymbol{\sigma}^{\mathrm{tr}}\end{bmatrix}, \tag{48}$$

and similarly for the time-dependent mean fields,

$$\delta\hat{\sigma}=\begin{bmatrix}e+\mathbf{e}\cdot\boldsymbol{\sigma} & (d+\mathbf{d}\cdot\boldsymbol{\sigma})i\sigma^2\\ i\sigma_2(\bar{d}+\bar{\mathbf{d}}\cdot\boldsymbol{\sigma}) & \bar{e}+\bar{\mathbf{e}}\cdot\boldsymbol{\sigma}^{\mathrm{tr}}\end{bmatrix}, \tag{49}$$

where the time-reversed functions are given by

$$\begin{aligned}&\bar{g}=g(-\hat{\mathbf{p}},\mathbf{R};-\epsilon,\omega),\ \ \bar{\mathbf{g}}=\mathbf{g}(-\hat{\mathbf{p}},\mathbf{R};-\epsilon,\omega)\ ,\\ &\bar{f}=-f(-\hat{\mathbf{p}},\mathbf{R};-\epsilon,-\omega)^*,\ \ \bar{\mathbf{f}}=\mathbf{f}(-\hat{\mathbf{p}},\mathbf{R};-\epsilon,-\omega)^*\ ,\\ &\bar{e}=e(-\hat{\mathbf{p}},\mathbf{R};\omega),\ \ \bar{\mathbf{e}}=\mathbf{e}(-\hat{\mathbf{p}},\mathbf{R};\omega)\ ,\\ &\bar{d}=d(\hat{\mathbf{p}},\mathbf{R};-\omega)^*,\ \ \bar{\mathbf{d}}=\mathbf{d}(\hat{\mathbf{p}},\mathbf{R};-\omega)^*\ .\end{aligned} \tag{50}$$

These equations eliminate redundancy contained in the 4×4 matrix formulation of the quasiclassical kinetic equations. It is convenient to introduce sum and difference functions, $A^{\pm}\equiv A\pm\bar{A}$, since these new functions have simple transformation rules under $(\hat{\mathbf{p}},\epsilon)\rightarrow(-\hat{\mathbf{p}},-\epsilon)$. For instance $e^{\pm}(-\hat{\mathbf{p}})=\pm e^{\pm}(\hat{\mathbf{p}})$, and $\mathbf{d}^+(\mathbf{d}^-)$ represents the real (imaginary) part of the time-dependent spin-triplet order parameter. Acoustic and magnetic modes of the superfluid correspond to natural oscillations of $e^{\pm}$ and $\mathbf{e}^{\pm}$. These modes are, in general, coupled to order parameter collective modes corresponding to oscillations of the time-dependent singlet and triplet gap functions $d^{\pm}$ and $\mathbf{d}^{\pm}$. Although much of our analysis may be applied to any of the collective oscillations of superfluid ^{3}He, we concentrate on the RSQ modes of ^{3}He-B which have been studied most thoroughly using ultrasound.[4] For zero wave vector, the RSQ modes correspond to coherent oscillations of the order parameter $\mathbf{d}^+$ and the spin-dependent mean field $\mathbf{e}^+$. The equations of motion governing the RSQ modes in a magnetic field are obtained from Eqs. (44)–(47), and the analytic continuation to real frequency of Eqs. (19) for the equilibrium Keldysh propagators,

$$\begin{aligned}&g_0^{\mathrm{eq}}=-2i\bar{N}\epsilon\tanh\left[\frac{\beta\epsilon}{2}\right],\\ &f_0^{\mathrm{eq}}=2i\bar{N}\epsilon\tanh\left[\frac{\beta\epsilon}{2}\right]\boldsymbol{\Delta}_0\cdot i\boldsymbol{\sigma}\sigma_2\ ,\\ &g_1^{\mathrm{eq}}=-2i\frac{\bar{N}^3}{\pi^2}\tanh\left[\frac{\beta\epsilon}{2}\right]\boldsymbol{\Delta}_0\cdot\boldsymbol{\sigma}\mathbf{h}_1\cdot\boldsymbol{\Delta}\ ,\\ &f_1^{\mathrm{eq}}=-2\frac{\bar{N}^3}{\pi^2}\epsilon\tanh\left[\frac{\beta\epsilon}{2}\right]\mathbf{h}_1\cdot\boldsymbol{\Delta}_0\sigma_2\ ,\\ &g_2^{\mathrm{eq}}=-2i\frac{\bar{N}^3}{\pi^2}\epsilon\tanh\left[\frac{\beta\epsilon}{2}\right]\left[\boldsymbol{\Delta}_0\cdot\boldsymbol{\Delta}_2+\frac{3}{2}\frac{\bar{N}^2}{\pi^2}(\mathbf{h}_1\cdot\boldsymbol{\Delta}_0)^2\right],\\ &f_2^{\mathrm{eq}}=2i\bar{N}\tanh\left[\frac{\beta\epsilon}{2}\right]\left[\boldsymbol{\Delta}_2+\frac{\bar{N}^2}{\pi^2}[\boldsymbol{\Delta}_0(\boldsymbol{\Delta}_0\cdot\boldsymbol{\Delta}_2)+\mathbf{h}_1(\mathbf{h}_1\cdot\boldsymbol{\Delta}_0)]\right.\\ &\qquad\left.+\frac{3}{2}\frac{\bar{N}^4}{\pi^4}\boldsymbol{\Delta}_0(\mathbf{h}_1\cdot\boldsymbol{\Delta}_0)^2\right]\cdot i\boldsymbol{\sigma}\sigma_2\ ,\end{aligned} \tag{51}$$

with

$$\bar{N}=\Theta(\epsilon^2-|\Delta_0|^2)\frac{\pi\,\mathrm{sign}(\epsilon)}{(\epsilon^2-|\Delta_0|^2)^{1/2}}\ . \tag{52}$$

The equations of motion for the RSQ modes are obtained from the solution to the transport equation (47), with Eqs. (51) and (52) as input, and the self-consistency equations (44) and (45) for the time-dependent mean field and order parameter. We can write the homogeneous eigenvalue equation governing the RSQ modes in the matrix form

$$\underset{\sim}{d}(\hat{\mathbf{p}})=\underline{L}(\hat{\mathbf{p}},\hat{\mathbf{p}}';\omega)*\underset{\sim}{d}(\hat{\mathbf{p}}')\ , \tag{53}$$

where $\underset{\sim}{d}^{\mathrm{tr}}=(\mathbf{d}^+,\mathbf{e}^+)$ is a six-component vector and the star product is defined as

$$[\underline{L}(\hat{\mathbf{p}},\hat{\mathbf{p}}';\omega)*\underset{\sim}{d}(\hat{\mathbf{p}}')]_k=\int\frac{d\Omega'}{4\pi}L(\hat{\mathbf{p}},\hat{\mathbf{p}}';\omega)_{kj}d_j(\hat{\mathbf{p}}')\ . \tag{54}$$

The matrix $\underline{L}$ is expressed in terms of four scalar operators,

$$\underline{L}(\hat{\mathbf{p}},\hat{\mathbf{p}}';\omega)=\begin{bmatrix}V^1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')L_1(\hat{\mathbf{p}}') & V^1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')L_2(\hat{\mathbf{p}}')\\ F^a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')L_3(\hat{\mathbf{p}}') & F^a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')L_4(\hat{\mathbf{p}}')\end{bmatrix}, \tag{55}$$

where $V^1(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')$ and $F^a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')$ are, respectively, the pairing interaction in the triplet channel and the exchange interaction in the particle-hole channel. The operators $L_i(\hat{\mathbf{p}}';\omega)$, given in Appendix A, are functions of the equilibrium order parameter, the effective field, and the frequency ω.

The eigenvalue equation (53) can be solved perturbatively in the magnetic field by expanding the operator $\underline{L}$, the eigenfunction $\underset{\sim}{d}$, and the eigenfrequency ω in powers of $(\gamma H/\Delta_0)$,

$$\begin{aligned}&\underline{L}(\omega)=\underline{L}^{(0)}+\underline{L}^{(1)}+\underline{L}^{(2)}\ ,\\ &\underset{\sim}{d}=\underset{\sim}{d}_0+\underset{\sim}{d}_1+\underset{\sim}{d}_2+\cdots\ ,\\ &\omega=\omega_0+\omega_1+\omega_2+\cdots\ ,\end{aligned} \tag{56}$$

where $|\omega_1| \sim (\gamma H/\Delta_0)|\omega_0|$, etc. Equation (53) separates order by order into the equations

$$\underset{\sim}{d}_0 = \underline{L}^{(0)}(\omega_0) * \underset{\sim}{d}_0 , \tag{57a}$$

$$\underset{\sim}{d}_1 = \underline{L}^{(0)}(\omega_0) * \underset{\sim}{d}_1 + \underline{L}^{(1)}(\omega_0) * \underset{\sim}{d}_0 + \omega_1 \frac{\partial \underline{L}^{(0)}}{\partial \omega_0} * \underset{\sim}{d}_0 , \tag{57b}$$

$$\underset{\sim}{d}_2 = \underline{L}^{(0)}(\omega_0) * \underset{\sim}{d}_2 + \underline{L}^{(1)}(\omega_0) * \underset{\sim}{d}_1 + \underline{L}^{(2)}(\omega_0) * \underset{\sim}{d}_0 + \omega_2 \frac{\partial \underline{L}^{(0)}}{\partial \omega_0} * \underset{\sim}{d}_0 + \omega_1 \frac{\partial \underline{L}^{(1)}}{\partial \omega_0} * \underset{\sim}{d}_0 + \frac{\omega_1^2}{2} \frac{\partial^2 \underline{L}^{(0)}}{\partial \omega_0^2} * \underset{\sim}{d}_0 + \omega_1 \frac{\partial \underline{L}^{(0)}}{\partial \omega_0} * \underset{\sim}{d}_1 , \tag{57c}$$

where the derivative terms result from the ω dependence of $\underline{L}(\omega)$. Because $\underline{L}$ is not Hermitian it is necessary to solve the transposed eigenvalue equation,

$$\underset{\sim}{b}(\hat{\mathbf{p}}) = \underset{\sim}{b}(\hat{\mathbf{p}}') * \underline{L}(\hat{\mathbf{p}}', \hat{\mathbf{p}}; \bar{\omega}) , \tag{58}$$

for the transposed eigenfunction $\underset{\sim}{b}(\hat{\mathbf{p}})$ (which is not $\underset{\sim}{d}^{\mathrm{tr}}$) and eigenfrequency $\bar{\omega}$. It is easily proven by induction, for solutions $\underset{\sim}{d}$ and $\underset{\sim}{b}$ with the same spin and orbital symmetry, that $\bar{\omega} = \omega$ to all orders in perturbation theory. Having solved Eq. (57a) and the corresponding equation for $\underset{\sim}{b}_0$, we can calculate the first-order quantities, $\omega_1, \underset{\sim}{d}_1, \underset{\sim}{b}_1$. The second-order correction to the mode frequency, ω_2, which gives the quadratic nonlinear Zeeman splitting of the collective modes, can then be computed:

$$\omega_1 = -N_1/D_\omega , \tag{59a}$$

$$N_1 = \underset{\sim}{b}_0 * \underline{L}^{(1)}(\omega_0) * \underset{\sim}{d}_0 , \tag{59b}$$

$$\omega_2 = -N_2/D_\omega , \tag{59c}$$

$$N_2 = \underset{\sim}{b}_0 * \underline{L}^{(2)}(\omega_0) * \underset{\sim}{d}_0 + \underset{\sim}{b}_0 * \underline{L}^{(1)}(\omega_0) * \underset{\sim}{d}_1 + \omega_1 \underset{\sim}{b}_0 * \frac{\partial \underline{L}^{(1)}}{\partial \omega_0} * \underset{\sim}{d}_0 + \frac{\omega_1^2}{2} \underset{\sim}{b}_0 * \frac{\partial^2 \underline{L}^{(0)}}{\partial \omega_0^2} * \underset{\sim}{d}_0 + \omega_1 \underset{\sim}{b}_0 * \frac{\partial \underline{L}^{(0)}}{\partial \omega_0} * \underset{\sim}{d}_1 , \tag{59d}$$

$$D_\omega = \underset{\sim}{b}_0 * \frac{\partial \underline{L}^{(0)}}{\partial \omega_0} * \underset{\sim}{d}_0 . \tag{59e}$$

The solutions to the zero-field eigenvalue equation [Eq. (57a)], given in Ref. 6, are classified by the total angular momentum J of the excited pair amplitude $\mathbf{d}_0^+$. The RSQ modes in zero field are superpositions of pair amplitudes with $l=1$ and $l=3$, both with $J=2$; the general form of the $J=2$ order parameter is

$$d_{0j}^+ = B_{ju}^{(2,1)} p_u + \tfrac{5}{3} B_{uv}^{(2,3)} p_u p_v p_j - \tfrac{2}{3} B_{uj}^{(2,3)} p_u , \tag{60}$$

where $B_{uv}^{(2,l)}$ are traceless and symmetric tensors reflecting the $J=2$ symmetry of $\mathbf{d}_0^+(\hat{\mathbf{p}})$. The $l=1$ and $l=3$ orbital components are related by

$$B_{uv}^{(2,3)} = B_{uv}^{(2,1)} \left[\frac{5\omega_0^2}{8(\Delta_0)^2} - 1 - \frac{\lambda F_2^a}{5} W \right] \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} , \tag{61}$$

where

$$\lambda(\omega,\beta) = \int_{\Delta_0}^{\infty} d\epsilon \frac{\tanh\left[\frac{\beta\epsilon}{2}\right]}{\epsilon^2 - \left[\frac{\omega}{2}\right]^2} \frac{(\Delta_0)^2}{[\epsilon^2 - (\Delta_0)^2]^{1/2}} ,$$

and $W \equiv 1 - (\omega/2\Delta_0)^2$. The technique for solving the eigenvalue equation follows closely the paper of Sauls and Serene;[6] we expand the spin-dependent mean field in spherical tensors.[25] Only the $J=2$ component is nonzero for the case of the RSQ modes, in which case $e_{0j}^+ = E_{j,uv} p_u p_v$ where $E_{j,uv}$ is related to the order parameter by

$$\check{E}_{uv} = \tfrac{1}{2}(\epsilon_{ujk} E_{k,jv} + \epsilon_{vjk} E_{k,ju}) , \tag{62}$$

$$\check{E}_{uv} = \frac{5i\omega_0}{4\Delta_0} B_{uv}^{(2,1)} \frac{\lambda F_2^a}{5} W \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} . \tag{63}$$

Equations (60)—(63) solve the zero-field eigenvalue problem for any traceless and symmetric tensor $B_{uv}^{(2,1)}$ (i.e., any $J=2$ order parameter) and yield the eigenfrequency

$$\omega_0^2 = \tfrac{8}{5}\Delta_0^2 \left\{ 1 + W\lambda \left[\tfrac{1}{5} F_2^a + \frac{5}{8} \left[\frac{\omega_0}{\Delta_0} \right]^2 x_3^{-1} \right] \right\} , \tag{64}$$

which reduces to the well-known limit $\omega_0^2 = \frac{8}{5}\Delta_0^2$ when interactions are neglected.

A similar analysis yields the solution to the zero-field transposed equation $\underset{\sim}{b}_0 = \underset{\sim}{b}_0 * \underline{L}^{(0)}$, with $\underset{\sim}{b} \equiv (\mathbf{b}^+, \boldsymbol{\rho}^+)$,

$$b_{0j}^+ = \bar{B}_{ju}^{(2,1)} p_u + \tfrac{5}{3} \bar{B}_{uv}^{(2,3)} p_u p_v p_j - \tfrac{2}{3} \bar{B}_{uj}^{(2,3)} p_u , \tag{65}$$

$$\bar{B}_{uv}^{(2,3)} = \frac{V_1}{V_3} \bar{B}_{uv}^{(2,1)} \left[\frac{5\omega_0^2}{8(\Delta_0)^2} - 1 - \frac{\lambda F_2^a}{5} W \right] \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} , \tag{66}$$

$$\rho_{0j}^+ = \bar{E}_{j,uv} p_u p_v , \tag{67}$$

$$\check{\bar{E}}_{uv} = \frac{5i\omega_0}{4\Delta_0} \lambda V_1 \bar{B}_{uv}^{(2,1)} W \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} . \tag{68}$$

From the zeroth-order solutions we obtain for the denominator, common to the frequency shifts ω_1 and ω_2,

$$D_\omega = \frac{5\omega_0}{24(\Delta_0)^2} V_1 \bar{B}_{uv}^{(2,1)} B_{uv}^{(2,1)} \times \left[\tfrac{5}{3}\omega_0 \frac{\partial\lambda}{\partial\omega_0} W \left[\tfrac{2}{5} - \frac{\omega_0^2}{4(\Delta_0)^2} \right] + \frac{\omega_0^2}{2(\Delta_0)^2} \lambda (1 - \tfrac{5}{3}\lambda W^2 x_3^{-1}) \right] \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-2} . \tag{69}$$

The quantization axis is determined by $\mathbf{H}$, so we label the $J=2$ tensors in (69) by the quantum number m_z. With the normalization, $\bar{B}_{uv}^{m_z} B_{uv}^{m_z'} = \frac{3}{2}\delta_{m_z m_z'}$, we have $\bar{B}_{uv} = B_{uv}^*$ and

$$
\begin{aligned}
&B_{33}^{m_J=0}=1\ ,\\
&B_{11}^{m_J=0}=B_{22}^{m_J=0}=-\tfrac{1}{2}\ ,\\
&B_{32}^{m_J=\pm1}=B_{23}^{m_J=\pm1}=\frac{-i}{2}(\tfrac{3}{2})^{1/2}\ ,\\
&B_{31}^{m_J=\pm1}=B_{13}^{m_J=\pm1}=\mp\tfrac{1}{2}(\tfrac{3}{2})^{1/2}\ ,\\
&B_{11}^{m_J=\pm2}=-B_{22}^{m_J=\pm2}=(\tfrac{3}{8})^{1/2}\ ,\\
&B_{12}^{m_J=\pm2}=B_{21}^{m_J=\pm2}=\pm i(\tfrac{3}{8})^{1/2}\ .
\end{aligned} \tag{70}
$$

The evaluation of Eq. (59b) for N_1 is equally straightforward; in terms of ω_L, the effective Larmor frequency defined in Eqs. (29), we obtain the linear Zeeman shift of the RSQ modes,

$$
\begin{aligned}
\omega_1=m_z\omega_L&\left[-\frac{W}{9}\right]\left[\lambda+1-y+\frac{\lambda^2F_2^a}{5}\left[2+y\frac{F_2^a}{5}\right]\right]\\
&\times\left[\frac{\omega_0^2}{2\Delta_0^2}\lambda(1-\tfrac{5}{3}\lambda W^2x_3^{-1})\right.\\
&\left.+\tfrac{5}{3}\omega_0\frac{\partial\lambda}{\partial\omega_0}W\left[\frac{2}{5}-\frac{\omega_0^2}{4\Delta_0^2}\right]\right]^{-1}\ ,
\end{aligned} \tag{71}
$$

which is the result of Sauls and Serene.[13]

The perturbation theory calculation of the quadratic Zeeman shift is complicated by the first-order corrections, $\underset{\sim}{d}_1$, to the eigenfunctions for the RSQ modes. The correction d_{1j}^{+} of the order parameter is not a pure $J=2$ state, but a superposition of amplitudes with $J=1$, 2, and 3.

$$
\begin{aligned}
d_{1j}^{+}=&C_{ju}^{(1,1)}p_u+C_{j,uvw}^{(3,3)}p_up_vp_w+C_{ju}^{(2,1)}p_u\\
&+\tfrac{5}{3}C_{uv}^{(2,3)}p_up_vp_j-\tfrac{2}{3}C_{uj}^{(2,3)}p_u\ .
\end{aligned} \tag{72}
$$

Similarly, the first-order correction to the spin-dependent mean field contains $l=2$ and $l=4$ terms,

$$
e_{1j}^{+}=D_{j,uv}^{(2)}p_up_v+D_{j,uvwx}^{(4)}p_up_vp_wp_x\ , \tag{73}
$$

that couple to the $J=1$, 2, and 3 order-parameter amplitudes. The first-order solutions to Eq. (57b) are lengthy, so we list the results in Appendix A. The point we emphasize regarding these solutions is that the RSQ modes in a finite magnetic field are no longer pair excitations with total angular momentum $J=2$; there is an admixture of Cooper-pair amplitude with $J=1$ and $J=3$. This mixing has an important effect on the nonlinear Zeeman shift of the RSQ mode frequencies. From Eqs. (59) and the solutions to Eqs. (57a) and (57b) we calculate the quadratic Zeeman shifts; the explicit formulas are given in Appendix A. There are three types of contributions: (i) terms directly proportional to ω_L^2 [Eqs. (A11) and (A12)], (ii) terms that arise from gap distortion—*i.e.*, Δ_2 in Eq. (A13), and (iii) contributions from mixing with the $J=1$ and $J=3$ amplitudes [in Eqs. (A14) and (A15)]. These latter contributions give rise to "mode repulsion," which reduces the effect of gap distortion.

The general form for the field-dependent Zeeman effect to order $(\gamma H/\Delta_0)^2$ is

$$
\omega=\omega_0+\left[\alpha m_z+\beta m_z^2\left[\frac{\gamma H}{\Delta_0(0)}\right]-\Gamma\left[\frac{\gamma H}{\Delta_0(0)}\right]\right]\gamma H\ , \tag{74}
$$

where $\Delta_0(0)=1.76T_c$ is the zero-temperature gap. Shivaram *et al.*[1] have analyzed their data using this quadratic formula to determine the coefficients α,β,Γ from experiment. Analytic expressions for these coefficients can be obtained from Appendix A. In Figs. 4–6 we exhibit the temperature and interaction dependence of the nonlinear Zeeman effect. Interactions, except F_0^a, drop out of the Zeeman shifts in the limit $T\to T_c$, just as they do for the nonlinear susceptibility,

$$
\begin{aligned}
&\alpha(T\to T_c)=0.0833(1+F_0^a)^{-1}\ ,\\
&\beta(T\to T_c)=0.0599(1+F_0^a)^{-2}(1-T/T_c)^{-1/2}\ ,\\
&\Gamma(T\to T_c)=0.1689(1+F_0^a)^{-2}(1-T/T_c)^{-1/2}\ .
\end{aligned} \tag{75}
$$

In principle measurements of these parameters, extrapolated to T_c, would yield unique determinations of F_0^a. For the region in temperature away from T_c, which is the relevant limit for comparison with experiment, there is significant dependence of the Zeeman shifts on the f-wave pairing interaction and Landau parameter F_2^a; β and Γ depend only weakly on F_4^a. The divergence of $\beta(T)$ and $\Gamma(T)$ for $T\to T_c$ reflects the breakdown of the perturbation expansion in $[\gamma H/\Delta_0(T)]$ for fixed H. Since experimental measurements of the RSQ modes necessarily require temperatures outside the Ginzburg-Landau region, comparison with results based on perturbation theory are typically valid for magnetic fields up to a few kilogauss. Nevertheless, there are numerous details, in particular, the dispersion corrections described in Sec. III B below, which complicate the comparison between theory and experiment. We discuss these matters and our numerical analysis of the data of Ref. 1 in a separate publication.[15]

Finally we comment on the relationship of this work to that of other researchers. At the time Shivaram *et al.*[1]

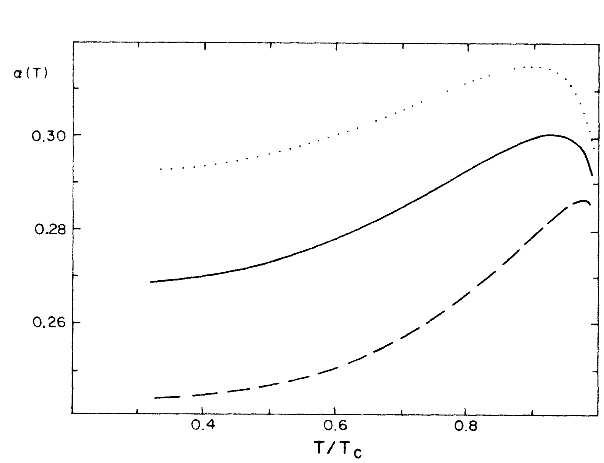

FIG. 4. Linear Zeeman coefficient. $\alpha(T)$ is calculated for the RSQ modes for (1) $F_2^a=x_3^{-1}=0$ (solid curve), (2) $F_2^a=-1.56,x_3^{-1}=0$ (long-dashed curve), and (3) $F_2^a=0,x_3^{-1}=-0.43$ (short-dashed curve). All curves were calculated with $F_0^a=-0.7$.

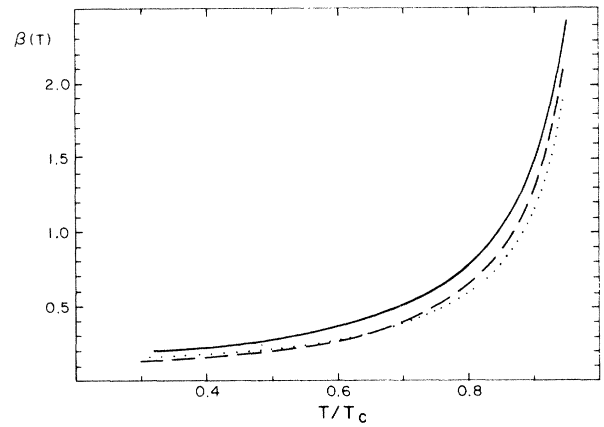

FIG. 5. Quadratic Zeeman coefficient $\beta(T)$ is calculated for the same three cases shown in Fig. 4.

observed the nonlinear Zeeman shift of the RSQ modes, Schopohl *et al.*[2] predicted this effect based on an earlier paper on the gap distortion.[12] They also noted that mode-repulsion effects reduce the nonlinearity resulting from gap distortion. We agree with Schopohl *et al.*[2] on these qualitative observations. However, in contrast to Schopohl *et al.*, we find no mixing of the RSQ modes with the $J=0$ order parameter, at least to leading order in $(\gamma H/\Delta_0)$. It is conceivable that mixing with $J=0$ occurs at higher order in $(\gamma H/\Delta)$. We do obtain mixing with the $J=3$ order parameter because we include f-wave pairing correlations. It is important to note that our expressions for the quadratic nonlinear Zeeman shift do not immediately reduce to those of Schopohl *et al.* in the limit $F_2^a=F_4^a=x_3^{-1}=0$; Eq. (1) of Schopohl *et al.* omits all nonlinear effects except gap distortion. We elaborate on this fact because Shivaram *et al.*[1] have recently used the formula derived by Schopohl *et al.* to analyze their data. Their procedure amounts to inserting the RSQ mode frequency ω_0 and the g factors for both $2+$ and $2-$ modes, calculated with interactions,[13] into the formula of Schopohl *et al.*, which omits interaction effects (except F_0^a) and mode repulsion. This procedure cannot yield reliable results for the material parameters F_2^a and x_3^{-1}.

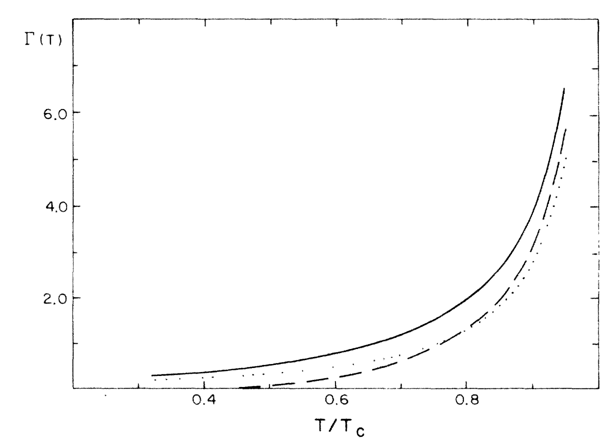

FIG. 6. Quadratic Zeeman coefficient $\Gamma(T)$ is calculated for the same three cases shown in Fig. 4.

B. Dispersion of the the RSQ modes

The RSQ modes have so far been observed only through their coupling to zero sound. Essentially all detailed comparisons between theory and the experimentally determined mode frequencies neglect the dispersion of the RSQ modes.[6,13,1] This is generally a good approximation because zero sound excites the RSQ mode at $q=c_0/\omega$, and the resulting dispersion correction to the mode frequency is small,

$$\omega_q \sim \tfrac{1}{2}c_{\rm RSQ}^2 q^2/\omega_0 \sim \tfrac{1}{2}(v_f/c_0)^2\omega_0 \ll \omega_0 \; . \tag{76}$$

The dispersion shift is not necessarily small compared with the Zeeman shift. Furthermore, the finite wave vector partially lifts the degeneracy of the RSQ modes in zero field;[4] i.e., the five RSQ modes travel with different group velocities when excited by zero sound. This dispersion splitting has recently been observed by Shivaram *et al.* at low temperatures $(T\sim 0.4T_c)$ and low fields $(H\sim 30\ \text{G})$. A detailed theoretical analysis of the Zeeman splittings of the RSQ modes necessarily requires an analysis of the dispersion splittings as a function of magnetic field. The effect of the dispersion splittings on the Zeeman splittings of the RSQ modes is shown in Fig. 7. The dispersion corrections lead to unequal Zeeman splittings which persist for fields $\gamma H \gg v_f^2 q^2/\Delta$.

To calculate the dispersion splittings of the RSQ modes we use the perturbation theory previously described. We start from linearized transport equation [Eq. (47)]; the gradient term generates the perturbation, $-\eta\delta\hat{g}$, with $\eta \equiv v_f \hat{\mathbf{p}}\cdot\mathbf{q}$. Combining the solutions to the transport equation with the self-consistency equations for the order parameter and mean fields we obtain a matrix eigenvalue equation,

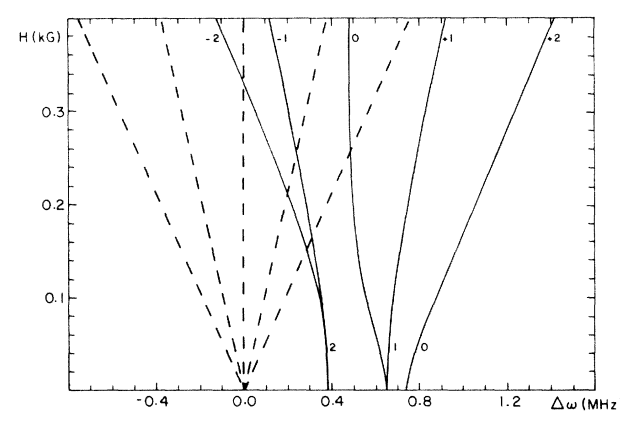

FIG. 7. Dispersion and Zeeman splitting of the RSQ modes. The dashed curves show the Zeeman splittings calculated for $T/T_c=0.65$, low pressure ($F_0^a=-0.7, F_2^a=-0.5, x_3^{-1}=-0.4$, $T_c=1.04$ mK), and $q=0$. The effect of dispersion of the RSQ modes on the Zeeman splittings is shown in the solid curves. The nonlinearity at low fields results from rotation of the quantization axis from $\mathbf{q}$ (at $\mathbf{H}=0$) to $\mathbf{H}$ ($\perp\mathbf{q}$) at intermediate fields ($H \gtrsim 0.3$ kG). The dispersions splittings were calculated with $(qv_f/\omega_0)^2=(v_f/c_0)^2\approx -0.10$, appropriate for $p=0$ bar. The states are labeled by $|m_q|$ at $H=0$, and by m_z in high fields.

$$\underset{\sim}{d}(\hat{\mathbf{p}})=\underline{L}(\hat{\mathbf{p}},\hat{\mathbf{p}}';\omega)*\underset{\sim}{d}(\hat{\mathbf{p}}') ,$$

$$\underline{L}=\begin{vmatrix} V^1L^1_{kj} & V^1L^2_{kj} & 0 & V^1L^5_{kj} \\ F^aL^3_{kj} & F^aL^4_{kj} & F^aM^1_k & F^aL^6_{kj} \\ 0 & V^0M^2_j & V^0N & V^0M^3_j \\ F^aL^7_{kj} & F^aL^8_{kj} & F^aM^4_k & F^aL^9_{kj} \end{vmatrix} , \quad (77)$$

for the ten-component eigenvector $\underset{\sim}{d}^{\,\mathrm{tr}}=(\mathbf{d}^+,\mathbf{e}^+,d^-,\mathbf{e}^-)$. The dimensionality of the eigenvalue equation increases for $\mathbf{q}\neq 0$ because of the coupling of the eigenfunction for the RSQ modes to the *singlet* order parameter, d^-, and the odd harmonics of the spin density $\mathbf{e}^-$. Explicit expressions for the matrix elements are given in Appendix B. The quadratic dispersion of the modes is calculated from

$$\begin{aligned} \omega_q &= -N_q/D_\omega , \\ N_q &= \underset{\sim}{b}_0*\underline{L}^{(2)}(\omega_0)*\underset{\sim}{d}_0+\underset{\sim}{b}_0*\underline{L}^{(1)}(\omega_0)*\underset{\sim}{d}_1 , \end{aligned} \quad (78)$$

where $\underline{L}^{(1)}$ ($\underline{L}^{(2)}$) is the perturbation of order η (η^2). The detailed expressions for the matrix products in Eq. (78) are also given in Appendix B; we find that the dispersion splittings depend on the interaction parameters F^a_1, F^a_3, and x_2^{-1}, in addition to F^a_2 and x_3^{-1}. The dependence of mode velocities on the d-wave pairing interaction results from mode repulsion between the RSQ modes and collective oscillations of the singlet, $l=2$ order parameter. The effect of d-wave mode repulsion is weak unless the d-wave collective mode frequency is nearly degenerate with the RSQ mode frequency ω_0, an unlikely situation.

In the absence of interaction effects we obtain for the $T=0$ RSQ mode dispersion, $\omega^2-\omega_0^2=c^2q^2$,

$$(c/v_f)^2=\begin{cases} \frac{65}{147}-m_q^2\frac{8}{147} , & \alpha\gamma H\omega_0 \ll q^2v_f^2 \\ \frac{41}{147}+m_z^2\frac{4}{147} , & \omega_0^2 \gg \alpha\gamma H\omega_0 \gg q^2v_f^2 . \end{cases} \quad (79)$$

Note that in the low-field limit, $\alpha\gamma H\omega_0 \ll q^2v_f^2$, the quantization axis is determined by $\hat{\mathbf{q}}$, and the dispersion splitting is determined by the eigenvalues $m_q=\hat{\mathbf{q}}\cdot\mathbf{J}$. In the intermediate-field limit, $\omega_0^2\gg\alpha\gamma H\omega_0\gg q^2v_f^2$, the quantization axis is determined by the magnetic field, so that the dispersion splitting is given by the eigenvalues of $m_z=\hat{\mathbf{z}}\cdot\mathbf{J}$. For zero-field several authors have calculated the RSQ mode dispersion neglecting interaction effects.[26–28] In this limit our calculation agrees with that of Brusov and Popov[28] at zero temperature. Combescot[29] has also calculated the RSQ mode dispersion for $F^a_2=F^a_3=x_3^{-1}=x_2^{-1}=0$, but $F^a_1\neq 0$. In this limit we find (at $T=0$ and $H=0$),

$$(c/v_f)^2=\tfrac{65}{147}+\tfrac{7}{125}F^a_1-m_q^2\left[\tfrac{8}{147}+\tfrac{7}{750}F^a_1\frac{1+7F^a_1/25}{1+14F^a_1/75}\right] , \quad (80)$$

which agrees with Combescot except for the $(F^a_1)^2$ terms which are absent from his equation for $|m_q|=2$. A quantitative comparison of the RSQ mode dispersion requires that the corrections from F^a_2 and x_3^{-1} be included since it is known that these interactions shift the $q=0$ mode frequency by 20–25% at low temperature.[6–9] However, Fig. 8 shows that the effects of F^a_2 and x_3^{-1} on the dispersion are relatively small; choosing $x_3^{-1}=-0.43$, $F^a_2=0$ we find only a 3% change in the $|m_q|=2$ mode dispersion. This small effect results from a near cancellation between the large shift in ω_0 from $\sqrt{8/5}\Delta_0$ and the explicit dependence of $(c/v_f)^2$ on F^a_2 and x_3^{-1}. In contrast the dispersion is more sensitive to the interactions $F^a_{1,3}$ which enter the dispersion relation explicitly. The effect of x_2^{-1} gives rise to a small shift in the mode dispersion. It is worth noting that for $\mathbf{H}=0$ the ratio $(\omega_{\pm 2}-\omega_0)/(\omega_{\pm 1}-\omega_0)\equiv\frac{1}{4}$ is dependent of interactions.

The crossover region, $q^2v_f^2\sim\alpha\gamma H\omega_0$, is slightly more complicated to analyze. With the condition that $\alpha\gamma H\ll\omega_0$, the first-order shifts in the RSQ mode frequencies, when both magnetic and dispersive perturbations are present, are given by

$$\Delta\omega=\omega-\omega_0=-[\underset{\sim}{b}_0*(\underline{L}^H+\underline{L}^q)*\underset{\sim}{d}_0]/D_\omega , \quad (81)$$

where $\underline{L}^H$ is the linear field contribution to Eq. (53), and $\underline{L}^q$ is the dispersive perturbation of order q^2 [we eliminated $\underset{\sim}{d}_1$ in favor of $\underset{\sim}{d}_0$ using Eqs. (B2) through (B7)]. In the crossover region the quantization axis, which defines the correct basis functions with which to compute $\Delta\omega$, is neither $\hat{\mathbf{q}}$ or $\hat{\mathbf{z}}$. To calculate the correct frequency shifts of the fivefold multiplet we consider the case $\hat{\mathbf{q}}\perp\hat{\mathbf{z}}$ and *choose* the basis of $J=2$ eigenfunctions with $\hat{\mathbf{z}}$ as the quantization axis. We then solve the determinantal equation,

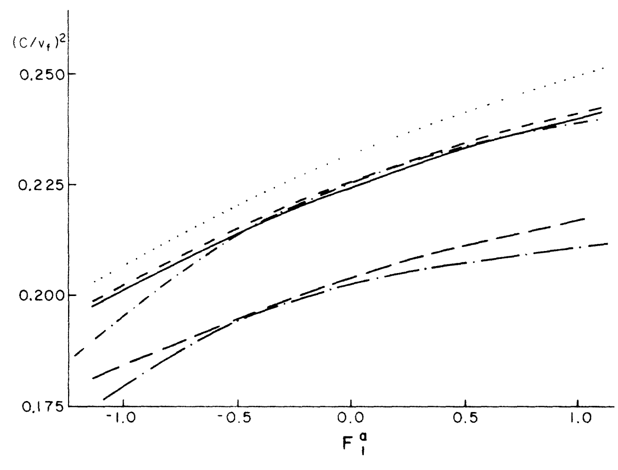

FIG. 8. Interaction dependence of the RSQ mode dispersion. The dispersion of the $|m_q|=2$ modes, $(c/v_f)^2$, is plotted as a function of F^a_1 at $T=0$. The solid curve corresponds to $F^a_2=x_3^{-1}=F^a_3=x_2^{-1}=0$. The short-dashed curve (dotted curve) was calculated with only $F^a_2=-1.56(x_2^{-1}=-0.43)$ nonzero. The short-dashed-dotted curve exhibits the effect of a d-wave interaction, $x_2^{-1}=-1.5$, on the curve with $F^a_2=-1.56$ and $F^a_3=x_3^{-1}=0$. The long-dashed curve, calculated with $F^a_2=-1.56, x_3^{-1}=x_2^{-1}=0$, and $F^a_3=-1.0$, illustrates the sizeable influence that the $l=3$ Landau parameter has on the mode dispersion; the long-dashed-dotted curve adds a finite d-wave interaction, $x_2^{-1}=-1.5$, to this calculation.

$$\det[\Delta\omega\delta_{m_z m_z'} D_\omega + \underset{\sim}{b}_0^{m_z} * (\underline{L}^H + \underline{L}^q) * \underset{\sim}{d}_0^{m_z'}] = 0 , \qquad (82)$$

which factorizes into subspaces for $m_z = \pm 1$ and $m_z = \pm 2,0$. For the $m_z = \pm 1$ subspace we obtain the solutions

$$\Delta\omega_{\pm 1} = -\frac{V_1}{D_\omega}[a + b \mp (c^2 + 9b^2)^{1/2}] , \qquad (83)$$

while for the $m_z = \pm 2,0$ subspace we obtain the cubic equation

$$\left[\frac{\Delta\omega D_\omega}{V_1} + a\right]\left[\left[\frac{\Delta\omega D_\omega}{V_1}\right]^2 + 2\frac{\Delta\omega D_\omega}{V_1}(a+4b) + (a+4b)^2 - 4c^2\right] - 12b^2\left[\frac{\Delta\omega D_\omega}{V_1} + a + 4b\right] = 0 , \qquad (84)$$

where a and b are given explicitly in Eqs. (B10) and $c = \alpha D_\omega \gamma H / V_1$ is proportional to the linear Zeeman coefficient α defined in Eq. (74). These solutions reveal the evolution of the q substates (for $q^2 v_f^2 \gg \alpha\gamma H\omega_0$) into the magnetic substates (for $\alpha\gamma H\omega_0 \gg q^2 v_f^2$). As illustrated in Fig. 7, the $m_q = 0$ state evolves into the $m_z = +2$ magnetic state, the $m_q = \pm 1$ states evolve into the $m_z = +1,0$ magnetic states, and the $m_q = \pm 2$ states evolve into the $m_z = -1,-2$ magnetic states. Note that the mode frequency of each state approaches its zero-field value, with zero slope $d\Delta\omega/dH = 0$, without crossing any other branch. Numerically we find the modes reach the intermediate-field region at about 0.3 kG, above which they exhibit the linear Zeeman effect until nonlinear field corrections become important.

The combined set of equations, given in this section and the appendices, for the nonlinear Zeeman effect and the dispersion splittings of the RSQ modes may be used to analyze experimentally determined collective-mode frequencies obtained using zero sound. A detailed analysis of the data obtained by Shivaram *et al.* will be published separately.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. DMR-8020263. R. Fishman acknowledges the support of Schlumberger-Doll. We thank W. P. Halperin, J. B. Ketterson, M. W. Meisel, B. S. Shivaram, and J. W. Serene for sending us their results and for numerous enlightening conversations on the collective excitations in ^{3}He.

APPENDIX A

In this Appendix we collect the relevant results for the (1) matrix elements of the eigenvalue equation [Eq. (55)] that defines the RSQ modes in a magnetic field, (2) the first-order corrections to the zero-field RSQ mode eigenfunctions, and (3) the results for the quadratic Zeeman shifts.

(1) The matrix elements appearing in Eq. (55) are

$$L_1(\hat{\mathbf{p}})_{kj} = \delta_{kj}\left[\frac{1}{V_1} + \frac{\omega^2}{4(\Delta_0)^2}\lambda + (\Sigma+\lambda)\frac{\mathbf{\Delta}_2\cdot\mathbf{\Delta}_0}{(\Delta_0)^2} + \tfrac{3}{2}(\Xi+\Sigma)\frac{(\mathbf{h}\cdot\mathbf{\Delta}_0)^2}{(\Delta_0)^4}\right] - (h_k\Delta_0^j + \Delta_0^k h_j)\frac{\mathbf{h}\cdot\mathbf{\Delta}_0}{(\Delta_0)^4}\Sigma + \frac{f(\lambda)}{(\Delta_0)^2}h_j h_k$$
$$- \frac{\lambda}{(\Delta_0)^2}(\Delta_0^k\Delta_0^j + \Delta_2^k\Delta_0^j + \Delta_0^k\Delta_2^j) - \Delta_0^k\Delta_0^j\frac{\mathbf{\Delta}_0\cdot\mathbf{\Delta}_2}{(\Delta_0)^4}\Sigma - \tfrac{3}{2}\Delta_0^k\Delta_0^j\frac{(\mathbf{h}\cdot\mathbf{\Delta}_0)^2}{(\Delta_0)^6}\Xi - \frac{i\omega}{2}\epsilon_{klj}\Delta_0^l\frac{\mathbf{h}\cdot\mathbf{\Delta}_0}{(\Delta_0)^4}\Sigma , \qquad (A1)$$

$$L_2(\hat{\mathbf{p}})_{kj} = -\delta_{kj}\frac{\mathbf{h}\cdot\mathbf{\Delta}_0}{(\Delta_0)^2}(\Sigma+\lambda) + \frac{i\omega\lambda}{2(\Delta_0)^2}\epsilon_{klj}\Delta_2^l + \frac{i\omega\mathbf{h}\cdot\mathbf{\Delta}_0}{(\Delta_0)^4}\Sigma\epsilon_{klj}h_l + i\omega\epsilon_{klj}\Delta_0^l\left[\frac{\mathbf{\Delta}_0\cdot\mathbf{\Delta}_2}{2(\Delta_0)^4}\Sigma + \frac{3(\mathbf{h}\cdot\mathbf{\Delta}_0)^2}{4(\Delta_0)^6}\Xi + \frac{\lambda}{2(\Delta_0)^2}\right]$$
$$+ \frac{\lambda}{(\Delta_0)^2}h_k\Delta_0^j + \frac{\mathbf{h}\cdot\mathbf{\Delta}_0}{(\Delta_0)^4}\Delta_0^j\Delta_0^k\Sigma - \frac{2i}{\omega(\Delta_0)^2}[\lambda + f(\lambda)]\epsilon_{lmj}h_k h_m\Delta_0^l , \qquad (A2)$$

$$L_3(\hat{\mathbf{p}})_{kj} = \frac{2i}{\omega(\Delta_0)^4}\Sigma\mathbf{h}\cdot\mathbf{\Delta}_0\epsilon_{klm}\Delta_0^l h_m\Delta_0^j$$
$$- i\epsilon_{klj}\Delta_0^l\left[\frac{\omega\lambda}{2(\Delta_0)^2}\left[1 + \frac{4h^2}{\omega^2}\right] + \frac{\omega}{2(\Delta_0)^4}\Sigma\mathbf{\Delta}_2\cdot\mathbf{\Delta}_0 + \frac{3\omega}{4(\Delta_0)^6}(\mathbf{h}\cdot\mathbf{\Delta}_0)^2\Xi + \frac{2}{\omega(\Delta_0)^4}\Sigma(\mathbf{h}\cdot\mathbf{\Delta}_0)^2\right]$$
$$- \frac{2i}{\omega(\Delta_0)^2}\epsilon_{klm}\Delta_0^l h_m h_j f(\lambda) - \frac{i\omega\lambda}{2(\Delta_0)^2}\epsilon_{klj}\Delta_2^l - \frac{2i}{\omega}Y_{3/2}\epsilon_{klj}h_l\mathbf{h}\cdot\mathbf{\Delta}_0$$
$$+ \delta_{kj}\frac{\mathbf{h}\cdot\mathbf{\Delta}_0}{(\Delta_0)^2}(\lambda+\Sigma) - \frac{\lambda}{(\Delta_0)^2}\Delta_0^k h_j - \frac{\mathbf{h}\cdot\mathbf{\Delta}_0}{(\Delta_0)^4}\Sigma\Delta_0^j\Delta_0^k + \frac{2i\lambda}{\omega(\Delta_0)^2}\epsilon_{lmj}h_k h_l\Delta_0^m + \frac{2i}{\omega(\Delta_0)^2}\Sigma\epsilon_{lmj}\Delta_0^k h_l\Delta_0^m\mathbf{h}\cdot\mathbf{\Delta}_0 , \qquad (A3)$$

$$L_4(\hat{\mathbf{p}})_{kj}=\frac{2i}{\omega(\Delta_0)^2}\epsilon_{klj}\Delta_0^l\mathbf{h}\cdot\boldsymbol{\Delta}_0\left[\lambda+\frac{\omega^2}{4(\Delta_0)^2}\Sigma\right]-\frac{2i\lambda}{\omega(\Delta_0)^2}\epsilon_{klm}\Delta_0^l\Delta_0^j h_m$$
$$-\frac{4}{\omega^2(\Delta_0)^2}\epsilon_{klm}\Delta_0^l h_m\epsilon_{jab}\Delta_0^a h_b[\lambda+f(\lambda)]+\frac{2i\lambda}{\omega(\Delta_0)^2}\Delta_0^k\epsilon_{jab}\Delta_0^a h_b$$
$$+\Delta_0^k\Delta_0^j\left[\frac{\boldsymbol{\Delta}_0\cdot\boldsymbol{\Delta}_2}{(\Delta_0)^2}\Sigma+\frac{3(\mathbf{h}\cdot\boldsymbol{\Delta}_0)^2}{2(\Delta_0)^6}\Xi\right]+\frac{\lambda}{(\Delta_0)^2}(\Delta_2^k\Delta_0^j+\Delta_0^k\Delta_2^j+\Delta_0^k\Delta_0^j)$$
$$+(\Delta_0^k h_j+h_k\Delta_0^j)\frac{\mathbf{h}\cdot\boldsymbol{\Delta}_0}{(\Delta_0)^4}\Sigma+\delta_{kj}\left[-\frac{4h^2}{\omega^2}(\lambda-1)-\lambda-\frac{2\boldsymbol{\Delta}_0\cdot\boldsymbol{\Delta}_2}{(\Delta_0)^2}\lambda-\left[\frac{\boldsymbol{\Delta}_0\cdot\boldsymbol{\Delta}_2}{(\Delta_0)^2}+2\frac{(\mathbf{h}\cdot\boldsymbol{\Delta}_0)^2}{(\Delta_0)^4}\right]\Sigma-\frac{3(\mathbf{h}\cdot\boldsymbol{\Delta}_0)^2}{2(\Delta_0)^4}\Xi\right]$$
$$-\frac{2i}{\omega}\epsilon_{klj}h_l+\frac{4}{\omega^2}h_k h_j(\lambda-1)\;, \tag{A4}$$

where the ω-dependent functions Σ, Ξ, and $f(\lambda)$ are simply related to $\lambda(\omega)$ defined below Eq. (61),

$$\Xi=[(\Delta_0)^4Y_{5/2}-\Sigma]/W\;,$$
$$\Sigma=-[(\Delta_0)^2Y_{3/2}+\lambda]/W\;, \tag{A5}$$
$$f(\lambda)=\lambda+2\omega\frac{\partial\lambda}{\partial\omega}\left[1-2\frac{(\Delta_0)^2}{\omega^2}\right]-2(\Delta_0)^2W\frac{\partial^2\lambda}{\partial\omega^2}\;.$$

(2) The first-order corrections to the order parameter, d_{1j}^+, and the mean field, e_{1j}^+, are represented by their tensor expansions in Eqs. (72) and (73). The nonzero components of the $J=1$ and $J=3$ tensors are related to the zeroth-order tensor $B_{uv}^{(2,1)}$ by,

$$i\epsilon_{ujk}C_{kj}^{(1,1)}=\frac{\gamma H}{\omega_0\lambda}B_{3u}^{(2,1)}W\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}\left[1+\frac{F_2^a}{5}\right]D^{-1}\left[\Sigma-2\lambda\frac{\lambda F_2^a}{5}-y\left[\frac{\lambda F_2^a}{5}\right]^2\right]\;, \tag{A6}$$

$$\frac{i}{3}(\epsilon_{ujk}C_{kjv3}^{(3,3)}+\epsilon_{vjk}C_{kju3}^{(3,3)}+\epsilon_{3jk}C_{kjuv}^{(3,3)})=\frac{\omega_0\gamma H}{9(\Delta_0)^2}Q^{-1}W\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}\left[1+\frac{F_2^a}{5}\right]D^{-1}$$
$$\times\left[\Sigma-2\lambda\frac{\lambda F_2^a}{5}-y\left[\frac{\lambda F_2^a}{5}\right]^2\right](B_{3u}^{(2,1)}\delta_{v3}+B_{3v}^{(2,1)}\delta_{u3}+\tfrac{5}{3}B_{uv}^{(2,1)})\;, \tag{A7}$$

$$D_u^{(1,2)}=\tfrac{2}{5}D_{j,ju}^{(2)}\;,$$
$$D_u^{(1,2)}=-\frac{\gamma H}{2\Delta_0}B_{3u}^{(2,1)}\frac{\lambda F_2^a}{5}W\left[1+y\frac{F_2^a}{5}\right]\left[1+\frac{F_2^a}{5}W\right]^{-1}\left[1+\frac{F_2^a}{5}\right]D^{-1}\;,$$
$$D_{uvw}^{(3,2)}=\tfrac{1}{3}[D_{u,vw}^{(2)}+D_{v,uw}^{(2)}+D_{w,uv}^{(2)}-\tfrac{2}{5}(D_{j,ju}^{(2)}\delta_{vw}+D_{j,jv}^{(2)}\delta_{uw}+D_{j,jw}^{(2)}\delta_{uv})]\;, \tag{A8a}$$
$$D_{uv3}^{(3,2)}=Q^{-1}\frac{\gamma H}{6\Delta_0}W\frac{F_2^a}{5}\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}\left[1+\frac{F_2^a}{5}\right]$$
$$\times D^{-1}\left\{x_3\left[\tfrac{4}{7}\left[\Sigma-\lambda\frac{\lambda F_2^a}{5}\right]\right.\right.$$
$$\left.\left.+\lambda\left[1+y\frac{F_2^a}{5}\right]\left[1+\frac{3}{7}\frac{\lambda F_4^a}{9}\right]\right]+\frac{\omega_0^2}{4(\Delta_0)^2}\lambda^2\left[1+y\frac{F_2^a}{5}\right]\right\}(B_{u3}^{(2,1)}\delta_{v3}+B_{v3}^{(2,1)}\delta_{u3}+\tfrac{5}{3}B_{uv}^{(2,1)})\;, \tag{A8b}$$

$$D_{uvw}^{(3,4)}=\tfrac{4}{9}D_{j,juvw}^{(4)}\;,$$
$$D_{uv3}^{(3,4)}=-Q^{-1}\frac{2\gamma H}{21\Delta_0}W\frac{F_4^a}{9}x_3\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}\left[1+\frac{F_2^a}{5}\right]D^{-1}\left[\Sigma-2\lambda\frac{\lambda F_2^a}{5}-y\left[\frac{\lambda F_2^a}{5}\right]^2\right](B_{u3}^{(2,1)}\delta_{v3}+B_{v3}^{(2,1)}\delta_{u3}+\tfrac{5}{3}B_{uv}^{(2,1)})\;, \tag{A8c}$$

where

$$Q = x_3 \left[1 + \frac{4}{7}\frac{\lambda F_2^a}{5} + \frac{3}{7}\frac{\lambda F_4^a}{9} \right] + \frac{\omega_0^2}{4(\Delta_0)^2}\lambda \ . \tag{A9}$$

The linear combinations of $J=2$ amplitudes that contribute to ω_2 are related to the zeroth order $J=2$ amplitudes by

$$\begin{aligned} C_{uv}^{(2,3)} = {} & C_{uv}^{(2,1)} \left[\frac{5}{8}\frac{\omega_0^2}{(\Delta_0)^2} - 1 \right] + \frac{i\omega_0}{2\Delta_0}\check{D}_{uv} + \frac{5}{4}\frac{\omega_1\omega_0}{(\Delta_0)^2} \left[1 + \frac{1}{2}\frac{\lambda F_2^a}{5} W \right] B_{uv}^{(2,1)} \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} \\ & - \frac{5i\gamma H\omega_0}{48(\Delta_0)^2\lambda} \left[\Sigma - \lambda\frac{\lambda F_2^a}{5} \right] W \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} \left[1 + \frac{F_2^a}{5} \right] D^{-1} (\epsilon_{u3j} B_{jv}^{(2,1)} + \epsilon_{v3j} B_{ju}^{(2,1)}) \ , \end{aligned} \tag{A10a}$$

$$\begin{aligned} \check{D}_{uv} \left[1 + \frac{\lambda F_2^a}{5} \right] = {} & \frac{3i\omega_0}{4\Delta_0}\frac{\lambda F_2^a}{5} (C_{uv}^{(2,1)} - \tfrac{2}{3} C_{uv}^{(2,3)}) + B_{uv}^{(2,1)} \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} W \frac{5}{4}\frac{i\omega_1}{\Delta_0}\frac{F_2^a}{5} \left[\omega_0 \frac{\partial\lambda}{\partial\omega_0} + \lambda \left[1 + \frac{\lambda F_2^a}{5} \right] \right] \\ & + \frac{5}{24}\frac{\gamma H}{\Delta_0} W \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} \frac{F_2^a}{5} \left[1 + \frac{F_2^a}{5} \right] D^{-1} \\ & \times \left[\Sigma \left[1 + \frac{\lambda F_2^a}{5} W \right] + \lambda \left[1 + \frac{F_2^a}{5} \right] \right] (\epsilon_{u3j} B_{jv}^{(2,1)} + \epsilon_{v3j} B_{ju}^{(2,1)}) \ , \end{aligned} \tag{A10b}$$

$$\begin{aligned} \left[x_3 + \frac{\lambda\omega_0^2}{4(\Delta_0)^2} \right] C_{uv}^{(2,3)} = {} & \tfrac{3}{5}\lambda (C_{uv}^{(2,1)} + C_{uv}^{(2,3)}) \\ & - \omega_1 \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} B_{uv}^{(2,1)} \left[-\frac{5\omega_0^2}{8(\Delta_0)^2} W \frac{\partial\lambda}{\partial\omega_0} + \lambda \frac{\omega_0}{(\Delta_0)^2} \left[\frac{5\omega_0^2}{16(\Delta_0)^2} - \frac{1}{2} - \frac{\lambda F_2^a}{20} W \right] \right] \\ & + \frac{i\omega_0\lambda}{5\Delta_0}\check{D}_{uv} + \frac{i\omega_0\gamma H}{24(\Delta_0)^2} W \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-1} \left[1 + \frac{F_2^a}{5} \right] D^{-1} \left[-\Sigma + \lambda \frac{\lambda F_2^a}{5} \right] (\epsilon_{u3j} B_{jv}^{(2,1)} + \epsilon_{v3j} B_{ju}^{(2,1)}) \ . \end{aligned} \tag{A10c}$$

(3) The quadratic nonlinear Zeeman shifts of the RSQ modes are calculated from Eqs. (59) and the equations above and in Sec. III. The contributions to N_2 are

$$\begin{aligned} \omega_1 \underset{\sim}{b}_0 * \frac{\partial \underline{L}^{(1)}}{\partial\omega_0} * \underset{\sim}{d}_0 = {} & -\tfrac{5}{144} V_1 m_z^2 \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-2} \left[1 + \frac{F_2^a}{5} \right]^2 D^{-2} \left[\frac{\gamma H}{\Delta_0} \right]^2 g W \\ & \times \left[(\lambda + 1 - y) \left[1 - \frac{2}{W} - 2\frac{\lambda F_2^a}{5} \right] + \left[\frac{\lambda F_2^a}{5} \right]^2 W(y - 2\lambda) - \omega_0 \frac{\partial\lambda}{\partial\omega_0} \left[1 + 2\frac{\lambda F_2^a}{5} W \right] \right] , \end{aligned} \tag{A11}$$

$$\begin{aligned} \frac{\omega_1^2}{2} \underset{\sim}{b}_0 * \frac{\partial^2 \underline{L}^{(0)}}{\partial\omega_0^2} * \underset{\sim}{d}_0 = {} & \tfrac{5}{4} V_1 m_z^2 \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-2} \left[1 + \frac{F_2^a}{5} \right]^2 D^{-2} \left[\frac{\gamma H}{\Delta_0} \right]^2 g^2 \\ & \times \left[-\frac{1}{6}\omega_0 \frac{\partial^2\lambda}{\partial\omega_0^2} W \left[\frac{5\omega_0^2}{8(\Delta_0)^2} - 1 \right] \right. \\ & \left. + \frac{\omega_0^2}{8(\Delta_0)^2} \left[\lambda + 2\omega_0 \frac{\partial\lambda}{\partial\omega_0} \right] (1 - \tfrac{5}{3} W^2 \lambda x_3^{-1}) + \tfrac{1}{3}\lambda W^2 \frac{\lambda F_2^a}{5} \left[1 + \frac{\lambda F_2^a}{5} \right] \right] , \end{aligned} \tag{A12}$$

$$\begin{aligned} \underset{\sim}{b}_0 * \underline{L}^{(2)} * \underset{\sim}{d}_0 = {} & \tfrac{5}{28} V_1 \left[\frac{\gamma H}{\Delta_0} \right]^2 W^{-1} \left[1 + \frac{\lambda F_2^a}{5} W \right]^{-2} \left[1 + \frac{F_2^a}{5} \right]^2 D^{-2} \\ & \times \Bigg[\Bigg\{ W \left[1 + \frac{\lambda F_2^a}{5} W \right]^2 (\Delta_0)^2 Y_{3/2} (1 - \tfrac{1}{2} A) - \tfrac{4}{3}\lambda \frac{\lambda F_2^a}{5} W^2 (1 - A)(1 - AW) - \tfrac{2}{3} W \left[\frac{\lambda F_2^a}{5} W \right]^2 (1 - A)^2 \\ & + \tfrac{1}{2}(\lambda - 1) W \left[\frac{\lambda F_2^a}{5} W \right]^2 + \tfrac{2}{3} f(\lambda) W \left[(1 - AW)^2 - \frac{3\omega_0^2}{8(\Delta_0)^2} \left[1 - \frac{5\omega_0^2}{16(\Delta_0)^2} - WA \right] \right] \end{aligned}$$

$$-\tfrac{1}{16}\omega_0^2 Y_{3/2}\left[\tfrac{5}{3}-\frac{7\omega_0^2}{8(\Delta_0)^2}-\tfrac{8}{3}AW+\frac{10}{3}\frac{\lambda F_2^a}{5}W^2(1-A)\right]$$

$$+\frac{\lambda\omega_0^2}{48(\Delta_0)^2}\left[31-\frac{53\omega_0^2}{8(\Delta_0)^2}+\frac{\omega_0^4}{16(\Delta_0)^4}+2W^2\frac{\lambda F_2^a}{5}(14+13W)\right]$$

$$-\frac{5\lambda\omega_0^2}{16}W\frac{Y_{5/2}}{Y_{3/2}}\left[1+\frac{\omega_0^2}{40(\Delta_0)^2}+\frac{8}{5}\frac{\lambda F_2^a}{5}W^2\right]-\frac{\omega_0^2}{4(\Delta_0)^2}\lambda WA\left[1-\frac{\omega_0^2}{24(\Delta_0)^2}+\frac{1}{6}\frac{\lambda F_2^a}{5}W(8+5W)\right]\Bigg\}$$

$$+m_z^2\Bigg\{-\tfrac{11}{36}W\left[1+W\frac{\lambda F_2^a}{5}\right]^2(\Delta_0)^2 Y_{3/2}(1-\tfrac{9}{11}A)+\tfrac{5}{18}\lambda\frac{\lambda F_2^a}{5}W^2(1-A)(1-AW)$$

$$+\tfrac{5}{36}W\left[\frac{\lambda F_2^a}{5}W\right]^2(1-A)^2-\tfrac{1}{18}(\lambda-1)W\left[\frac{\lambda F_2^a}{5}W\right]^2$$

$$-\tfrac{5}{36}Wf(\lambda)\left[(1-AW)^2-\frac{\omega_0^2}{5(\Delta_0)^2}\left[1-\frac{5\omega_0^2}{16(\Delta_0)^2}-WA\right]\right]$$

$$+\tfrac{1}{48}\omega_0^2 Y_{3/2}\left[\tfrac{11}{3}-\frac{7\omega_0^2}{6(\Delta_0)^2}-4AW+5\frac{\lambda F_2^a}{5}W^2(1-A)\right]$$

$$+\tfrac{1}{12}\omega_0^2 W\lambda\frac{Y_{5/2}}{Y_{3/2}}\left[1-\frac{\omega_0^2}{16(\Delta_0)^2}+\frac{5}{4}\frac{\lambda F_2^a}{5}W^2\right]$$

$$+\frac{\omega_0^2}{8(\Delta_0)^2}\lambda WA\left[1-\frac{\omega_0^2}{24(\Delta_0)^2}+\frac{5}{18}\frac{\lambda F_2^a}{5}W(3W+2)\right]$$

$$+\frac{\omega_0^2}{144(\Delta_0)^2}\lambda\left[-29+\frac{31\omega_0^2}{4(\Delta_0)^2}-\frac{5\omega_0^4}{16(\Delta_0)^4}-\frac{\lambda F_2^a}{5}W^2(14+25W)\right]\Bigg\}\Bigg], \qquad (A13)$$

$$\underset{\sim}{b}_0 * \left[\underline{L}^{(1)}+\omega_1\frac{\partial\underline{L}^{(0)}}{\partial\omega_0}\right]*\underset{\sim}{d}_1=V_1\left[\frac{\gamma H}{\Delta_0}\right]^2\left[1+\frac{\lambda F_2^a}{5}W\right]^{-2}\left[1+\frac{F_2^a}{5}\right]^2 D^{-2}(R_1+R_2+Q^{-1}R_3), \qquad (A14)$$

$$R_1=\tfrac{5}{24}m_z^2\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}\Bigg[g^2\left[\lambda\left[1+\frac{\lambda F_2^a}{5}\right]\left[\frac{\omega_0^2}{(\Delta_0)^2}\left[\frac{5\omega_0^2}{8(\Delta_0)^2}-1\right]-\frac{\lambda F_2^a}{5}W^2\left[1+\frac{\lambda F_2^a}{5}W\right]\right]\right.$$

$$-2\omega_0\frac{\partial\lambda}{\partial\omega_0}W\left\{\left[1+\frac{\lambda F_2^a}{5}W\right]^2+\left[\frac{5\omega_0^2}{8(\Delta_0)^2}-1\right]\left[1+\frac{\lambda F_2^a}{5}\left[1+\frac{\omega_0^2}{4(\Delta_0)^2}\right]\right]\right\}$$

$$\left.+\left[\omega_0\frac{\partial\lambda}{\partial\omega_0}\right]^2\frac{F_2^a}{5}W^2\left[\frac{5\omega_0^2}{8(\Delta_0)^2}-1\right]\right]$$

$$-\tfrac{1}{12}g\left[-\frac{\omega_0^2}{(\Delta_0)^2}\left[\frac{5\omega_0^2}{3(\Delta_0)^2}-2\right]\right]\left[\lambda+1-y+2\lambda\frac{\lambda F_2^a}{5}W+y\left[\frac{\lambda F_2^a}{5}\right]^2W\right]$$

$$+4\frac{\lambda F_2^a}{5}W\left[1+\frac{\lambda F_2^a}{5}W\right]\left[\left[1+\frac{\lambda F_2^a}{5}W\right](\lambda+1-y)-W\lambda\left[1+\frac{F_2^a}{5}\right]\right]$$

$$+\frac{\omega_0}{\lambda}\frac{\partial\lambda}{\partial\omega_0}W\left\{\frac{5\omega_0^2}{4(\Delta_0)^2}(\lambda+1-y)+4\frac{\lambda F_2^a}{5}W\left[\lambda\frac{5\omega_0^2}{8(\Delta_0)^2}+1-y\right]\right.$$

$$+2\left[\frac{\lambda F_2^a}{5}\right]^2 W\left[2\lambda W+y\left[\frac{5\omega_0^2}{8(\Delta_0)^2}-1\right]\right]\Bigg\}\Bigg]$$

$$+\frac{1}{12\lambda}\left[1+\frac{\lambda F_2^a}{5}\right]^{-1}\Bigg\{-\left[1-y+\lambda\left[1+\frac{\lambda F_2^a}{5}W\right]\right]^2$$

$$\times\left[\frac{\omega_0^2}{12(\Delta_0)^2}+\frac{5}{7}\frac{\lambda F_2^a}{5}\left[1+\frac{\lambda F_2^a}{5}W\right]\right]$$

$$-\lambda W\frac{\lambda F_2^a}{5}\left[1-y+\lambda\left[1+\frac{\lambda F_2^a}{5}W\right]\right]\left[1+y\frac{F_2^a}{5}\right]$$

$$\times\left[\frac{\omega_0^2}{6(\Delta_0)^2}-\tfrac{12}{7}\left[1+\frac{\lambda F_2^a}{5}W\right]\right]$$

$$-\lambda^2\frac{\lambda F_2^a}{5}W^2\left[1+y\frac{F_2^a}{5}\right]^2\left[1+\frac{\lambda F_2^a}{5}\left[1-\frac{\omega_0^2}{6(\Delta_0)^2}\right]\right]\Bigg\}\Bigg], \tag{A15a}$$

$$R_2=-\frac{1}{36\lambda}\left[1+\lambda F_2^a\right]^{-1}(1-\tfrac{1}{4}m_z^2)\Bigg\{\tfrac{3}{2}\left[1-y+\lambda\left[1+\frac{\lambda F_2^a}{5}W\right]\right]^2\left[1+\frac{15}{7}\frac{\lambda F_2^a}{5}\right]$$

$$-\tfrac{33}{7}\lambda\frac{\lambda F_2^a}{5}W\left[1-y+\lambda\left[1+\frac{\lambda F_2^a}{5}W\right]\right]\left[1+y\frac{F_2^a}{5}\right]$$

$$+\tfrac{15}{2}\lambda^2\frac{\lambda F_2^a}{5}W^2\left[1+y\frac{F_2^a}{5}\right]^2\left[1+\frac{3}{5}\frac{\lambda F_2^a}{5}\right]\Bigg\}, \tag{A15b}$$

$$R_3=\tfrac{1}{49}\left[1+\frac{\lambda F_2^a}{5}\right]^{-1}(1-\tfrac{1}{9}m_z^2)\left\{1+\lambda\left[1+2\frac{\lambda F_2^a}{5}W\right]-y\left[1-\left[\frac{\lambda F_2^a}{5}\right]^2 W\right]\right\}$$

$$\times\Bigg[\left[x_3\left[\frac{F_2^a}{5}-\frac{F_4^a}{9}\right]-\frac{7\omega_0^2}{12(\Delta_0)^2}\right]\left\{1+\lambda\left[1+\frac{11}{4}\frac{\lambda F_2^a}{5}W\right]-y\left[1-\frac{7}{4}\left[\frac{\lambda F_2^a}{5}\right]^2 W\right]\right\}$$

$$-\frac{7}{4}x_3\frac{\lambda F_2^a}{5}W\left[1+\frac{\lambda F_2^a}{5}\right]\left[1+y\frac{F_2^a}{5}\right]\Bigg], \tag{A15c}$$

where g is the g factor for the linear Zeeman effect defined by $\omega_1=m_z g\omega_L$, and $A=(\Delta_0)^2Y_{3/2}A_2^a/5$.

APPENDIX B

In this appendix we collect (1) the matrix elements of the eigenvalue equation that determine the RSQ modes for finite wave vector, (2) the first-order corrections to the $\mathbf{q}=0$ eigenfunctions, and (3) the coefficients that determine the quadratic dispersion of the RSQ modes in a magnetic field.

(1) The matrix elements appearing in Eq. (77) are

$$L_{kj}^1=\delta_{kj}\left[\frac{1}{V_1}+\frac{\omega^2}{4(\Delta_0)^2}\left[\lambda+\frac{\eta^2}{\omega^2}f(\lambda)\right]\right]-p_kp_j\left[\lambda+\frac{\eta^2}{\omega^2}[\lambda+f(\lambda)]\right],$$

$$L_{kj}^2=-L_{kj}^3=\frac{i\omega}{2\Delta_0}\epsilon_{kij}p_i\left[\lambda+\frac{\eta^2}{\omega^2}[\lambda+f(\lambda)]\right],$$

$$L_{kj}^4=\delta_{kj}\left[-\lambda+\frac{\eta^2}{\omega^2}[1-2\lambda-f(\lambda)]\right]+p_kp_j\left[\lambda+\frac{\eta^2}{\omega^2}[\lambda+f(\lambda)]\right],$$

$$L_{kj}^5=-L_{kj}^7=i\eta\epsilon_{kij}p_i\frac{\lambda}{2\Delta_0},\quad L_{kj}^6=L_{kj}^8=\frac{\eta}{\omega}\delta_{kj}(1-\lambda), \tag{B1}$$

$$L_{kj}^{9}=\delta_{kj}\frac{\eta^2}{\omega^2}(1-\lambda)-p_k p_j\left[\lambda+\frac{\eta^2}{\omega^2}[\lambda+f(\lambda)]\right],$$

$$M_l^1=-M_l^2=\hat{p}_l\eta\frac{\lambda}{2\Delta_0},\quad M_j^3=-p_j\omega\frac{\lambda}{2\Delta_0},\quad M_k^4=\frac{\omega}{2\Delta_0}p_k\left[\lambda+\frac{\eta^2}{\omega^2}[\lambda+f(\lambda)]\right],$$

$$N=\frac{1}{V_1}+\frac{\omega^2}{4(\Delta_0)^2}\left[\lambda+\frac{\eta^2}{\omega^2}f(\lambda)\right],$$

where $f(\lambda)$ is defined in Eq. (A5), and $\eta=v_f\hat{\mathbf{p}}\cdot\mathbf{q}$.

(2) The first-order correction to the $\mathbf{q}=0$ eigenvector is $\underset{\sim}{d}{}_1^{\text{tr}}=(0,0,d^-,\mathbf{e}^-)$ with

$$d^-=\rho^0+\rho_{uv}^2 p_u p_v\ ,$$
$$e_i^-=\phi_{i,u}^1 p_u+\phi_{i,uvw}^3 p_u p_v p_w\ . \tag{B2}$$

For the $J=0$ contribution we find $\rho^0=0$, while the $J=2$ amplitudes for $\mathbf{q}=q\hat{y}$ are given by

$$\rho_{uv}^2=\frac{\lambda\omega_0}{2\Delta_0}\left[x_2+\frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right]^{-1}(\phi_{uv}^{(2,1)}+\phi_{uv}^{(2,3)})\ , \tag{B3}$$

$$\phi_{uv}^{(2,1)}=-\frac{iqv_F}{12\Delta_0}\frac{\lambda F_1^a}{3}W\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}P^{-1}\left[1+\frac{F_2^a}{5}\right]\left[x_2\left[1+\frac{\lambda F_3^a}{7}\right]+\frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right](\epsilon_{u2j}B_{jv}^{(2,1)}+\epsilon_{v2j}B_{ju}^{(2,1)})\ , \tag{B4}$$

$$\phi_{uv}^{(2,3)}=\frac{iqv_F}{12\Delta_0}\frac{\lambda F_3^a}{7}W\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}P^{-1}\left[1+\frac{F_2^a}{5}\right]\left[x_2\left[1+\frac{\lambda F_1^a}{3}\right]+\frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right](\epsilon_{u2j}B_{jv}^{(2,1)}+\epsilon_{v2j}B_{ju}^{(2,1)})\ ,$$

with

$$P=x_2\left[1+\frac{2}{5}\frac{\lambda F_1^a}{3}+\frac{3}{5}\frac{\lambda F_3^a}{7}\right]+\frac{\omega_0^2}{4(\Delta_0)^2}\lambda\ . \tag{B5}$$

The quantity $P(\omega)$ that enters the $J=2$ amplitudes has a simple physical interpretation. This factor reflects the coupling between the $S=1$, $J=2$ RSQ modes and the $S=0$, $l=2$ (d-wave) order parameter; $P(\omega)$ is the denominator for the d-wave order-parameter response, and $P(\omega^*)=0$ defines the excitation energy for d-wave Cooper pairs. This eigenvalue equation has a solution with $\omega^*<2\Delta$ provided $x_2^{-1}<0$; i.e., the d-wave pairing interaction is attractive. In the unlikely event of near degeneracy between the RSQ modes and the d-wave modes ($\omega^*=\omega_0$) the perturbation expansion for the mode dispersion breaks down. The more likely case is that the modes are well separated, in which case the d-wave modes contribute weakly to the dispersion by repelling the RSQ modes. The quantity $Q(\omega)$ defined in (A9) has a similar interpretation as the denominator for the $J=3+$ order-parameter response.

The $J=1$ and $J=3$ amplitudes also contribute and are found to be

$$\phi_{uv}^{(1,1)}=-\frac{iqv_F}{12\Delta_0}\frac{\lambda F_1^a}{3}\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}W\left[1+\frac{F_2^a}{5}\right](\epsilon_{u2j}B_{jv}^{(2,1)}-\epsilon_{v2j}^{(2,1)}B_{ju}^{(2,1)}+2\epsilon_{uvj}B_{j2}^{(2,1)})\ , \tag{B6}$$

$$\begin{aligned}\epsilon_{ujk}\phi_{kjv2}^{(3,3)}+\epsilon_{vjk}\phi_{kju2}^{(3,3)}=i\frac{qv_F}{\Delta_0}\frac{\lambda F_3^a}{7}W\left[1+\frac{\lambda F_2^a}{5}W\right]^{-1}P^{-1}\left[1+\frac{F_2^a}{5}\right]\\ \times\left\{B_{uv}^{(2,1)}\left[x_2\left[1+\frac{5}{9}\frac{\lambda F_1^a}{3}+\frac{4}{9}\frac{\lambda F_3^a}{7}\right]+\frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right]\right.\\ \left.+\tfrac{1}{4}(B_{u2}^{(2,1)}\delta_{v2}+B_{v2}^{(2,1)}\delta_{u2}-\tfrac{2}{3}B_{22}^{(2,1)}\delta_{uv})\left[x_2\left[1-\frac{1}{15}\frac{\lambda F_1^a}{3}+\frac{16}{15}\frac{\lambda F_3^a}{7}\right]+\frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right]\right\}\ .\end{aligned} \tag{B7}$$

(3) The above relations for the first-order corrections to the $\mathbf{q}=0$ eigenfunctions are used to evaluate the matrix elements in Eq. (78) for the quadratic dispersion of the RSQ modes,

$$\underset{\sim}{b}_0 * \underline{L}^{(2)} * \underset{\sim}{d}_0 = \frac{5\lambda}{126} V_1 \left[\frac{qv_F}{\Delta_0}\right]^2 \left[1+\frac{\lambda F_2^a}{5} W\right]^{-2}$$
$$\times \left\{ \bar{B}_{ju}^{(2,1)} B_{ju}^{(2,1)} \left[-\frac{5}{3} W^2 \frac{\lambda F_2^a}{5}\left[1+\frac{1}{2}\frac{F_2^a}{5}\right] - \frac{3\omega_0^2}{16(\Delta_0)^2} + \frac{5f(\lambda)}{6\lambda}\left[1-\frac{29\omega_0^2}{40(\Delta_0)^2}+\frac{19\omega_0^2}{160(\Delta_0)^2}\right]\right]\right.$$
$$\left. + \bar{B}_{j2}^{(2,1)} B_{j2}^{(2,1)} \left[-2 W^2 \frac{\lambda F_2^a}{5}\left[1+\frac{1}{2}\frac{F_2^a}{5}\right] - \frac{3\omega_0^2}{4(\Delta_0)^2} + \frac{f(\lambda)}{\lambda}\left[1-\frac{5\omega_0^2}{4(\Delta_0)^2}+\frac{\omega_0^4}{4(\Delta_0)^4}\right]\right]\right\}, \tag{B8}$$

$$\underset{\sim}{b}_0 * \underline{L}^{(1)} * \underset{\sim}{d}_1 = -\frac{\lambda}{18} V_1 \left[\frac{qv_F}{\Delta_0}\right]^2 \left[1+\frac{F_2^a}{5}\right]^2 \left[1+\frac{\lambda F_2^a}{5} W\right]^{-2} W^2 P^{-1}$$
$$\times \left[\bar{B}_{ju}^{(2,1)} B_{ju}^{(2,1)} \left[\left[x_2 + \frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right]\left[\frac{1}{6}\frac{\lambda F_1^a}{3}+\frac{3}{7}\frac{\lambda F_3^a}{7}\right] + \frac{1}{21} x_2 \frac{\lambda F_3^a}{7}\left[4\frac{\lambda F_3^a}{7}+\frac{17}{2}\frac{\lambda F_1^a}{3}\right]\right]\right.$$
$$+ \bar{B}_{2u}^{(2,1)} B_{2u}^{(2,1)} \left\{ \left[x_2 + \frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right]\left[\frac{1}{2}\frac{\lambda F_1^a}{3}+\frac{3}{14}\frac{\lambda F_3^a}{7}\right]\right.$$
$$\left.\left. + \tfrac{1}{5} x_2 \left[\frac{3}{2}\left[\frac{\lambda F_1^a}{3}\right]^2 + \frac{13}{14}\frac{\lambda F_1^a}{3}\frac{\lambda F_3^a}{7} + \frac{8}{7}\left[\frac{\lambda F_3^a}{7}\right]^2\right]\right\}\right]. \tag{B9}$$

The above equations are also used to obtain the field-dependent mode dispersion given by Eqs. (83) and (84) with

$$a = \frac{1}{18}\left[\frac{qv_F}{\Delta_0}\right]^2 \lambda \left[1+\frac{\lambda F_2^a}{5} W\right]^{-2} \left[\frac{5}{7}\left[-3W^2\frac{\lambda F_2^a}{5}\left[1+\frac{1}{2}\frac{F_2^a}{5}\right] - \frac{15\omega_0^2}{32(\Delta_0)^2} + \frac{3f(\lambda)}{2\lambda}\left[1-\frac{13\omega_0^2}{16(\Delta_0)^2}+\frac{9\omega_0^4}{64(\Delta_0)^4}\right]\right]\right.$$
$$- \left[1+\frac{F_2^a}{5}\right]^2 W^2 P^{-1} \left\{ \left[x_2 + \frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right]\left[\frac{3}{8}\frac{\lambda F_1^a}{3}+\frac{39}{56}\frac{\lambda F_3^a}{7}\right]\right.$$
$$\left.\left. + x_2\left[\frac{3}{40}\left[\frac{\lambda F_1^a}{3}\right]^2 + \frac{183}{280}\frac{\lambda F_1^a}{3}\frac{\lambda F_3^a}{7} + \frac{12}{35}\left[\frac{\lambda F_3^a}{7}\right]^2\right]\right\}\right], \tag{B10}$$

$$b = \frac{1}{144}\left[\frac{qv_F}{\Delta_0}\right]^2 \lambda \left[1+\frac{\lambda F_2^a}{5} W\right]^{-2} \left[\frac{5}{7}\left[-2W^2\frac{\lambda F_2^a}{5}\left[1+\frac{1}{2}\frac{F_2^a}{5}\right] - \frac{3\omega_0^2}{4(\Delta_0)^2} + \frac{f(\lambda)}{\lambda}\left[1-\frac{5\omega_0^2}{4(\Delta_0)^2}+\frac{\omega_0^4}{4(\Delta_0)^4}\right]\right]\right.$$
$$- \left[1+\frac{F_2^a}{5}\right]^2 W^2 P^{-1} \left\{ \left[x_2 + \frac{\omega_0^2}{4(\Delta_0)^2}\lambda\right]\left[\frac{1}{2}\frac{\lambda F_1^a}{3}+\frac{3}{14}\frac{\lambda F_3^a}{7}\right]\right.$$
$$\left.\left. + x_2\left[\frac{3}{10}\left[\frac{\lambda F_1^a}{3}\right]^2 + \frac{13}{70}\frac{\lambda F_1^a}{3}\frac{\lambda F_3^a}{7} + \frac{8}{35}\left[\frac{\lambda F_3^a}{7}\right]^2\right]\right\}\right].$$

These equations neglect small field-dependent corrections of order $(qv_f/\Delta_0)^2(\gamma H/\Delta_0)$.

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