Acoustical activity in the framework of the rotation-gradient theory of elasticity

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The phenomenon of acoustical activity is examined within the framework of the rotation-gradient theory of elasticity. The nonvanishing independent components of the fourth-rank pseudotensor responsible for acoustical activity are worked out and tabulated for all the crystal classes. In the context of acoustic wave propagation the concept of pure mode axes is discussed when the acoustical activity tensor has nonzero components. The point-group symmetries allowing acoustical activity are identified. These results are compared with those based on the theory of acoustical activity due to Portigal and Burstein.

I. INTRODUCTION

The concept of acoustical activity was introduced in 1960 independently by Andronov[1] and Silin.[2] In an acoustically active crystal the plane of polarization of a plane-polarized transverse acoustic wave traveling along an acoustic axis gets progressively rotated. This phenomenon was observed in α quartz by Pine.[3]

In literature, the phenomenon of acoustical activity has found a variety of explanations. For example, Portigal and Burstein[4] explained it on the basis of spatial dispersion of an elastic stiffness tensor, in much the same way as the optical activity has been explained by the spatial dispersion of the dielectric constant. The rotation-gradient theory of elasticity by Toupin[5] also leads to this effect. A linearized version of this theory has been developed by Mindlin and Tiersten.[6] We shall be primarily concerned with Mindlin's approach in this paper. The polarization-gradient theory of elastic dielectrics by Mindlin and Toupin[7] also leads to the explanation of both optical and acoustical activity in one single formulation.

Portigal and Burstein[4] used a fifth-rank tensor $\underline{d}$ for describing this effect, very crudely speaking, by allowing a wave-vector dependence to the elastic stiffness tensor $\underline{c}$:

$$c_{ijkl}(q,\omega)=c_{ijkl}(0,\omega)+iq_m d_{ijklm} \ . \tag{1.1}$$

The fifth-rank tensor $\underline{d}$ is called the acoustic gyrotropic tensor.[4] From considerations of causality, as well as those of invariance of the crystal Hamiltonian under time reversal, $\underline{d}$ has been shown[4] to have the following symmetry properties:

$$d_{ijklm}=d_{jiklm}=d_{ijlkm} \ , \tag{1.2}$$

$$d_{ijklm}=-d_{klijm} \ . \tag{1.3}$$

Recently, we have introduced[8,9] a fourth-rank pseudotensor $\underline{G}$ which is completely equivalent to the fifth-rank tensor $\underline{d}$.

$$G_{qkmn}=\tfrac{1}{2}(e_{ilq}d_{iklmn}) \ . \tag{1.4}$$

The above relation can be inverted to yield

$$d_{iklmn}=e_{ilq}G_{qkmn}+e_{kmq}G_{qiln} \ . \tag{1.5}$$

Here, e_{ilq} is the Levi-Civeta tensor, a totally antisymmetric tensor of rank 3. The tensor $\underline{G}$ has the following two properties:[8]

$$G_{qkmn}=G_{qmkn} \ , \tag{1.6}$$

$$\sum_k G_{kkmn}=0 \ , \tag{1.7}$$

for each $m,n=1,2,3$. Both the tensors $\underline{G}$ and $\underline{d}$ have a maximum of only 45 independent components. In Ref. 9 we have examined the wave propagation along an acoustic axis in terms of the tensor $\underline{G}$ and have also identified the acoustically active crystal classes, where our results were found to be in complete agreement with those of Portigal and Burstein, who worked with the fifth-rank tensor $\underline{d}$.[4] It would therefore be of interest to see what crystal symmetries would allow acoustical activity if one follows other descriptions of this phenomenon. In this paper we examine this question employing the viewpoint of the rotation-gradient theory of linear elasticity,[6] which also describes acoustical activity in terms of a fourth-rank pseudotensor. The paper is organized as follows.

In Sec. II we briefly recapitulate the basic results of the rotation-gradient theory.[6] The nonvanishing independent components of the fourth-rank pseudotensor responsible for the acoustical activity are worked out and listed in Table I. Section III deals with the wave propagation along an acoustic axis. We examine the validity, in the presence of a rotation gradient, of the sufficiency conditions given by Waterman[10] for a given propagation direction to be a pure mode axis. We also derive an expression for the acoustical rotatory power in terms of the components of the pseudotensor. On the basis of these results and Table I, the crystal symmetries allowing acoustical activity are identified and tabulated (Table II). In Sec. IV, our results are compared with earlier results based on the use of the tensor $\underline{d}$ for describing acoustical activity[4,9] and conclusions are summarized in Sec. V.

TABLE I. Nonvanishing independent components of the fourth-rank acoustic gyrotropic tensor $\underline{b}$. This tensor has the symmetry expressed by Eq. (2.6a) and is subject to the constraint expressed by Eq. (2.6b). Therefore components of the form b_{ij33} are not listed in this table because they can always be obtained from b_{ijkk}, $k \neq 3$. The tensor $\underline{b}$ is identically equal to zero for all 11 centrosymmetric crystal classes. For noncentrosymmetric crystal classes the zero components of this tensor are indicated by a blank. The nonzero components b_{ijkl} are simply written as $ijkl$. An independent component is indicated by writing its symbol in the appropriate place, and a dependent component is expressed in terms of other independent components. Two additional symbols are introduced: (A) denotes $b_{2211}-2b_{1212}$ and (B) denotes $\frac{1}{2}(b_{1112}+b_{1121})$. For the monoclinic system, the twofold axis is taken along x_3.

	Triclinic	Monoclinic		Orthorhombic		Tetragonal				
	1	m	2	$2mm$	222	4	$\bar{4}$	$4mm$	$\bar{4}2m$	422
Component	C_1	C_s	C_2	C_{2v}	D_2	C_4	S_4	C_{4v}	D_{2d}	D_4
1111	1111		1111		1111	1111	1111		1111	1111
2211	2211		2211		2211	2211	2211		2211	2211
3311	3311		3311		3311	3311	3311		3311	3311
2311	2311	2311								
1311	1311	1311								
1211	1211		1211	1211		1211	1211	1211		
1122	1122		1122		1122	2211	−2211		−2211	2211
2222	2222		2222		2222	1111	−1111		−1111	1111
3322	3322		3322		3322	3311	−3311		−3311	3311
2322	2322	2322								
1322	1322	1322								
1222	1222		1222	1222		−1211	1211	−1211		
1123	1123	1123								
2223	2223	2223								
3323	3323	3323								
2323	2323		2323		2323	2323	2323		2323	2323
1323	1323		1323	1323		1323	1323	1323		
1223	1223	1223								
1132	1132	1132								
2232	2232	2232								
3332	3332	3332								
2332	2332		2332		2332	2332	2332		2332	2332
1332	1332		1332	1332		1332	1332	1332		
1232	1232	1232								
1113	1113	1113								
2213	2213	2213								
3313	3313	3313								
2313	2313		2313	2313		−1323	1323	−1323		
1313	1313		1313		1313	2323	−2323		−2323	2323
1213	1213	1213								
1131	1131	1131								
2231	2231	2231								
3331	3331	3331								
2331	2331		2331	2331		−1332	1332	−1332		
1331	1331		1331		1331	2332	−2332		−2332	2332
1231	1231	1231								
1112	1112		1112	1112		1112	1112	1112		
2212	2212		2212	2212		2212	2212	2212		
3312	3312		3312	3312		3312	3312	3312		
2312	2312	2312								
1312	1312	1312								
1212	1212		1212		1212	1212	1212		1212	1212
1121	1121		1121	1121		−2212	2212	−2212		
2221	2221		2221	2221		−1112	1112	−1112		
3321	3321		3321	3321		−3312	3312	−3312		
2321	2321	2321								
1321	1321	1321								
1221	1221		1221		1221	1212	−1212		−1212	1212

TABLE I. (*Continued*).

	Trigonal			Hexagonal					Cubic		
	3				6	$\bar{6}m2$	$6mm$	622	23	$42m$	432
Component	C_3	$3m$	32	$\bar{6}$	C_6	D_{3h}	C_{6v}	D_6	T	T_d	0
1111	(A)		(A)		(A)			(A)	1111		1111
2211	2211		2211		2211			2211	2211	2211	2211
3311	3311		3311		3311			3311	3311	−2211	2211
2311	2311	2311	2311	2311		2311					
1311	1311			1311							
1211	$-(B)$	$-(B)$			$-(B)$		$-(B)$				
1122	2211		2211		2211			2211	3311	−2211	2211
2222	(A)		(A)		(A)			(A)	1111		1111
3322	3311		3311		3311			3311	2211	2211	2211
2322	−2311	−2311	−2311	−2311		−2311					
1322	−1311			−1311							
1222	(B)	(B)			(B)		(B)				
1123	1123	1123	1123	1123		1123					
2223	−1123	−1123	−1123	−1123		−1123					
3323											
2323	2323		2323		2323			2323	2323	2323	2323
1323	1323	1323			1323		1323				
1223	1223			1223							
1132	1132	1132	1132	1132		1132					
2232	−1132	−1132	−1132	−1132		−1132					
3332											
2332	2332		2332		2332			2332	2332	−2232	2323
1332	1332	1332			1332		1332				
1232	1232			1232							
1113	−1223			−1223							
2213	1223			1223							
3313											
2313	−1323	−1323			−1323		−1323				
1313	2323		2323		2323			2323	2332	−2323	2323
1213	1123	1123	1123	1123		1123					
1131	−1232			−1232							
2231	1232			1232							
3331											
2331	−1332	−1332			−1332		−1332				
1331	2332		2332		2332			2332	2323	2323	2323
1231	1132	1132	1132	1132		1132					
1112	1112	1112			1112		1112				
2212	2212	2212			2212		2212				
3312	3312	3312			3312		3312				
2312	−1311			−1311							
1312	2311	2311	2311	2311		2311					
1212	1212		1212		1212			1212	2323	2323	2323
1121	−2212	−2212			−2212		−2212				
2221	−1112	−1112			−1112		−1112				
3321	−3312	−3312			−3312		−3312				
2321	−1311			−1311							
1321	2311	2311	2311	2311		2311					
1221	1212		1212		1212			1212	2332	−2332	2323

II. BASIC FORMULATION

In the rotation gradient theory of elasticity[5,6] one represents elastic deformation by two tensors, the usual strain tensor $\underline{\epsilon}$ and the rotation gradient $\underline{\chi}$. To first order in the displacement $\mathbf{u}$, its gradient $\nabla\mathbf{u}$ and the second gradient $\nabla\nabla\mathbf{u}$, one has the following expressions[6] for $\underline{\epsilon}$ and $\underline{\chi}$:

TABLE II. Classification with regard to acoustical activity of noncentrosymmetric crystal classes containing a proper axis of threefold or higher rotational symmetry. All the centrosymmetric classes are acoustically inactive. The cubic systems 23 (T) and 432 (0) allow acoustical activity along both the threefold and fourfold symmetry axes.

	Tetragonal	Trigonal	Hexagonal	Cubic
Active	4, 422	3, 32	6, 622	23, 432
Inactive	$4mm$, $\bar{4}$, $\bar{4}2m$	$3m$	$6mm$, $\bar{6}$, $\bar{6}m2$	$\bar{4}2m$

$$\underline{\epsilon}=\tfrac{1}{2}(\nabla\mathbf{u}+\mathbf{u}\nabla)\ , \tag{2.1a}$$

$$\underline{\chi}=\tfrac{1}{2}\nabla(\nabla\times\mathbf{u})\ . \tag{2.1b}$$

The elastic energy density W in the presence of strain and the rotation gradient can be expressed as[6]

$$W=\tfrac{1}{2}\underline{\chi}{:}\underline{a}{:}\underline{\chi}+\underline{\epsilon}{:}\underline{b}{:}\underline{\chi}+\tfrac{1}{2}\underline{\epsilon}{:}\underline{c}{:}\underline{\epsilon}\ , \tag{2.2}$$

and the constitutive relations for $\underline{\tau}_s$, the symmetric part of the stress, and $\underline{\mu}_D$, the nonscalar part of the couple stress, take the form[6]

$$\underline{\tau}_s=\left[\frac{\partial W}{\partial\underline{\epsilon}}\right]=\underline{c}{:}\underline{\epsilon}+\underline{b}{:}\underline{\chi}\ , \tag{2.3a}$$

$$\underline{\mu}_D=\left[\frac{\partial W}{\partial\underline{\chi}}\right]=\underline{\epsilon}{:}\underline{b}+\underline{\chi}{:}\underline{a}\ . \tag{2.3b}$$

The equation of motion for small displacement $\mathbf{u}$ is given by[6]

$$\ddot{\mathbf{u}}=\nabla\cdot\underline{\tau}_s+\tfrac{1}{2}\nabla\times(\nabla\cdot\underline{\mu}_D)\ . \tag{2.4}$$

It should be noted that the tensors $\underline{a}$ and $\underline{c}$ are polar tensors while $\underline{b}$ is a pseudotensor. Furthermore, they have the following symmetry properties:[6]

$$a_{ijkl}=a_{klij}\ , \tag{2.5a}$$

$$\sum_k a_{ijkk}=0\ , \tag{2.5b}$$

$$b_{ijkl}=b_{jikl}\ , \tag{2.6a}$$

$$\sum_k b_{ijkk}=0\ , \tag{2.6b}$$

$$c_{ijkl}=c_{jikl}=c_{klij}\ . \tag{2.7}$$

Using the expression (2.2) for W, the constitutive relations (2.3), the definitions (2.1), and the symmetry properties (2.5)–(2.7), we can write the equation of motion (2.4) explicitly in the component form:

$$\begin{aligned}\rho\ddot{u}_i=c_{ijkl}\frac{\partial^2 u_k}{\partial x_j\,\partial x_1}&+\tfrac{1}{2}b_{ijkl}e_{lmn}\frac{\partial^3 u_n}{\partial x_j\,\partial x_k\,\partial x_m}\\&+\tfrac{1}{2}b_{kljn}e_{imn}\frac{\partial^3 u_k}{\partial x_j\,\partial x_l\,\partial x_m}\\&+\tfrac{1}{4}a_{kljn}e_{imn}e_{lpr}\frac{\partial^2 u_r}{\partial x_j\,\partial x_k\,\partial x_m\,\partial x_p}\ .\end{aligned} \tag{2.8}$$

The nonvanishing independent components of the tensor $\underline{b}$ for all of the crystallographic point groups are listed in Table I. A similar exercise has been carried out for the tensor $\underline{a}$, but we do not give a detailed table. However, the relevant results are quoted in the text.

III. WAVE PROPAGATION ALONG A PURE MODE AXIS

We shall now seek a wavelike solution of Eq. (2.8) in the form of plane waves

$$\mathbf{u}=\mathbf{v}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\ , \tag{3.1}$$

substituting the solution (3.1) in (2.8) we get a system of algebraic equations for the polarization amplitudes v_i:

$$\begin{aligned}\rho\omega^2 v_i=c_{ijkl}q_jq_lv_k&+(i/2)b_{ijkl}e_{lmn}q_jq_kq_mv_n\\&+(i/2)b_{kljn}e_{imn}q_jq_mq_lv_k\\&-(\tfrac{1}{4})a_{kljn}e_{imn}e_{lpr}q_jq_mq_kq_pv_r\ .\end{aligned} \tag{3.2}$$

It is advantageous to transform to a new coordinate frame such that the z axis (axis 3) is along the direction of propagation $\mathbf{q}$. We shall denote the transformed components as $\tilde{v}_i$, $\tilde{c}_{ijkl}$, etc. If q, β, α are the spherical polar coordinates of the wave vector $\mathbf{q}$ with respect to the standard crystal frame of reference,[11] the transformation relation for $(\tilde{v}_i)$ is

$$\begin{bmatrix}\tilde{v}_1\\\tilde{v}_2\\\tilde{v}_3\end{bmatrix}=\begin{bmatrix}\cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta\\ -\sin\alpha & \cos\alpha & 0\\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\ . \tag{3.3}$$

Under the above transformation Eq. (3.2) simplifies into

$$\begin{aligned}\rho\omega^2\tilde{v}_i=q^2\tilde{c}_{i3k3}\tilde{v}_k&+(iq^3/2)\tilde{b}_{i33l}e_{l3n}\tilde{v}_n\\&+(iq^3/2)\tilde{b}_{k33n}e_{i3n}\tilde{v}_k\\&-(q^4/4)\tilde{a}_{3l3n}e_{i3n}e_{l3r}\tilde{v}_r\ .\end{aligned} \tag{3.4}$$

When written out explicitly, Eq. (3.4) reads as

$$\rho\omega^2\tilde{v}_1 = q^2\tilde{c}_{13k3}\tilde{v}_k - (iq^3/2)(\tilde{b}_{1331}+\tilde{b}_{2332})\tilde{v}_2 - (iq^3/2)\tilde{b}_{3332}\tilde{v}_3 + (q^4/4)(\tilde{a}_{3232}\tilde{v}_1 - \tilde{a}_{3132}\tilde{v}_2) , \quad (3.5a)$$

$$\rho\omega^2\tilde{v}_2 = q^2\tilde{c}_{23k3}\tilde{v}_k + (iq^3/2)(\tilde{b}_{1331}+\tilde{b}_{2332})\tilde{v}_1 + (iq^3/2)\tilde{b}_{3331}\tilde{v}_3 - (q^4/4)(\tilde{a}_{3231}\tilde{v}_1 - \tilde{a}_{3131}\tilde{v}_2) , \quad (3.5b)$$

$$\rho\omega^2\tilde{v}_3 = q^2\tilde{c}_{33k3}\tilde{v}_k + (iq^3/2)(\tilde{b}_{3332}\tilde{v}_1 - \tilde{b}_{3331}\tilde{v}_2) . \quad (3.5c)$$

If a given propagation direction x_3 is to be a pure mode axis, the longitudinal-polarization component $\tilde{v}_3$ must not depend on the transverse-polarization components $\tilde{v}_1$ and $\tilde{v}_2$, and vice versa. This requires that the crystal symmetry be such that certain components of the tensor $\underline{\tilde{c}}$ and $\underline{\tilde{b}}$ in Eq. (3.5) vanish identically. It is easily seen that this leads to the following conditions:

$$\tilde{c}_{1333}(\omega) = \tilde{c}_{2333}(\omega) = \tilde{c}_{3313}(\omega) = \tilde{c}_{3323}(\omega) = 0 , \quad (3.6a)$$

$$\tilde{b}_{3331}(\omega) = \tilde{b}_{3332}(\omega) = 0 . \quad (3.6b)$$

We shall now examine the compatibility of conditions (3.6) with those given by Waterman.[10] He derived the following sufficient conditions for a propagation direction x_3 in a crystal to be a pure mode axis (*in the absence of rotation gradients*).

(i) x_3 should be a proper axis of twofold or higher rotational symmetry.

(ii) x_3 should be normal to a mirror plane, or normal to a proper axis of sixfold symmetry.

When the wave propagation is along a twofold axis, the minimum point symmetry of the crystal must be monoclinic (C_2). It is straightforward to verify (cf. Ref. 11) that Eqs. (3.6a) are compatible with both of Waterman's sufficiency requirements.

In the presence of rotation gradients, Eqs. (3.6b) represent the additional requirements for a propagation direction to be a pure mode axis. Referring to Table I for the crystal class 2 (C_2), we find that Eqs. (3.6b) are satisfied under Waterman's condition (i). We now examine condition (ii).

If a propagation direction is normal to a mirror plane, the minimum symmetry of the crystal must be m (C_m). Table I shows that b_{3331} ($=\tilde{b}_{3331}$) and b_{3332} ($=\tilde{b}_{3332}$) are not zero for this crystal class. Similarly, when an acoustic wave is propagating normal to an axis of sixfold symmetry, it is implied that the minimum symmetry of the crystal is 6 (C_6). Suppose a wave is propagating in a crystal with this symmetry along the negative x axis of the standard crystallographic frame of reference;[11] that is x_3 is along $-x$. Then $\alpha=\pi$ and $\beta=\pi/2$ in Eq. (3.3) and the component b_{1112} (which is nonzero according to Table I) transforms to $-\tilde{b}_{3332}$. Therefore $\tilde{b}_{3332}\neq 0$ for this situation. [However, the component b_{1113}, which goes into $\tilde{b}_{3331}$ under the transformation of Eq. (3.3), does happen to be zero for C_6 and higher symmetries.]

In summary, pure mode axes continue to be pure mode axes even in the presence of rotation gradients if they are along axes of twofold or higher rotational symmetry. But if the pure mode axis in the absence of rotation gradients is normal to a mirror plane or normal to a sixfold axis, it may not remain a pure mode axis when rotation gradients are explicitly taken into account.

A. Degenerate pure mode axis

A pure mode axis is called a *degenerate pure mode axis* if the pure transverse-acoustic waves traveling along this axis are not only completely decoupled from the pure longitudinal mode but also have the same phase velocity in the absence of rotation-gradient effects. It is clear that, if we neglect the rotation gradient, the transverse modes $\tilde{v}_1$ and $\tilde{v}_2$ would be uncoupled if

$$\tilde{c}_{1323}(\omega) = \tilde{c}_{2313}(\omega) = 0 . \quad (3.7)$$

Furthermore, the two pure, completely uncoupled, transverse waves would have the same velocity $V=\omega/q$ if

$$\tilde{c}_{1313}(\omega) = \tilde{c}_{2323}(\omega) . \quad (3.8)$$

The conditions (3.7) and (3.8) are satisfied if the direction of propagation is a proper axis of threefold or higher rotational symmetry.[11] This is in agreement with the analysis of Waterman.[10] For such a situation, i.e., if the minimum point symmetry is a proper axis of threefold or higher rotational symmetry, (i) the components $\tilde{a}_{3132}$ and $\tilde{a}_{3231}$ vanish identically so that the tensor $\underline{a}$ does not introduce any coupling between the two transverse waves, and (ii) the components $\tilde{a}_{3131}$ and $\tilde{a}_{3232}$ are equal. The tensor $\underline{b}$, however, introduces a coupling between the transverse components. Thus if the direction of wave propagation is an acoustic axis, Eq. (3.5) leads to the conclusion that the two uncoupled transverse waves are circularly polarized and travel with speeds V_+ (V_-) given by

$$\rho V_\pm^2 = \tilde{c}_{2323}(\omega) + \frac{q^2}{4}\tilde{a}_{3232} \pm \frac{q}{2}(\tilde{b}_{1331}+\tilde{b}_{2332}) . \quad (3.9)$$

Thus acoustical activity along an acoustic axis is possible *only if* $\tilde{b}_{1331}+\tilde{b}_{2332}\neq 0$ for the crystal. When such is the case, the plane of polarization of a plane-polarized transverse acoustic wave will be rotated, on traversing a length l along the acoustic axis, by an angle ϕ, given by[4]

$$\varphi = \tfrac{1}{2}\omega l (V_-^{-1} - V_+^{-1}) \quad (3.10)$$

which on using Eq. (3.9) becomes

$$\varphi = \frac{\omega^2 l\rho(\tilde{b}_{1331}+\tilde{b}_{2332})}{4c_{2323}^2} . \quad (3.11)$$

B. Occurrence of acoustical activity

We shall consider the acoustical activity along a degenerate pure mode axis (i.e., an acoustic axis). It can occur only in crystal classes with a threefold or higher rotational symmetry provided that $(\tilde{b}_{1331}+\tilde{b}_{2332})\neq 0$. For the uniaxial systems, viz., tetragonal, trigonal, and hexagonal, the z axis (x_3 axis) of the standard crystallographic frame coincides with the acoustic axis (which is also the direction of wave propagation under consideration), we have $\tilde{\underline{b}}=\underline{b}$ and $\tilde{\underline{c}}=\underline{c}$, etc. For the cubic system, for wave propagation along the x_3 axis which is a fourfold axis of symmetry, we again have $\tilde{\underline{b}}=\underline{b}$, etc. There are also four threefold axes along the $\langle 111\rangle$ directions, and in this case, for the [111] direction, for example,

$$\tilde{b}_{1331}+\tilde{b}_{2332}=\tfrac{1}{3}(b_{1331}+b_{2332})+\tfrac{2}{3}b_{111} .$$

Referring to Table I we find that, in addition to the 11 centrosymmetric crystal classes, the following eight classes cannot exhibit acoustical activity along an acoustic axis: $\bar{4}$ (S_4), $4mm$ (C_{4v}), $\bar{4}2m$ (D_{2d}), $3m$ (C_{3V}), $\bar{6}$ (C_{3h}), $6mm$ (C_{6v}), $\bar{6}m2$ (D_{3h}), and $\bar{4}3m$ (T_d). The following set of eight point groups can exhibit pure acoustical activity: 4 (C_4), 422 (D_4), 3 (C_3), 32 (D_3), 6 (C_6), 622 (D_6), 23 (T), and 432 (O). These results are summarized in Table II.

IV. COMPARISON WITH EARLIER RESULTS

The tensor $\underline{b}$ describing the acoustical activity in the rotation gradient theory has symmetry different from that of the tensor $\underline{G}$ of the earlier theory based on spatial dispersion of the elastic stiffness tensor. The maximum number of nonvanishing independent components is also different in the two cases, viz., 48 in the case of the tensor $\underline{b}$ and 45 for the tensor $\underline{G}$. However, there is no difference in the prediction with regard to the acoustically active crystal classes. The crystal classes allowing acoustical activity with respect to the tensor $\underline{b}$ are precisely those listed earlier on the basis of the theory which employs the tensor $\underline{G}$ (Ref. 9) or equivalently the fifth-rank tensor $\underline{d}$ (Ref. 4). Even the sufficiency condition for a direction of propagation to be a pure mode axis is the same in both the descriptions of acoustical activity.

V. CONCLUSIONS

In the linearized rotation-gradient theory of elasticity, the energy density is specified in terms of three tensors $\underline{c}$, $\underline{b}$, and $\underline{a}$, representing the interactions between strain-strain, strain-rotation gradient, and the rotation gradient with itself. As far as the wave propagation along a pure mode axis is concerned, the tensor $\underline{a}$ does not play any significant role. But the tensor $\underline{b}$ gives rise to acoustical activity for the following eight crystal symmetries: 4 (C_4), 422 (D_4), 3 (C_3), 32 (D_3), 6 (C_6), 622 (D_6), 23 (T), and 432 (O). The acoustical activity tensor $\underline{b}$ of the rotation-gradient theory cannot be directly compared with the tensor $\underline{G}$ of the theory of spatial dispersion even though both are pseudotensors of rank 4. The final results, regarding the possible pure mode axes as well as the occurrence of acoustical activity in a given crystal class, are identical in both the cases.

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