Invariant-imbedding approach to resistance fluctuations in disordered one-dimensional conductors

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The invariant-imbedding method provides a first-order differential equation for the complex reflection amplitude $R(L)$ of a one-dimensional conductor of length L, in terms of the random potential $V(L)$ at the edge of incidence for a particle of energy E. The Landauer formula is used to express the resistance $\rho(L)$ and the conductance $\rho^{-1}(L)$ in terms of $R(L)$ and to derive an exact second-order differential equation for $\rho(L)$, from the above equation for $R(L)$. This equation for $\rho(L)$ emphasizes important aspects of the problem but has not been solved explicitly. However, two types of explicit solutions, referred to as (a) and (b), have been derived, starting from the differential equation for $R(L)$. Solution (a) is valid in the special case where typical fluctuations of $V(L)$ about an average *barrier* $V>0$ are comparable with $2E-V$. Solution (b) is obtained by using an intuitively justified averaging over the phase angle of $R(L)$ to derive an approximate first-order differential equation for the resistance, valid for both signs of V and for arbitrary E. This equation yields an expression for $\rho(L)$ in terms of $V(L)$ which is formally similar to that obtained in case (a). Averaging solutions (a) and (b) for $\rho(L)$ over Gaussian, δ-correlated variables $V(L)-V$ yields generalized Landauer expressions for $\langle\rho(L)\rangle$ and infinite average conductances $\langle\rho^{-1}(L)\rangle$, for arbitrary L. Closed-form expressions for the higher moments $\langle\rho^n(L)\rangle$ can also be given and reveal that $\rho(L)$ has no central limit. The exact probability distributions of $R(L)$ in case (a) and of both $\rho(L)$ and $\rho^{-1}(L)$ in cases (a) and (b) are obtained for arbitrary L, using a moment method. It is found that the variable $\ln[(1-iR)(1+iR)^{-1}]$ in case (a) ($V>0$) and the variables $\ln\rho(\ln\rho^{-1})$, $\rho\gg 1$, for $V<0$ in case (b) have Gaussian distributions with mean values and variances that scale linearly with L. In particular, this confirms $\ln\rho$ as the correct scaling variable for large L in the case $V<0$, but shows that the analogous scaling variable for $V>0$ is the quantity $\ln[(1-iR)(1+iR)^{-1}]$. On the other hand, the distributions of $\ln\rho(\ln\rho^{-1})$, $\rho\gg 1$, for $V>0$, for both solutions (a) and (b) are rapidly decaying exponentials, corresponding to weak Gaussian tails. Averages of $|R(L)|$ and of $|R(L)|^2$, which determine the average electron density outside the conductor, are also studied and their asymptotic, length-dependent rates of exponential growth are obtained. In an appendix solution (a) is formally generalized to include the effect of small deviations from the electron-energy range considered above.

I. INTRODUCTION

It is well established that the slightest amount of disorder in a perfect lattice suffices to localize all electronic states in one dimension. It follows, therefore, that a disordered chain of infinite length has infinite electrical resistance at zero temperature. On the other hand, due to the exponential localization of the states, the average residual resistance of long finite chains grows exponentially with length L, as was first shown by Landauer.[1,2] In the limit of very short chains, the Landauer result reduces to the linear length dependence which characterizes the classical additive behavior of resistance. Landauer's analysis is based on a general formula for the zero-temperature resistance G^{-1} of a one-dimensional conductor, the so-called Landauer formula[1]

$$G^{-1}=\frac{\pi\hbar}{e^2}\rho,\quad \rho=\frac{r}{t},\quad t=1-r\ , \tag{1}$$

where the dimensionless resistance ρ is the ratio of the reflection and transmission coefficients of the system. In recent years, the Landauer formula has been discussed and rederived by many authors[2-9] in attempts to strengthen its theoretical basis.

Along with the exponential growth of the average resistance with length, one finds that the relative mean-square fluctuations also grow exponentially at an even faster rate than the average value. This would indicate that viewing a macroscopic chain of length L as an ensemble of statistically equivalent, independent subunits of smaller size, as in the usual ensemble description, is actually meaningless. Indeed, since the relative mean dispersion of resistances of such subunits would always be smaller than the relative mean dispersion of resistances of different configurations of the actual chain, many of the latter configurations of high probability would not be properly represented in the ensemble of independent subunits. This shows that, in addition to having a *nonadditive* average value corresponding to the sum of average resistances of subunits, the resistance of the chain is also *non-self-averaging* in that it may not be represented, in general, as a sum of resistances of an ensemble corresponding to a large number of independent shorter units (which would imply that the actual physical resistance would correspond very closely to the ensemble-averaged value). Equivalently, one may say that the failure of the usual ensemble description, due to the rapid growth of fluctuations with length, means that the resistance does not obey the central-limit theorem. Due to the effect of these very large fluctuations, one expects differences, e.g., in the rates of exponential change for averages of various quantities related to ρ.[3]

In an important paper by Anderson *et al.*[3] it was observed that the problem of resistance of a disordered linear conductor could actually be reformulated in terms of a new variable (related to ρ), such that, for it, the concepts of additivity, self-averaging, and central limit do remain valid in the usual ensemble sense. Using a scale transformation analogous to the one introduced earlier by Landauer,[1] Anderson *et al.*[3] showed that for long samples the new scaling variable of physical significance is $\ln\rho$.

The main purpose of subsequent work by Abrikosov and Ryzhkin[10] and Abrikosov,[11] Mel'nikov,[12] Peres *et al.*,[13] and Kumar[14] has been to derive the above form of the proper scaling variable from first principles, thus avoiding the somewhat *ad hoc* scale transformation of Refs. 1 and 3. In these treatments the interest is focused on finding the probability distribution of the resistance rather than average values. Among the mentioned studies, only those of Abrikosov[11] and Mel'nikov[12] yield Gaussian distributions for $\ln\rho$ centered at the value L/L_c (L_c=localization length) and of variance $2L/L_c$ as a consequence of assuming a Gaussian δ-correlated random potential. These results imply indeed that $\ln\rho$ has the property of additive mean and is a self-averaging quantity since its relative rms deviation goes to zero as $L^{-1/2}$ for $L\rightarrow\infty$. In the other treatments,[13,14] similar results are obtained by assuming, for mathematical convenience, that quantities indirectly related to the potential, rather than the potential itself, are Gaussian random variables.

On the other hand, the various analyses[11–14] do reproduce the Landauer result for the average resistance for both large and small sample lengths. Actually, however, the reduction of the results of Abrikosov[11] and of Mel'nikov[12] to the Landauer expression requires replacing their resistance formula $\rho'=t^{-1}$ by the Landauer formula $\rho=\rho'-1$ of Eq. (1).[2] In their set of papers, which are mathematically rigorous but somewhat difficult to comprehend, Abrikosov and Rizhkin,[10] Abrikosov,[11] and Mel'nikov[12] also studied the average conductance $\langle g'\rangle$ and its higher moments, using the definition $\langle g'\rangle=\langle t\rangle$. Unlike their results for $\langle\rho'\rangle$, those for $\langle g'\rangle$ cannot be simply related to the correct Landauer expression $\langle g\rangle=\langle t(1-t)^{-1}\rangle$. Part of our initial interest in this problem arose in connection with the quantity $\langle g\rangle$ which, though being important, has not been analyzed in detail.

A particularly direct, first-principles approach to the one-dimensional (1D) resistance problem is the one applied recently by Kumar.[14] This approach is based on the invariant-imbedding method, which yields a differential equation of the Riccati type for the reflection amplitude of an electron impinging on an inhomogeneous one-dimensional medium, described by the Schrödinger wave equation. The method of invariant imbedding which developed from the work of G. G. Stokes on the intensity of the light reflected and transmitted through a layered medium (with layers of different refractive indices), and some further basic work by Chandrasekhar,[15] are fully described in the excellent book by Bellman and Wing.[16] The detailed form of the Riccati equation and of the associated boundary condition depend, however, on the behavior of the wave number (potential) in a small neighborhood on both sides of the edge of incidence. In this regard, the form of the Riccati equation used by Kumar[14] involves continuity and differentiability assumptions about the potential near the incidence edge which are not expected to be met, in general, in a disordered conductor connected to a current source via nondisordered conducting wires.

The purpose of this paper is to discuss an extensive treatment of resistance (conductance) of a disordered 1D conductor, starting from a Riccati equation for the reflection amplitude which is adapted to the physical situation at hand. A considerable advantage of the Riccati equation used here is that it depends linearly on the random potential, while the equation used by Kumar involved both the potential and its derivative in a nonlinear, nonanalytic form. We also wish to consider the reflection properties of the conductor which determine the electron density in scattering states in the outside regions. The reflection amplitude and coefficient are more difficult to analyze than the resistance but should reflect more directly the actual localization behavior of electronic states in the disordered system. As in Refs. 10–14, emphasis is put on finding the probability distributions of the quantities of interest, rather than just their mean values, which are less meaningful as discussed above.

The paper is organized as follows. In Sec. II A we discuss the model and recall the stochastic Riccati equation for the reflection amplitude $R(L)$ of the disordered linear conductor. A drawback of the latter equation is its dependence on the phase of $R(L)$ which is not of direct interest since the definition of $\rho(L)$ involves only the reflection coefficient $r=|R(L)|^2$. Therefore, using the Landauer formula (1), we have converted the equation for $R(L)$ into an exact second-order differential equation for the resistance $\rho(L)$ itself. While this general equation for $\rho(L)$ has some interesting implications it is difficult to solve explicitly. Thus, in Sec. II B we concentrate on the simpler equation for $R(L)$ to obtain explicit solutions for $R(L)$ and/or $\rho(L)$ as a function of the random potential. In Sec. II B 1 we discuss the exact solution for $R(L)$ [referred to as solution (a)] in a limiting case where the incident energy is defined within some interval about an average value $E_0=V/2$, such that $2E-V$ is typically of the order of the most probable potential fluctuations about a systematic *barrier* $V>0$. In this case E is less than the random potential, which implies that conduction takes place by tunneling only and that exponential attenuation of electronic wave functions arises not only from localizing effects of the random potential but also from barrier penetration. As shown by our later analysis, the above special solution does not lead to a scaling behavior for $\ln\rho(L)$ of the form predicted by previous authors.[3,10–14]

This leads us, in Sec. II B 2, to look for a more general approximate solution [solution (b)] for $\rho(L)$ which is valid, in particular, for potential fluctuations about an average *well*, i.e., $V<0$. Solution (b) is obtained by first averaging an appropriate form of the general equation for $R(L)=|R(L)|\exp[i\theta(L)]$ over the random phase angle $\theta(L)$. This approximation amounts to transforming the equation for $R(L)$ into an exactly soluble, first-order differential equation for $\rho(L)$, which is valid for both signs of V and for arbitrary energy $E>0$. The averaging over $\theta(L)$ is justified, in part, by the fact that Eq. (1) for $\rho(L)$ does not directly depend on $\theta(L)$. On the other hand, we note that various types of phase averages have also been invoked in previous analyses of resistance fluctuations.[1,3,14]

In Sec. III the explicit solutions (a) and (b) are used to construct exact probability distributions $P_\rho(\rho,L)$ $[P_g(\rho^{-1},L)]$ of ρ $[\rho^{-1}]$ by a moment method, and to discuss the mean values $\langle\rho\rangle$, $\langle\rho^2\rangle$, and $\langle\rho^{-1}\rangle$, assuming Gaussian potential fluctuations with spatial δ correlations. While both solutions yield similar results for the above mean values [which, in the case of solution (b), are, in fact, independent of the sign of V], the results for $P_\rho(\rho,L)$ are quite different depending on whether V is positive or negative. In fact, only the form of $P_\rho(\rho,L)$ obtained for the case $V<0$ of solution (b) yields a scaling behavior for $\ln\rho$ analogous to that predicted previously.[3,11,14] Detailed discussion of these results and comparison with previous work is also presented in Sec. III.

In Sec. IV we perform a similar analysis of the probability distribution of the reflection amplitude $R(L)$ for solution (a), and discuss in some detail the averages of the real variable $iR(L)$ and of $|R(L)|^2$, which determine the average electron density outside the conductor. In particular, we find that the variable $\ln\{[1-iR(L)][1+iR(L)]^{-1}\}$ has a Gaussian distribution with additivity, and self-averaging properties similar to $\ln\rho$, $\rho\gg 1$, in the case $V<0$. It follows therefore that $\ln\{[1-iR(L)][1+iR(L)]^{-1}\}$ replaces $\ln\rho$ as the appropriate scaling variable (with the property of additive mean and additive variance) in the case of an average barrier $V>0$. Section V contains some concluding remarks. Finally, in the Appendix we discuss the formal generalization of solution (a) of Sec. II to include small deviations from the limiting energy range defined above.

II. REFLECTION AMPLITUDE AND RESISTANCE OF A DISORDERED LINEAR CONDUCTOR

A. Differential equations

We consider electrons of kinetic energy E moving along the x axis and impinging on a linear random conductor of length L, confined to the region $0\le x\le L$. The conductor is described by a potential $V(x)$ which fluctuates randomly about an average value V defined with respect to a zero energy placed at the fixed constant potential of the outside medium. The electron motion is described by the one-electron Schrödinger equation

$$\frac{d^2\psi}{dx^2}+K^2(x)\psi=0\ , \tag{2}$$

where

$$K^2(x)\equiv k^2(x)=\frac{2m}{\hbar^2}[E-V(x)],\ \ 0\le x\le L\ , \tag{3}$$

$$K^2(x)\equiv k_0^2=\frac{2m}{\hbar^2}E,\ \ x<0 \text{ or } x>L\ . \tag{4}$$

In the region $x>L$, we choose a scattering state corresponding to an incident plane wave of unit amplitude moving from $x=\infty$ towards the conductor. Thus,

$$\psi(x)=e^{-ik_0(x-L)}+R(L)e^{ik_0(x-L)},\ \ x>L\ , \tag{5}$$

$$\psi(x)=T(L)e^{-ik_0x},\ \ x<0\ , \tag{6}$$

where $R(L)$ and $T(L)$ [with $|R(L)|^2+|T(L)|^2=1$ and $|R(L)|^2=r$] denote the complex reflection and transmission amplitudes, which depend on the length of the conductor. In the method of invariant imbedding, one expresses the reflection (transmission) amplitude in a form which does not involve the wave function inside the conductor explicitly. In particular, $R(L)$ is found to be given by the Riccati equation[17] (see Sec. 10 of Chap. VI in Ref. 16)

$$ik_0\frac{dR(L)}{dL}=V(L)[1+R^2(L)]-2[k_0^2-V(L)]R(L)\ , \tag{7}$$

subject to the obvious boundary condition, $R(0)=0$. The potential $V(L)$ at the far end of the conductor is the sum of a mean value V and a random part $v(L)$ [with $\langle v(L)\rangle=0$],

$$V(L)=V+v(L)\ . \tag{8}$$

We note that the equation for $R(L)$ used by Kumar[14] [Eq. (6.31) of Ref. 16] is quite different from Eq. (7) and describes a system where the wave vector $K(x)$ and its first derivative may be assumed to be continuous across the incidence edge, $x=L$, of the conductor. One may doubt that the latter equation would provide an adequate description of our system, where $K(x)$ is random for $x=L-0^+$ and takes a fixed value k_0 for $x=L+0^+$.

Equation (7) may be transformed into a close differential equation for the resistance $\rho(L)$ which is the quantity of direct physical interest. Such a transformation is quite desirable, in principle, as Eq. (7) includes detailed information about the evolution of the phase of $R(L)$,

$$\theta\equiv\theta(L)=\tan^{-1}\left[\frac{\mathrm{Im}R(L)}{\mathrm{Re}R(L)}\right]\ , \tag{9}$$

which is not of direct interest, since it does not enter explicitly in the definition (1) of $\rho(L)$. From Eq. (7) and its complex conjugate one easily arrives at the following result, using Eq. (1):

$$\mathrm{Im}R(L)=\sqrt{r}\,\sin[\theta(L)]=-\frac{k_0}{2V(L)(\rho+1)}\frac{\partial\rho}{\partial L}\ , \tag{10}$$

which defines $\theta(L)$ in terms of $\rho(L)$ and $V(L)$. By inserting this expression, together with

$$\mathrm{Re}R(L)=s\left[\frac{\rho}{\rho+1}-[\mathrm{Im}R(L)]^2\right]^{1/2} \tag{11}$$

(where the sign $s=1$ if $-\pi/2<\theta<\pi/2$ and $s=-1$ if $\pi/2<\theta<3\pi/2$), in, e.g., the real part of Eq. (7) one obtains the following exact differential equation for $\rho\equiv\rho(L)$:

$$k_0^2(\rho+1)\frac{\partial}{\partial L}\left[\frac{1}{V(L)(\rho+1)}\frac{\partial\rho}{\partial L}\right]=V(L)\left[4\rho+2-\frac{1}{\rho+1}\left[\frac{k_0}{V(L)}\frac{\partial\rho}{\partial L}\right]^2\right]$$
$$-4s[k_0^2-V(L)]\left[\rho(\rho+1)-\left[\frac{k_0}{2V(L)}\frac{\partial\rho}{\partial L}\right]^2\right]^{1/2}. \quad (12)$$

The differential equation (12) differs from the simpler equation (7) for $R(L)$ by the fact that it is of second order and that it involves not only the potential $V(L)$ but also its derivative. This latter feature only shows that the resistance is more sensitive to details of the potential than $R(L)$ itself, which suggests that reasonable approximations for $R(L)$ may yield poorer results for $\rho(L)$. On the other hand, the explicit solution of Eq. (12) requires an additional boundary condition as compared to the first-order equation (7). This additional boundary condition is fixed by the value of the classical resistivity,

$$\left[\frac{d\rho}{dL}\right]_{L=0}\sim\frac{1}{l_e}, \quad (13)$$

describing the classical, linearly additive, resistance of sufficiently short samples (such that $\rho\ll 1$). The classical resistivity thus enters as a basic requirement for a general treatment of resistance based on Eq. (12), in a similar way as it does in the scaling theory of Anderson *et al.*[3] In Eq. (13) l_e denotes the mean free path for elastic backscattering which may be expected to differ from the localization length L_c of wave functions at energy E, although existing first-principles treatments[11,12,14] (including the one of Sec. III), all lead to $(d\rho/dL)_{L=0}=L_c^{-1}$ in the classical limit.

B. Detailed solutions

The solution of Eq. (12) for an arbitrary potential $V(L)$ is clearly very difficult and so we return to Eq. (7) for detailed analysis. Equation (7) may be solved analytically, for arbitrary $E>0$, only in the absence of disorder, i.e., for $v(L)=0$. In this case $R(L)$ reduces to the well-known result for scattering off a rectangular potential of width L. In particular, for energies such that $E\gg|V|$ one obtains $R(L)=0$ from the condition $R(0)=0$, as required. In Secs. II B 1 and II B 2 we present two types of solutions incorporating the effect of the random part of $V(L)$ in cases of interest.

1. Solution (a): special exact solution

For the purpose of simplifying Eq. (7) we consider the special case where E is defined within some energy interval around a value $E_0=V/2$ such that for the most typical values of $V(L)$ in Eq. (8) (i.e., those occurring with sufficiently high probability) one has

$$k_0^2\simeq V(L)=V+v(L)$$

i.e.,

$$2E-V\simeq v(L). \quad (14)$$

In this case, where V is necessarily a *potential barrier* (i.e., $V>0$), Eq. (7) becomes, in lowest approximation

$$ik_0\frac{dR(L)}{dL}=V(L)[1+R^2(L)], \quad (15)$$

whose solution subject to $R(0)=0$ is

$$R(L)=-i\tanh\left[\frac{1}{k_0}\int_0^L V(L')dL'\right]. \quad (16)$$

Using Eq. (16), the dimensionless resistance in Eq. (1) takes the simple form

$$\rho(L)=\frac{1}{2}\left[\cosh\left[\frac{2}{k_0}\int_0^L V(L')dL'\right]-1\right]. \quad (17)$$

For later reference it is useful to note that $\rho(L)$ obeys the differential equation

$$\left[\frac{\partial\rho}{\partial L}\right]^2=\frac{4V^2(L)}{k_0^2}\rho(\rho+1), \quad (18)$$

which follows from Eqs. (1) and (10) by observing that Eq. (15) implies $\mathrm{Re}R(L)=0$. On the other hand, we note that the validity of Eq. (17) extends outside the range (14) in the case of small L. In Sec. III we show indeed that, for small L, it coincides with the general solution of (7) for arbitrary $V(L)$, for energies such that $k_0L\ll 1$.

Solutions (16) and (17) play an important role in the analyses of Secs. III and IV where they are used to discuss the probability distributions of $R(L)$ and of $\rho(L)$, as well as averages of these and related quantities of interest, for $V>0$. An important result is that, in the case $V>0$, the variable $\ln\{[1-iR(L)][1+iR(L)]^{-1}\}$ (rather than $\ln\rho$) is the physically significant scaling variable[3] whose mean value and variance scale additively with length L. In view of the importance of Eqs. (16)—(17) it is of interest to consider their generalization in the case of small departures from the energy range (14). Such a generalization, valid for small values of $2E-V(L)$, is discussed in the Appendix where relatively simple formal expressions for the corrections to Eqs. (16) and (17) are derived. Before concluding this subsection, it may be useful to recall the effects of a finite barrier in the absence of disorder, at energy $E=V/2$. We obtain the limiting expressions (with $\alpha=2V/k_0$),

$$|R(L)|=\alpha L,\ \ \rho(L)=\frac{\alpha^2L^2}{2},\ \ \alpha L\ll 1,$$

$$|R(L)|=1,\ \ \rho(L)=\tfrac{1}{2}e^{\alpha L},\ \ \alpha L\gg 1,$$

as expected as a result of the penetration of the finite barrier to a depth of the order of α^{-1}.

2. Solution (b): approximate general solution

While solution (a) is restricted to the case of an average potential barrier and describes pure tunneling conduction, it is clearly desirable to derive an analogous solution valid for a *potential well,* $V<0$. Furthermore, since the solution (17) yields a distribution for $\ln\rho$ (Sec. III) which does not have the properties predicted by the phenomenological scaling theory,[3] the question arises as to whether these properties could be justified within an analysis for the case $V<0$. For these reasons, we now discuss an approximate solution for $\rho(L)$ which is valid for both signs of V and for arbitrary energy $E>0$.

Returning to Eq. (10) for the phase angle $\theta(L)$, we note that the latter is a random quantity which is completely determined, in terms of the random variables $V(L)$ and $dV(L)/dL$, by the solution of Eq. (12). In the following we shall make the crude assumption that, due to the dependence on $dV(L)/dL$, $\theta(L)$ may be treated effectively as a new random variable, independent of $V(L)$, and, for convenience, we take its distribution to be uniform in the range $0\le\theta\le 2\pi$. On the other hand, as observed earlier, $\theta(L)$ does not enter directly in the definition (1) of the resistance although it enters, of course, indirectly in determining the form of Eq. (12), via Eq. (7) for $R(L)$ or, equivalently, the coupled Eqs. (7) for $\sqrt{r}$ and θ. This implies, in particular, that averaging over $\theta(L)$ would leave Eq. (1) invariant. Both the randomness of $\theta(L)$ discussed above and its nonappearance in explicit form in Eq. (1) suggest that an intuitively reasonable procedure would be to average appropriately over phase angles $\theta(L)$. Since Eq. (1) involves $(\mathrm{Im}R)^2$, the simplest such averaging consists in averaging the equation obtained by squaring both sides of (10). Since the average of $\sin^2\theta$ over the assumed uniform distribution of θ is $\frac{1}{2}$ we thus obtain, from Eqs. (1) and (10),

$$\left[\frac{\partial\rho}{\partial L}\right]^2=\frac{2V^2(L)}{k_0^2}\rho(\rho+1)\ . \tag{19}$$

This first-order equation defines an effective resistance via the averaging over phases discussed above. Its solution such that $\rho(0)=0$ is

$$\rho(L)=\frac{1}{2}\left[\cosh\left[\frac{\sqrt{2}}{k_0}\int_0^L V(L')dL'\right]-1\right]\ . \tag{20}$$

We emphasize that Eqs. (19)—(20) for solution (b), while differing only by numerical factors from the corresponding equations (17)—(18) for solution (a), are valid for arbitrary energy and for both signs of V. We note that the averaging over phase angles θ used in the present solution is similar to the one employed by Kumar[14] in his derivation of a Fokker-Planck equation for the probability distribution of resistance, $P_\rho(\rho,L)$.

III. PROBABILITY DISTRIBUTION OF RESISTANCE

The random potential fluctuations in Eq. (8) are assumed to be Gaussian and to be δ correlated in L space:

$$\langle v(L)v(L')\rangle=\frac{1}{\xi}\delta(L-L')\ . \tag{21}$$

As shown by Anderson *et al.*,[3] the average resistance of a long sample is not a representative physical quantity because fluctuations prevent the central-limit theorem from being obeyed. This conclusion readily follows from Eqs. (17) and (20), using a well-known result for averages over Gaussian, δ-correlated variables:[18]

$$\left\langle\exp\left[\pm a\int_0^L v(L')dL'\right]\right\rangle=\exp\left[\frac{a^2L}{2\xi}\right]\ . \tag{22}$$

By averaging successively Eqs. (17), (20), and their squares, using Eq. (15), we get

$$\langle\rho(L)\rangle=\tfrac{1}{2}[e^{2l}\cosh(\kappa L)-1] \tag{23}$$

and

$$\langle\rho^2(L)\rangle=\tfrac{1}{8}[e^{8l}\cosh(2\kappa L)-4e^{2l}\cosh(\kappa L)+3]\ . \tag{24}$$

where

$$l=\frac{L}{L_c}\ , \tag{25}$$

$$L_c=k_0^2\xi,\ \ \kappa=\frac{2V}{k_0}\ \text{ for solution (a)}\ , \tag{26a}$$

$$L_c=2k_0^2\xi,\ \ \kappa=\frac{\sqrt{2}V}{k_0}\ \text{ for solution (b)}\ , \tag{26b}$$

L_c being the localization length, which is seen to be twice as large for solution (b) than for solution (a). Equation (23) for $\kappa=0$ coincides with the Landauer result,[1] which is generally regarded as a direct expression of exponential localization of electronic states when L is much larger than the localization length. We shall return to this point in Sec. IV. When $\kappa\neq 0$, Eq. (23) reduces to the Landauer form only in a range of large but finite sample lengths, such that $|\kappa|L\ll 1$. However, purely exponential growth of resistance is obtained when L is large compared to both L_c and $|\kappa|^{-1}$. Finally, for very short samples, such that $L\ll L_c$ and $L\ll|\kappa|^{-1}$, Eq. (23) leads exactly to the Landauer result $\langle\rho(L)\rangle=l$, in agreement with other treatments[3,11,14] (see Ref. 19). We note that the Eq. $\langle\rho(L)\rangle=l$ with L_c given by (26a) is, in fact, more general than solution (a) from which it has been derived, since it remains valid for all energies E, independent of the random potential $V(L)$, such that $k_0L\ll 1$. This is shown by transforming Eq. (7) in the form

$$ik_0\frac{dQ}{dL}=V(L)(e^{-ik_0L}+e^{ik_0L}Q)^2,\ \ Q=e^{-2ik_0L}R \tag{27}$$

and noting that the solution for $k_0L\ll 1$ [with $Q(L=0)=0$],

$$Q(L)=-\frac{i}{k_0}\int_0^L V(L')dL'\left[1+\frac{i}{k_0}\int_0^L V(L')dL'\right]^{-1}\ , \tag{28}$$

leads to an expression for $\rho(L)$ which coincides with the lowest-order term of the expansion of (17) for $L\to 0$. The Landauer result $\langle\rho(L)\rangle=l$ readily follows from this lowest-order term, using (22).

Comparison of Eqs. (23) and (24) shows that $\langle\rho^2(L)\rangle$ grows faster than $\langle\rho(L)\rangle$, which implies the non-self-averaging property of resistance and the nonexistence of a central limit for large L. Therefore, a proper description of the system requires knowledge of the actual probability distribution $P_\rho(\rho(L),L)$ of ρ rather than its lowest-order moments. The latter may be obtained as follows. We rewrite Eqs. (17) and (20), using (8) and (26a) and (26b) in terms of the quantity

$$y(L)=\exp\left[-\frac{\kappa}{V}\int_0^L v(L')dL'\right], \tag{29}$$

i.e.,

$$\rho(L)=\tfrac{1}{4}[y^{-1}(L)e^{\kappa L}+y(L)e^{-\kappa L}-2], \tag{30}$$

and first obtain the probability density $P_{\ln y}(\ln y,L)$ of the logarithm of $y(L)$. The latter may be calculated by constructing the characteristic or moment-generating function $\phi(k)$. Since $\langle v(L)\rangle=0$, the odd moments vanish and the even moments are given by

$$\langle[\ln y(L)]^{2n}\rangle=\left[\frac{\kappa}{V}\right]^{2n}\left\langle\left[\int_0^L v(L')dL'\right]^{2n}\right\rangle =\left[\frac{\kappa^2 L}{V^2\xi}\right]^n(2n-1)!!\,. \tag{31}$$

It then follows that for both solutions (a) and (b),

$$\phi(k)=\sum_{n=0}^{\infty}\frac{(ik)^{2n}}{n!}\left[\frac{\kappa^2 L}{2V^2\xi}\right]^n=\exp(-2lk^2)\,, \tag{32}$$

whose inverse Fourier transform yields $P_{\ln y}(\ln y,L)=yP_y(y,L)$, with

$$P_y(y(L),L)=[2\sqrt{2\pi l}\,y(L)]^{-1}\exp\left[-\frac{[\ln y(L)]^2}{8l}\right], \tag{33}$$

which obeys the proper normalization condition. Finally, $P_\rho(\rho(L),L)$, for both solutions (a) and (b), is given by

$$P_\rho(\rho(L),L)=\int_0^\infty dy\,P_y(y,L)\delta\left[\rho-\frac{1}{4y}e^{\kappa L}-\frac{y}{4}e^{-\kappa L}+\tfrac{1}{2}\right] =\left[\frac{2}{\pi l}\right]^{1/2}\left[\frac{y_+}{|y_+^2-1|}\exp\left[-\frac{(\ln y_+ +\kappa L)^2}{8l}\right]+\frac{y_-}{|y_-^2-1|}\exp\left[-\frac{(\ln y_- +\kappa L)^2}{8l}\right]\right], \tag{34}$$

where

$$y_\pm(L)=2\rho(L)+1\pm2\{\rho(L)[\rho(L)+1]\}^{1/2}\,. \tag{35}$$

The limiting forms of $P_\rho(\rho(L),L)$ are

$$P_\rho(\rho(L),L)=(2\sqrt{2\pi l}\,)^{-1}\rho^{-1}\exp\left[-\frac{1}{8l}(\ln\rho+\kappa L)^2\right],\quad \rho\gg1\,, \tag{36a}$$

$$=\frac{e^{2l-\kappa L}}{2\sqrt{2\pi l}}\frac{1}{\rho^2}\exp\left[-\frac{1}{8l}(\ln\rho-4l+\kappa L)^2\right], \tag{36b}$$

and

$$P_\rho(\rho(L),L)=(8\pi l\rho)^{-1/2}\left[\exp\left[-\frac{1}{8l}(2\sqrt{\rho}-\kappa L)^2\right]+\exp\left[-\frac{1}{8l}(2\sqrt{\rho}+\kappa L)^2\right]\right],\quad \rho\ll1\,. \tag{37}$$

Equation (36a) reveals a qualitatively different behavior depending on whether the average potential V (related to κ) describes a barrier ($V>0$) or a well ($V<0$). Thus in the case of solution (a) and in the case $V>0$ of solution (b), the probability density of $\ln\rho$,

$$P_{\ln\rho}(\ln\rho,L)=\rho P_\rho(\rho,L)\,,$$

for large L ($\rho\gg1$) is a rapidly decaying exponential having the form of a weak Gaussian tail. On the other hand, Eq. (36a) in the case of solution (b) for $V<0$ (i.e., $\kappa<0$), shows that $\ln\rho$, rather than ρ itself, is a proper scaling variable of physical significance for large L ($\rho\gg1$), as suggested originally by Anderson *et al.*[3]: In this case $P_{\ln\rho}(\ln\rho,L)$, $\rho\gg1$, is a Gaussian centered at a mean (most probable) value[20]

$$\{\ln\rho\}=\langle\ln\rho\rangle=-\kappa L\gg1\,, \tag{38}$$

which shows that $\langle\ln\rho\rangle$ scales additively with length for $\rho\gg1$; furthermore, the relative rms deviation

$$\frac{[\langle\ln^2\rho\rangle-(\langle\ln\rho\rangle)^2]^{1/2}}{\langle\ln\rho\rangle}=-\frac{2}{\kappa(LL_c)^{1/2}}\,, \tag{39}$$

decreases with increasing length L, which means that $\ln\rho$ has a central limit.

Our results for the distribution of resistance for large L, in the case $V<0$ ($\kappa<0$) of solution (b), agree only qualitatively with previous work.[3,11,13,14] This is because our scaling parameter for $\langle\ln\rho\rangle$ is $-\kappa$, while being L_c^{-1} (i.e., the scaling parameter of the variance) in earlier analyses.[3,11,14,21] In particular, the most probable value (38) of $\ln\rho$ ($\rho\gg1$) yields a typical or scale resistance (in the language of Anderson *et al.*[3])

$$\rho_s=e^{-\kappa L}\,, \tag{40}$$

whose rate of exponential growth does not involve the localization length, in contrast to the asymptotic form of the average resistance (23). Different scaling parameters for scale and mean resistances have also been found by Peres *et al.*[13] in their strong scattering case and by Kumar and Mello.[22] We note that, in a sense, it is quite natural that if the variance of $\ln\rho$ is determined by the variance of the potential fluctuations (related to L_c^{-1}) its average should correspondingly be determined by the average potential (related to κ), as we do indeed find. However, this is contrary to the findings in Refs. 11, 12, and 14. Finally, the probability density (27) for $\rho \ll 1$ leads to the classical behavior $\langle\rho(L)\rangle \sim L/L_c$, in agreement with the Landauer result.

Our results for $P_\rho(\rho(L),L)$, Eqs. (34)–(37), provide the first detailed analytical description of the crossover from the strong localization, large resistance regime for sufficiently long samples to the classical low resistance regime for short samples. The qualitatively different behavior of the distribution of resistance for long samples in the cases of average potential wells or barriers is also shown for the first time. We note that, by combining our results for large and small ρ, for $V<0$, one might introduce the quantity $\ln(1+\rho)$ as an effective scaling variable, such that both its mean value and its variance scale linearly, for both small and large L, as in the treatment of Anderson *et al.*[3] However, while the distribution of $\ln(1+\rho)$, for solution (b) with $V<0$, is Gaussian for $\rho \gg 1$, it deviates markedly from a Gaussian for $\rho \ll 1$, as shown by Eq. (37).

We now turn to the discussion of the dimensionless conductance $g(L)=\rho^{-1}(L)$ whose probability distribution is given in terms of Eq. (34) by

$$P_g[g(L),L]=g^{-2}P_\rho[g^{-1}(L),L] \ .$$

We shall reach the surprising conclusion that the average conductance $\langle g\rangle$ is infinite as a direct consequence of the Landauer definition of resistance, rather than of the existence of large statistical fluctuations. Indeed the exact expression (30) for $\rho(L)$, which is based on Eq. (1) and yields the generalized Landauer expression (23) for $\langle\rho(L)\rangle$, now leads to the result

$$\langle g(L)\rangle=\infty \ , \tag{41}$$

for samples of finite length and for arbitrary κ. This follows immediately from Eqs. (30) and (33) and is a consequence of the second-order pole at $y=e^{\kappa L}$ in $\rho^{-1}(L)$, which is itself due to the form of Landauer's equation (1). In fact, Eqs. (30) and (33) imply that the higher moments $\langle g^n\rangle$ are also infinite. An infinite ensemble-averaged conductance was, however, also found in the early treatment by Landauer.[1] Now, for $\kappa=0$, in the absence of disorder, $g(L)$ is infinite, as is well known. The result (41) then implies that the *average* conductance is not affected by the disorder for $\kappa=0$. On the other hand, for $\kappa\neq 0$ the ordered conductance has the finite value $g(L)=2[\cosh(\kappa L)-1]^{-1}$. In this case, therefore, the average conductance is made infinite by the effect of the disorder. Again, this is a consequence of the singular behavior of the Landauer formula, as revealed by Eq. (30).

Equations (30) and (33) show that the infinite average conductance is due to a narrow range of y values near the pole of $\rho^{-1}(L)$. In the case $\kappa=0$, these values correspond to most probable values, while for $\kappa\neq 0$, they correspond to values of increasingly smaller probabilities as L is increased. In contrast to this, the average resistance samples a region of the order of the width of the y distribution, which increases linearly with L. On the other hand, the corrections to $\rho(L)$ studied in the Appendix [for solution (a)] are expected to shift the pole of $\rho^{-1}(L)$ somewhat, without basically changing the above conclusions.

Finally, we recall that, due to inelastic processes, the average conductance is not generally expected to be observable in actual systems.[3] Rather, the observed quantity in sufficiently long samples is the scale conductance $\rho_s^{-1}(L)$, which is well behaved. On the other hand, we also note that the finite value for $\langle g(L)\rangle$ obtained by Abrikosov[11] is due entirely to his use of $g(L)=1-|R(L)|^2$ as the definition of conductance.[19] In contrast to the Landauer definition, the latter expression remains nonsingular as seen, e.g., by substituting Eq. (16) [solution (a)] into it.

IV. PROBABILITY DENSITY OF REFLECTION AMPLITUDE

The interesting effects concerning the resistance of a 1D conductor discussed in the preceding section are related to the fact that statistical fluctuations are very large and grow faster with sample length than the mean resistance itself. These features are not restricted to the resistance and should also reflect in related physical quantities. In addition, a direct consequence of the large fluctuations is that averaging of different quantities over the random potential will lead to different rates (which may be length dependent) of exponential change for long samples.[3] It is therefore of interest to study physical quantities other than the resistance as well. Here, we apply solution (a) of Sec. II to analyze in some detail the reflection amplitude (coefficient) $R(L)$ [$R(L)^2$], which determines the electron probability density in the region outside the conductor:

$$|\psi(x,L)|^2=1+|R(L)|^2+2\,\mathrm{Re}[R(L)e^{2ik_0(x-L)}] \ , \quad x>L \ . \tag{42}$$

Because of this interpretation of $R(L)$, one might expect the localization aspects of electronic states to reflect more directly in the averages $\langle R(L)\rangle$ and $\langle|R(L)|^2\rangle$ than in the average resistance itself. However, due to the unlimited growth of the relative fluctuations with L, these averages, like the average resistance, are not otherwise of physical interest.

It is straightforward to obtain the probability density of the reflection amplitude whose expression has been found explicitly in solution (a) valid for $V>0$. From Eq. (16), and by defining $R'(L)=iR(L)$, we have

$$P_{R'}(R'(L),L)=\int_0^\infty dy\, P_y(y,L)\delta\left[R'-\frac{1-e^{-\kappa L}y}{1+e^{-\kappa L}y}\right] , \tag{43}$$

and by inserting Eq. (33) and performing the integral, we obtain

$$P_{R'}(R'(L),L)=\frac{1}{\sqrt{2\pi l}}\,\frac{1}{1-R'^2}\times\exp\left\{-\frac{1}{8l}\left[\ln\left[\frac{1-R'}{1+R'}\right]+\kappa L\right]^2\right\} . \tag{44}$$

We thus find that the quantity $B\equiv\ln[(1-R')(1+R')^{-1}]$, which may take all values from $-\infty$ to ∞, has a Gaussian distribution,

$$P_B(B,L)=\tfrac{1}{2}(1-R'^2)P_{R'}(R',L)\ , \tag{45}$$

centered about a mean value

$$\langle B\rangle=-\kappa L\ , \tag{46}$$

and of variance equal to $4L/L_c$, where κ and L_c are given by (26a). It follows therefore that in the case of an average barrier ($V>0$) the variable $\ln[(1-iR)(1+iR)^{-1}]$ unlike $\ln\rho$ itself, is a proper scaling variable such that its mean and its variance scale additively with length L. By analogy with Eq. (40) we are led, in the present case ($V>0$), to define a scale amplitude R_s by

$$\frac{1-iR_s}{1+iR_s}=e^{-\kappa L}\ , \tag{47}$$

from which we deduce a scale resistance

$$\rho_s'=\frac{|R_s|^2}{1-|R_s|^2}=\tfrac{1}{2}(\cosh\kappa L-1)\ . \tag{48}$$

While the definition (48) is valid for arbitrary L it reduces, for $\kappa L\gg 1$, to the form (40) up to the factor $\sqrt{2}$ between the scaling parameters (26a) and (26b).

Using Eq. (43), the mean values of $R'(L)$ and $R'^2(L)$ may be written in the form

$$\langle R'(L)\rangle=\frac{1}{2\sqrt{2\pi l}}\int_{-\infty}^{\infty}dt\left[\frac{1-e^t}{1+e^t}\right]\exp\left[-\frac{1}{8l}(t+\kappa L)^2\right]\ , \tag{49}$$

$$\langle R'^2(L)\rangle=\frac{1}{2\sqrt{2\pi l}}\int_{-\infty}^{\infty}dt\left[\frac{1-e^t}{1+e^t}\right]^2\times\exp\left[-\frac{1}{8l}(t+\kappa L)^2\right]\ . \tag{50}$$

Asymptotic forms of these expressions may be obtained using the saddle-point method.[23] The saddle points of the integrands are given by the transcendental equation

$$(t+\kappa L)\sinh t=4\mu l\ , \tag{51}$$

where $\mu=1$ in the case of (49) and $\mu=2$ in the case of (50). Since our main interest lies in the effects of localization induced by disorder, and for the sake of brevity, we restrict ourselves to the situation corresponding to $L/L_c\gg 1$ and $\kappa L\ll 1$, where localization effects should be most prominent. In this case, the solution of (51) may be written in the convergent form

$$t\equiv t_0=\ln\left[\frac{8\mu l}{\ln\left[\frac{8\mu l}{\ln\left[\frac{8\mu l}{\ln(\cdots)}\right]}\right]}\right]\ , \tag{52}$$

of which an approximation valid to better than a few percents, for $\mu l>5$, is

$$t_0\simeq\ln\left[\frac{8\mu l}{\ln[8\mu l/\ln(8\mu l)]}\right]\ . \tag{53}$$

The asymptotic forms of (49) and (50) valid for $l\gg 1$, $\kappa L\ll 1$ are then

$$\langle R'(L)\rangle\simeq -r_1(l)\exp\left\{-\frac{1}{8l}\left[\ln\left[\frac{8l}{\ln(8l/\ln 8l)}\right]\right]^2\right\}\ , \tag{54}$$

$$\langle R'^2(L)\rangle\simeq r_2(l)\exp\left\{-\frac{1}{8l}\left[\ln\left[\frac{16l}{\ln(16l/\ln 16l)}\right]\right]^2\right\}\ , \tag{55}$$

where

$$r_\mu(l)\simeq\left[\ln\left[\frac{8\mu l}{\ln 8\mu l}\right]\right]^{-1/2}\ ,\quad \mu=1,2\ . \tag{56}$$

These results show that $\langle R'^2\rangle$ increases less rapidly than $\langle R'\rangle$ as l is increased, which implies that the relative rms deviation increases with l, within the above range. Equations (42), (54), and (55) completely determine the averaged probability density $\langle|\psi(x,L)|^2\rangle$ in the region $x\geq L$ and, in particular, the average density $1+\langle|R(L)|^2\rangle$ at the incidence edge of the 1D conductor. We note, however, that their relatively complicated dependence on sample length (involving length-dependent rather than constant rates of exponential increase) is less suggestive of simple exponential localization than Landauer's expression for the mean resistance. This suggests that the interpretation of various physical quantities for such systems, in terms of exponential localization, may not be as straightforward as is usually believed.

Finally, a quantity far more significant than $\langle|\psi(x,L)|^2\rangle$ for describing the particle density is the statistical density of $|\psi(x,L)|^2$. For completeness, we display the relatively simple form of this probability density at $x=L$. Putting $A=|\psi(L,L)|^2$, we obtain from Eq. (44),

$$P_A(A(L),L)=\int_{-1}^{1}dR'P_{R'}(R',L)\delta(A-1-R'^2)=\frac{1}{2\sqrt{2\pi l}}\frac{1}{(2-A)\sqrt{A-1}}\left[\exp\left\{-\frac{1}{8l}\left[\ln\left[\frac{1-\sqrt{A-1}}{1+\sqrt{A-1}}\right]+\kappa L\right]^2\right\}-\exp\left\{-\frac{1}{8l}\left[\ln\left[\frac{1-\sqrt{A-1}}{1+\sqrt{A-1}}\right]-\kappa L\right]^2\right\}\right]\ . \tag{57}$$

V. CONCLUDING REMARKS

We have applied the method of invariant imbedding to study resistance fluctuations of a 1D disordered conductor of length L, in the framework of the Landauer formula. We have derived explicit expressions for the reflection amplitude $R(L)$ and the dimensionless resistance $\rho(L)$ [or conductance $g(L)=\rho^{-1}(L)$] in terms of a random potential with an average barrier, for a certain range of electron energies [solution (a)]. A similar expression for $\rho(L)[\rho^{-1}(L)]$ valid for arbitrary energy and for both average barriers and wells has been derived by averaging over phases of $R(L)$ [solution (b)]. From these expressions we have obtained the general forms of the probability distributions of $\rho(L)$ and of $R(L)$, as well as various averages of interest, for Gaussian δ-correlated potential fluctuations. In particular, we have derived generalized Landauer expressions for $\langle \rho(L)\rangle$ (and corresponding infinite conductances) for both signs of the average potential.

Our results for the case of an average *well* agree qualitatively with previous analytical results for the distribution of resistance. An important difference lies in the fact that our results require two length scales, L_c and $|\kappa|^{-1}$ in Eq. (26b), while previous ones involve only L_c.[3,11,14] In particular, the scaling parameter ensuring the property of additive mean for $\ln\rho^3$ is $|\kappa|^{-1}$ in our treatment while being L_c in the previous ones. On the other hand, in the case of solution (a) for an average potential *barrier*, our results differ qualitatively from those of previous analyses[11,14] in that the proper scaling variable is now $\ln[(1-iR)(1+iR)^{-1}]$ rather than $\ln\rho$. These properties of the distribution of the reflection amplitude, as well as the moments $\langle R(L)\rangle$ and $\langle R^2(L)\rangle$ discussed in Sec. IV would seem to deserve further study in the framework of the perturbation formalism discussed in the Appendix.

Finally, a further significant difference between the present treatment and the ones of Abrikosov[11] and of Kumar[14] is seen by considering the evolution equation for the distribution of resistance, $P_\rho(\rho,L)$, as a function of the generalized coordinates ρ and L. Using an elaborate perturbation-theoretic analysis Abrikosov derived a Fokker-Planck equation for $P_\rho(\rho,L)$ valid for arbitrary ρ. On the other hand, starting from the stochastic Liouville equation associated with the "dynamical equation (7)" in the framework of Van Kampen's general formalism, Kumar[14] obtained a Fokker-Planck equation, similar to that of Abrikosov, by using the same type of averaging over phase angles $\theta(L)$ as in solution (b) above. However, it appears that our explicit distributions of resistance [given by Eq. (34)] for solutions (a) and (b) do not obey a simple Fokker-Planck equation in general. Indeed, by recalling that the variable $z\equiv\ln y(L)$ in Eq. (29) has a Gaussian distribution so that

$$\frac{\partial P_z}{\partial L}=\frac{2}{L_c}\frac{\partial^2 P_z}{\partial z^2}\ , \tag{58}$$

one easily verifies that $P_y(y,L)$ obeys the following exact generalized Fokker-Planck equation:

$$\frac{\partial P_y}{\partial L}=\frac{2}{L_c}\left[P_y+3y\frac{\partial P_y}{\partial y}+y^2\frac{\partial^2 P_y}{\partial y^2}\right]\ , \tag{59}$$

and the distribution of resistance is expressed in terms of $P_y(y,L)$ by the first line of Eq. (34). From the nonlinear relation (30) between the variables ρ and y it then follows that Eq. (59) cannot be converted into an ordinary partial differential equation (of the Fokker-Planck type) for $P_\rho(\rho,L)$ except for $\rho\gg 1$, where Eqs. (34)–(35) yield

$$P_\rho(\rho,L)=4e^{\kappa L}P_y(4e^{\kappa l}\rho,L)\ ,$$

which leads to

$$\frac{\partial P_\rho}{\partial L}=\left[\frac{2}{L_c}+\kappa\right]P_\rho+\frac{2\rho}{L_c}\left[3\frac{\partial P_\rho}{\partial\rho}+\rho\frac{\partial^2 P_\rho}{\partial\rho^2}\right],\quad \rho\gg 1\ . \tag{60}$$

This equation differs, however, from the limit for $\rho\gg 1$ of the Fokker-Planck equations derived in Refs. 11 and 14. The difference between the results for $P_\rho(\rho,L)$ based on angular averaging in the present work and in Ref. 14 might be due to the fact that the angular average of the Liouville equation could only be performed approximately.[14]

APPENDIX: GENERALIZATION OF SOLUTION FOR $R(L)$

Here we generalize the solution (a) for the reflection amplitude $R(L)$ obtained in Sec. II when the last term in Eq. (7) is assumed to be nonzero but small. We recall that this solution was obtained for energies

$$2E\simeq V+v(L)\ , \tag{A1}$$

for which the term in question was negligible. For our purposes it is convenient to look for a solution of Eq. (7) in the form

$$R(L)=-i\tanh\left[\frac{1}{2}\left[\frac{2}{k_0}a(L)+g(L)\right]\right]\ , \tag{A2a}$$

$$a(L)=VL+\int_0^L v(L')dL'\ , \tag{A2b}$$

so that $g(L)=0$ when the perturbation parameter

$$\epsilon(L)=k_0\left[1-\frac{V(L)}{2E}\right] \tag{A3}$$

is zero. The function $g(L)$ obeys the exact equation

$$\frac{dg(L)}{dL}=2i\epsilon(L)\sinh\left[\frac{2}{k_0}a(L)+g(L)\right]. \tag{A4}$$

The solution of this equation to second order in $\epsilon(L)$ is

$$g(L)=g_1(L)+g_2(L)\ , \tag{A5a}$$

where

$$g_1(L)=2i\int_0^L dL'\epsilon(L')\sinh\left[\frac{2}{k_0}a(L')\right]\ , \tag{A5b}$$

$$g_2(L)=-4\int_0^L dL'\int_0^{L'} dL''\epsilon(L')\epsilon(L'')\cosh\left[\frac{2}{k_0}a(L')\right]\times\sinh\left[\frac{2}{k_0}a(L'')\right]\ . \tag{A5c}$$

The evaluation of $\rho(L)=|R(L)|^2[1-|R(L)|^2]^{-1}$ to second order in ϵ then yields

$$\rho(L)=\frac{1}{2}\left[\cosh\left[\frac{2}{k_0}a(L)\right]-1\right]-\frac{1}{4}\left[g_1^2(L)\cosh\left[\frac{2}{k_0}a(L)\right]-2g_2(L)\sinh\left[\frac{2}{k_0}a(L)\right]\right]. \tag{A6}$$

Equation (A6) shows that $\rho(L)$ is affected only to second order by small deviations from Eq. (A1). We have not been able to obtain explicit results for the effect of the correction terms in (A6) on the average resistance $\langle\rho(L)\rangle$ for a Gaussian distribution of $v(L)$. However, Eq. (A6) might provide a useful starting point for numerical studies of these corrections.

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[17]From now on we use units such that $\hbar=m=1$.

[18]See, e.g., N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[19]Using the definition $\rho(L)\equiv\rho'(L)=[1-|R(L)|^2]^{-1}$, Abrikosov (Ref. 11) actually finds $\langle\rho'(L)\rangle=\frac{1}{2}(e^{2l}+1)$. In particular, this expression does not show the proper behavior for $l\ll 1$. However, by noting that Eq. (1) may be written $\rho=\rho'-1$, one immediately recovers the Landauer form for $\langle\rho(L)\rangle$ from Abrikosov's result (Ref. 2).

[20]We use the notation $\{f(\rho)\}$ to denote averages with respect to the probability density of $\ln\rho$ where, of course, $\{f(\rho)\}=\langle f(\rho)\rangle$.

[21]We present the equivalent form (36b) of our asymptotic distribution of $\rho(L)$ to emphasize that the presence of the inverse localization length inside the Gaussian factor, as well as its appearance in the rate of growth of $\langle\rho(L)\rangle$ in Eq. (23), are not sufficient conditions for obtaining linear scaling in terms of L_c^{-1}, as shown by Eq. (38).

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[23]As a check on the accuracy of the saddle-point method for obtaining asymptotic expressions, we note that its application in the evaluation of $\langle\rho\rangle$ from the asymptotic distribution (36a) yields $\langle\rho\rangle=\exp(2l+\kappa L)$, which agrees up to the numerical coefficient with Eq. (23) for $\rho\gg 1$.