Magnetoplasmons in a two-dimensional electron fluid: Disk geometry

Alexander L. Fetter
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California 94305-2196
(Received 27 August 1985)

A hydrodynamic model is used to study the magnetoplasma modes of a two-dimensional electron fluid confined to a thin circular disk and screened by two parallel grounded planes. For arbitrary values of the screening parameter, the equations are reduced to a single integral equation that is solved by expanding in a complete set of polynomials. The smallest resulting eigenfrequency for each angular quantum number represents the corresponding edge mode for this geometry.

I. INTRODUCTION

Recent experiments[1–3] on two-dimensional (2D) electron systems have found a new and unexpected magnetoplasma mode. In contrast to the bulk resonance modes whose squared frequency increases linearly with the squared cyclotron frequency, these anomalous modes have a frequency that decreases with increasing magnetic field. Theoretical analyses[2–4] suggest that these modes are localized at the boundary of the 2D system, and they have therefore been called "edge modes"[2,4] or "perimeter waves."[3]

In practice, comparison between experiment and theory is complicated by several special features. Most important is that the electrons on the surface of liquid He are not charge compensated and thus require external electric fields to maintain their equilibrium configuration. As a result, the actual system contains electrodes above and below the surface of the He; for simplicity, these can be taken as grounded and symmetrically placed a distance h from the liquid's surface. A second important feature of the geometry is the lateral dimension R of the electron fluid. Much of the previous theoretical analysis has assumed a semi-infinite halfplane,[2,4] but this idealization may allow charge flow from infinity, in contrast to the fixed total charge of the actual bounded 2D system. Thus the effect of finite R must be included, and the ratio h/R determines the character of the magnetoplasma motions. For $h/R \gg 1$, the electrodes are unimportant and the motion can be considered unscreened. For $h/R \ll 1$, however, the electrodes screen the electrostatic interaction, significantly altering the dynamics; this latter limit is easiest to treat theoretically,[3] but the actual physical configuration involves important corrections arising from the small but nonzero value of h/R. Consequently, detailed analysis of the experiments has required considerable numerical work.

The present paper provides an essentially exact treatment of magnetoplasmons for a 2D electron fluid confined to a disk of radius R. The single approximation is to assume a uniform rigid charge-compensating positive background, but the resulting formulation is valid for arbitrary values of the dimensionless ratio h/R, allowing a uniform study of the effect of screening by the grounded electrodes. Section II formulates the problem and reduces it to a single integral equation for the induced electron density. An expansion in a suitable complete set of polynomials (Sec. III) yields an equivalent matrix problem, involving coefficients that are either known explicitly (for $h/R=0$ or ∞) or readily calculated as one-dimensional numerical integrals. A truncation scheme is studied in Sec. IV, where the properties of the magnetoplasma modes are studied in detail. Comparison with the exact results for $h/R=0$ supports this approximation scheme. Section V considers the special features associated with axisymmetric modes.

II. GENERAL FORMULATION

Consider a thin disk of radius R, placed in a perpendicular magnetic field $\mathbf{B}$. The disk contains a rigid positive background with areal charge density en_0 and a compressible electron fluid with areal charge density $-e(n_0+n)$, where n is a small perturbation with a time dependence $e^{-i\omega t}$. Charge neutrality requires that the integral of n over the disk vanish. It is natural to introduce cylindrical polar coordinates (r,ϕ,z), with the disk placed symmetrically in the x-y plane. In addition, assume two infinite grounded planes parallel to the disk and located at $z=h_1$ and $-h_2$. I use an undamped linearized hydrodynamic model for the constitutive equations characterizing the conservation of particle number and momentum:

$$-i\omega n+n_0\nabla\cdot\mathbf{v}=0 \tag{1}$$

$$-i\omega\mathbf{v}+s^2n_0^{-1}\nabla n-em^{-1}\nabla\Phi\,|_0-\omega_c\hat{\mathbf{z}}\times\mathbf{v}=0\ . \tag{2}$$

Here, $\mathbf{v}$ is the local velocity in the x-y plane, $\Phi\,|_0$ is the electrostatic potential evaluated at the plane of the charge $(z=0)$, $\omega_c=eB/mc$ is the cyclotron frequency, s is an effective wave speed obtained from the compressibility of the fluid, and the gradients in Eq. (2) involve only the x and y components. It is easy to see that the vertical component of vorticity $\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}$ is proportional to ω_c, so that the presence of a static magnetic field qualitatively alters the hydrodynamic flow. A little manipulation readily leads to the single dynamical equation

$$\nabla_2^2(n_0em^{-1}\Phi\,|_0-s^2n)+(\omega_c^2-\omega^2)n=0\ , \tag{3}$$

relating the electron density n and the electrostatic potential $\Phi|_0$ at the plane of the disk, where ∇_2^2 is the 2D Laplacian.

The remaining dynamical equation comes from Maxwell's equations, which will here be treated in the electrostatic (nonretarded) limit. Assume that uniform dielectric materials surround the plane $z=0$, with dielectric constants ϵ_1 and ϵ_2 above and below, respectively. For $z\neq 0$, the electrostatic potential satisfies Laplace's equation, and it obeys the usual boundary conditions that Φ be continuous and that its normal derivative have the following discontinuity at $z=0$:

$$\epsilon_1 \frac{\partial \Phi}{\partial z}\bigg|_{0+} - \epsilon_2 \frac{\partial \Phi}{\partial z}\bigg|_{0-} = 4\pi e n \Theta(R-r) \ , \tag{4}$$

where Θ denotes the unit step function.

The axial symmetry of the problem allows the classification of the normal modes according to the angular dependence $e^{il\phi}$, where l is an integer. The modes with $l=0$ require a slightly different treatment (Sec. V), so that the analysis in Secs. II—IV will assume $l\neq 0$. For $z\neq 0$, Laplace's equation becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{l^2}{r^2} + \frac{\partial^2}{\partial z^2}\right]\Phi(r,z)=0 \ . \tag{5}$$

Take a Hankel transform[5] in the variable r:

$$\overline{\Phi}(p,z)= \int_0^\infty dr\, rJ_l(pr)\Phi(r,z) \ . \tag{6a}$$

This function obeys the ordinary differential equation $(z\neq 0)$

$$\left[\frac{\partial^2}{\partial z^2} - p^2\right]\overline{\Phi}(p,z)=0 \tag{6b}$$

along with the boundary conditions that it vanish at $z=h_1$ and $-h_2$. A combination with the Hankel transform of Eq. (4) readily yields the explicit solution

$$\overline{\Phi}(p,z=0)=\frac{-4\pi e\bar{n}(p)}{p\,[\epsilon_1 \coth(ph_1)+\epsilon_2\coth(ph_2)]} \ , \tag{7a}$$

where

$$\bar{n}(p)= \int_0^R dr\, rJ_l(pr)n(r) \tag{7b}$$

is the Hankel transform of the electron density. The inverse Hankel transform then gives the desired expression for the potential at the plane of the disk

$$\Phi(r)\equiv\Phi(r,z=0)= \int_0^\infty dp\, pJ_l(pr)\overline{\Phi}(p,z=0) \ . \tag{8}$$

For simplicity, I specialize to a symmetrical geometry $(h_1=h_2=h)$, with liquid He below $(\epsilon_2=\epsilon)$ and vacuum above $(\epsilon_1=1)$, and introduce dimensionless units $x=r/R$ and the quantity

$$N(x)=4\pi eR(1+\epsilon)^{-1}n(r)\tanh(h/R) \ . \tag{9}$$

In this way, the "density" $N(x)$ determines the potential $\Phi(x)$ through a nonlocal integral relation

$$\Phi(x)+ \int_0^1 dx'x'K(x,x')N(x')=0 \ , \tag{10a}$$

where the kernel is real and symmetric,

$$K(x,x')=\coth(h/R) \int_0^\infty dp\,\tanh(ph/R)J_l(px)J_l(px') \ . \tag{10b}$$

In terms of the same variables, Eq. (3) becomes

$$\left[\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x} - \frac{l^2}{x^2}\right]\left[\Omega_0^2\Phi - \frac{s^2}{R^2}N\right] + (\omega_c^2-\omega^2)N=0 \ , \tag{11a}$$

where Ω_0 is a characteristic frequency given by

$$\Omega_0^2=\frac{4\pi n_0 e^2\tanh(h/R)}{mR(1+\epsilon)} \ . \tag{11b}$$

These last equations constitute a set of coupled integro-differential equations for the two functions $\Phi(x)$ and $N(x)$. Note that Φ is defined for all positive x, whereas $N(x)$ is defined only for $0<x<1$. It is sufficient to solve Eqs. (10a) and (11a) for $x<1$, however, because the potential for $x>1$ then follows directly from Eq. (10a).

In addition, it is necessary to impose a boundary condition at the edge of the disk, and I assume that there is no net flux of charge across the boundary,[6] taking $v_r=0$ at $x=1^-$. Straightforward manipulations with Eqs. (1) and (2) lead to the explicit condition

$$\left[\left[\omega\frac{\partial}{\partial x}+l\omega_c\right]\left[\Omega_0^2\Phi-\frac{s^2}{R^2}N\right]\right]_{x=1^-}=0 \ , \tag{12}$$

which completes the specification of the problem. It is notable that Eqs. (10) and (11) are invariant under the separate transformations $l\rightarrow -l$ and $\omega_c\rightarrow -\omega_c$, so that the basic dynamical equations do not distinguish right and left helicity. In contrast, the boundary condition (12) is invariant only under the simultaneous sign reversal of both l and ω_c. Furthermore, the appearance of ω in Eq. (12) renders the problem non-self-adjoint. These features imply that a finite magnetic field will split the normal modes, lifting the degeneracy in $|\omega|$.[2–4] To avoid the need for absolute value signs, it is convenient to introduce the notation $L=|l|$ and $\Omega_c=\omega_c\,\mathrm{sgn}l$. It is clear from inspection that Eqs. (10)—(12) can be rewritten directly in terms of these new variables.

Before proceeding with the analysis, it is instructive to examine the limiting forms for large and small values of h/R. In the unscreened case $(h/R\gg 1)$, the hyperbolic functions can be replaced by 1, leading to the approximate expressions

$$K(x,x')= \int_0^\infty dp\,J_L(px)J_L(px') \ , \tag{13a}$$

$$\Omega_0^2=4\pi n_0 e^2/mR(1+\epsilon) \ . \tag{13b}$$

This integral can be evaluated, but its explicit form is unnecessary here. In contrast, the fully screened limit $(h/R\ll 1)$ yields

$$K(x,x')= \int_0^\infty dp\,pJ_L(px)J_L(px')=x^{-1}\delta(x-x') \ , \tag{14a}$$

$$\Omega_0^2=4\pi n_0 e^2 h/mR^2(1+\epsilon)\equiv c_p^2/R^2 \ , \tag{14b}$$

where c_p is the characteristic wave speed introduced in Ref. 3. Since the integral kernel now reduces to a δ function, the interaction in the fully screened limit becomes local. As a result, Φ is proportional to N and hence vanishes for $x>1$, suggesting that the limit $h/R=0$ is singular. This simple proportionality between Φ and N permits an exact solution for all normal modes in any applied field.[3] In particular, the fully screened zero-field plasma oscillations have frequencies given by

$$\omega=\pm\alpha'_{Lm}\Omega_0\ , \tag{15}$$

where α'_{Lm} is the mth zero of $J'_L(x)$. For a given $L\neq 0$, which determines the number of wavelengths around the circumference and hence identifies the azimuthal wave number as L/R, the integer m equals the number of radial nodes plus 1.

For an arbitrary value of h/R, Eqs. (11) and (12) have the important feature that the same quantity appears both in the boundary condition and in the differential equation. This suggests recasting them in a single integral form that automatically incorporates the boundary condition. To be specific, introduce a Green's function $G(x,x')$ that satisfies the differential equation on the interval [0,1]

$$\left[\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}-\frac{L^2}{x^2}\right]G(x,x')=-x^{-1}\delta(x-x') \tag{16}$$

and the two boundary conditions that $G(0,x')$ be bounded and that

$$\left[\left[\omega\frac{\partial}{\partial x}+L\Omega_c\right]G(x,x')\right]_{x=1^-}=0\ . \tag{17}$$

With these definitions, Eqs. (11) and (12) together are equivalent to the single equation

$$\Omega_0^2\Phi(x)-(s/R)^2N(x)+(\omega^2-\Omega_c^2)\int_0^1 dx'\,x'G(x,x')N(x')=0\ . \tag{18}$$

The explicit construction of G is straightforward.[7] With the definitions of the auxiliary functions

$$\gamma(x,x')=(2L)^{-1}(xx')^L\ , \tag{19a}$$

$$g(x,x')=(2L)^{-1}(x_</x_>)^L\ , \tag{19b}$$

where $x_<$ and $x_>$ are the smaller and larger of x and x', the full solution becomes

$$G(x,x')=\left[\frac{\omega-\Omega_c}{\omega+\Omega_c}\right]\gamma(x,x')+g(x,x')\ . \tag{20}$$

Equations (10a) and (18) can now be combined to give a single integral equation for $N(x)$ that holds throughout the interval $0<x<1$,

$$(s/R)^2N(x)+\Omega_0^2\int_0^1 dx'\,x'K(x,x')N(x')-(\omega^2-\Omega_c^2)\int_0^1 dx'\,x'G(x,x')N(x')=0\ . \tag{21}$$

Any solution of this equation determines the corresponding potential for all positive x through Eq. (10a).

It is evident that Eq. (21) constitutes an eigenvalue problem, for solutions exist only for certain allowed frequencies. In contrast to the usual Hermitian small-amplitude problem, however, a combination with Eq. (20) shows that the eigenfrequency appears both *linearly* and *quadratically* for any finite magnetic field

$$(s/R)^2N(x)+\Omega_0^2\int_0^1 dx'\,x'K(x,x')N(x')-(\omega-\Omega_c)^2\int_0^1 dx'\,x'\gamma(x,x')N(x')-(\omega^2-\Omega_c^2)\int_0^1 dx'\,x'g(x,x')N(x')=0\ . \tag{22}$$

This unusual feature reflects the non-Hermitian character of the problem for $\omega_c\neq 0$.[8] As will be seen, the magnetic field splits each degenerate normal mode. Thus the positive and negative frequencies represent distinct physical motions with different phase velocities and can no longer be combined to form standing waves. Equation (22) also shows that the eigenfrequencies change sign under the separate sign reversal of l or ω_c (which reverses the sign of $\Omega_c=\omega_c\,\mathrm{sgn}l$), and that they are invariant under the simultaneous reversal of both of them (which leaves Ω_c unchanged).

III. REDUCTION TO A MATRIX EIGENVALUE PROBLEM

Equation (22) reduces the problem to an integral equation over a finite domain, and there are several ways to attempt a solution. In a previous study of magnetoplasmons in a half-plane,[2,4] the integral kernel analogous to K was replaced by an approximate one that had the same area and second moment, leading to an exactly soluble model problem. Unfortunately, that approximation is rather uncontrolled, for there is no simple way to improve the solution systematically. Here, in contrast, I expand the unknown function $N(x)$ in a complete set of orthogonal polynomials, reducing the basic equation to one involving matrices; truncations that include successively more terms should provide a systematic method to estimate the accuracy of the procedure.

The basic question is the choice of the polynomials, and it turns out to be convenient to use a special form of Jacobi polynomials[9] $P_j^{(L,0)}(1-2x^2)$. For $L=0$, they reduce to the usual Legendre polynomials, and they can be considered suitable generalizations for $L=1,2,\ldots$. They have the following explicit form for $j=0,1,2$:

$$\begin{aligned}
&P_0^{(L,0)}(1-2x^2)=1\ ,\\
&P_1^{(L,0)}(1-2x^2)=L+1-(L+2)x^2\ ,\\
&P_2^{(L,0)}(1-2x^2)=\tfrac{1}{2}(L+1)(L+2)-(L+2)(L+3)x^2\\
&\qquad+\tfrac{1}{2}(L+3)(L+4)x^4\ ,
\end{aligned} \tag{23}$$

and obey the following orthogonality relations:

$$\int_0^1 dx\,x^{2L+1}P_i^{(L,0)}(1-2x^2)P_j^{(L,0)}(1-2x^2)=\tfrac{1}{2}\delta_{ij}(L+2j+1)^{-1}\ . \tag{24}$$

The reason for choosing this particular set of polynomials is the following remarkable identity:[10,11]

$$\int_0^1 dx\, x^{L+1} J_L(px) P_j^{(L,0)}(1-2x^2) = p^{-1} J_{L+2j+1}(p) \,, \tag{25}$$

which expresses its weighted definite integral with a Bessel function as another Bessel function.

I now assume that the unknown function $N(x)$ has an expansion in this complete set

$$N(x) = \sum_{j=0}^{\infty} c_j x^L P_j^{(L,0)}(1-2x^2) \,, \tag{26}$$

where $\{c_j\}$ is a set of coefficients to be determined from Eq. (22). Direct substitution and use of Eq. (24) yields the following set of linear algebraic equations:

$$\sum_{j=0}^{\infty} \left[\frac{s^2 \delta_{ij}}{2R^2(L+2j+1)} + \Omega_0^2 K_{ij} - (\omega - \Omega_c)^2 \gamma_{ij} - (\omega^2 - \Omega_c^2) g_{ij} \right] c_j = 0 \,, \tag{27}$$

involving the matrix

$$K_{ij} = \int_0^1 dx\, x^{L+1} \int_0^1 dy\, y^{L+1} P_i^{(L,0)}(1-2x^2) K(x,y) \times P_j^{(L,0)}(1-2y^2) \,, \tag{28}$$

with similar definitions for γ_{ij} and g_{ij}. Use of Eqs. (10b) and (25) reduces. Eq. (28) to a single definite integral

$$K_{ij} = \coth(h/R) \int_0^\infty dp\, p^{-2} \tanh(ph/R) J_{L+2i+1}(p) \times J_{L+2j+1}(p) \,, \tag{29}$$

whose limiting values are known[12,13] for $h/R \to \infty$,

$$K_{ij} = \int_0^\infty dp\, p^{-2} J_{L+2i+1}(p) J_{L+2j+1}(p) = \frac{(-1)^{i-j+1}}{\pi[4(i-j)^2-1](L+i+j+\frac{1}{2})(L+i+j+\frac{3}{2})} \,, \tag{30a}$$

and for $h/R=0$,

$$K_{ij} = \int_0^\infty dp\, p^{-1} J_{L+2i+1}(p) J_{L+2j+1}(p) = \tfrac{1}{2}\delta_{ij}(L+2j+1)^{-1} \,. \tag{30b}$$

The integrals for intermediate values of h/R can be evaluated numerically with little difficulty.

In a similar way, Eqs. (19a), (24), and the first of Eq. (23) show that γ_{ij} vanishes unless $i=j=0$, in which case it has the value

$$\gamma_{00} = [8L(L+1)^2]^{-1} \tag{31}$$

The remaining matrix g_{ij} is most simply evaluated with the integral representation[12]

$$g(x,x') = (2L)^{-1}(x_</x_>)^L = \int_0^\infty dp\, p^{-1} J_L(px) J_L(px') \,, \tag{32}$$

and use of Eq. (25) then yields the result[12] that g_{ij} is symmetric and tridiagonal with nonzero elements

$$g_{ii} = [4(L+2i)(L+2i+1)(L+2i+2)]^{-1} \,, \tag{33a}$$

$$g_{i,i+1} = g_{i+1,i} = [8(L+2i+1)(L+2i+2)(L+2i+3)]^{-1} \,. \tag{33b}$$

In this way, the original integral eigenvalue equation has been reduced to a matrix equation with known elements. The eigenvalue condition for a solution is that the determinant of coefficients in Eq. (27) vanish. A truncation that includes the first m terms in the expansion (26) evidently yields a polynomial in ω of order $2m$. In zero field, this system reduces to a polynomial in ω^2 of order m, so that the corresponding roots occur in positive and negative pairs. The application of a magnetic field splits these roots, and I shall concentrate on the pair of modes that develops from the smallest squared frequency in zero field. These roots are the edge modes; the negative one (for positive Ω_c) is anomalous in that its absolute value *decreases* for a large applied magnetic field.

IV. APPROXIMATE SOLUTION

For simplicity, the dispersive correction associated with the wave speed s in Eq. (2) will be omitted entirely, which corresponds to studying the long-wavelength limit.[2,4] Thus, I drop the term proportional to $(s/R)^2$ in Eqs. (22) and (27). The simplest approximation to the full matrix equation is to retain only the lowest term in Eq. (26). The resulting determinant is a quadratic equation in ω:

$$(g_{00}+\gamma_{00})\omega^2 - 2\gamma_{00}\Omega_c\omega - K_{00}\Omega_0^2 - (g_{00}-\gamma_{00})\Omega_c^2 = 0 \,. \tag{34}$$

It is easy to see that the positive root increases monotonically with increasing field (assuming $\Omega_c > 0$); the negative root also increases initially in accordance with the conclusions of Refs. 1–4, but it reaches a maximum and then decreases to $-\infty$ in the high-field limit (since $g_{00}-\gamma_{00} > 0$). This latter behavior reflects the present severe truncation, and the inclusion of more terms shifts this maximum to steadily higher fields. Such behavior is to be expected, because the edge mode becomes more localized with increasing magnetic field, requiring many terms in the polynomial expansion. Thus any finite truncation will provide a better description for small fields than for large ones. In fact, even the very crude approximation in Eq. (34) does surprisingly well at reproducing the known zero-field value for complete screening ($h/R=0$). For $L=1$, for example, Eq. (34) yields ≈ 1.8516 for the fully screened dimensionless frequency, which should be compared to the exact value $\alpha'_{11} \approx 1.8412$. The corresponding approximate values for $L=2$, 3, and 4 are all within a few percent of the exact ones.

Equation (27) with $s=0$ has been solved in successive approximations including 1, 2, and 3 terms in the expansion. For the unscreened ($h/R \to \infty$) and fully screened ($h/R=0$) cases, the exact values of K_{ij} [Eq. (30)] were used, and for other cases ($h/R=0.1$, 0.2, 0.5, and 1.0), the relevant integrals in Eq. (29) were evaluated numerically. Figure 1 shows the results of a three-term truncation for

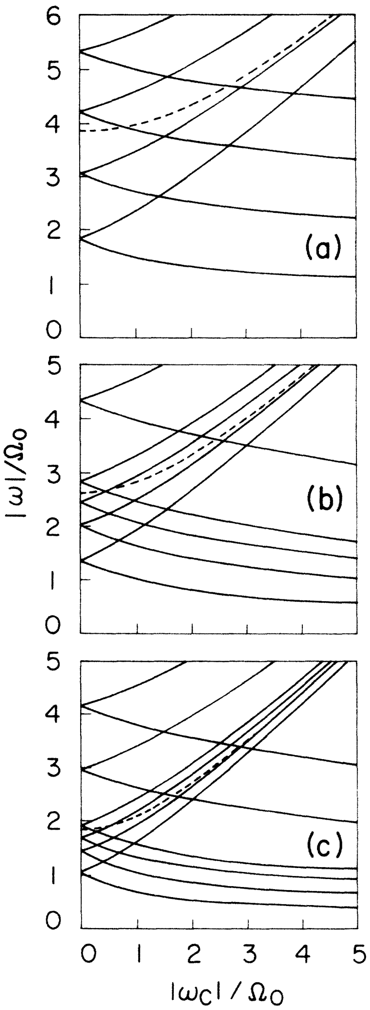

FIG. 1. Magnetic field dependence of the absolute value of the lowest magnetoplasma modes for several low values of the angular momentum quantum number $L=|l|$. Modes with $L\neq 0$ are split in finite field, whereas the axisymmetric mode ($l=0$, shown as dashed curves) remains degenerate. (a) Complete screening ($h/R\approx 0$)—solid curves denote $L=1$, 2, 3, and 4, respectively. (b) Partial screening ($h/R=0.5$)—solid curves denote $L=1$, 2, 3, 4, and 10, respectively. (c) No screening ($h/R\rightarrow\infty$)—solid curves denote $L=1$, 2, 3, 4, 10 and 20, respectively.

$h/R=0$, 0.5, and ∞, giving the magnetic field dependence of the absolute value of the frequency of the lowest radial modes for several small values of L. It is evident that the spacing of these edge modes for different L varies significantly with the screening parameter h/R. Figure 1(a) (complete screening) is very similar to the exact solution shown in Fig. 2 of Ref. 3, and my approximate zero-field values reproduce the exact ones to better than 10^{-4}.

For small L, the finite radius of the disk is significant, but for large L, the edge mode should behave like one on a semi-infinite halfplane. Since the effective wave number q for propagation along the circumference is given by L/R, finite-size effects should be small if $qR=L$ far exceeds 1. Previous work[2,4] indicates that the zero-field edge modes on a half-plane have a frequency that scales with $[q\tanh(qh)]^{1/2}$. Thus the corresponding large-L limit for a disk should have zero-field frequencies that scale with L (for $Lh/R\ll 1$) or $L^{1/2}$ (for $Lh/R\gg 1$). My numerical studies bear out this expectation. For complete screening ($h/R\approx 0$), the zero-field ratio $|\omega|/\Omega_0 L$ had the value 1.064 and 1.046 for $L=50$ and 100, respectively; these approximate values should be compared with the exact ones 1.060 and 1.038 obtained from the asymptotic expansion of α'_{L1}/L.[14] It is also worth noting that for any nonzero L in the fully screened limit, the high-field edge mode frequency has the field-independent value[3] $|\omega|/\Omega_0 L=1$. Since this is the same as the corresponding large-L zero-field value, such fully screened modes will exhibit no field dependence. An independent study of fully screened edge modes on a halfplane confirms this conclusion.[15] In the opposite limit of no screening ($h/R\rightarrow\infty$), the relevant zero-field ratio $|\omega|/\Omega_0 L^{1/2}$ was 0.922 and 0.919. An independent numerical treatment of the unscreened half-plane[15] yields the value ≈ 0.91, which agrees well with that found here. Both of these values are somewhat higher than the approximate one $(\frac{2}{3})^{1/2}\approx 0.816$ found in Refs. 2 and 4.

It is also interesting to consider the behavior as the screening parameter h/R varies. In view of Eq. (14a), the limit $h/R=0$ is special, in that the two distinct functions N and Φ become identical. Equation (29) might suggest that the corrections to this limiting case would involve an expansion in powers of $(h/R)^2$, but the integral of the first correction term in the expansion of the hyperbolic functions diverges linearly, indicating that the first correction is actually linear in h/R (as found in Ref. 3) and hence nonanalytic. Figure 2 shows this behavior for the low-lying zero-field edge modes, where the dots are the computed values and the lines are to guide the eye.

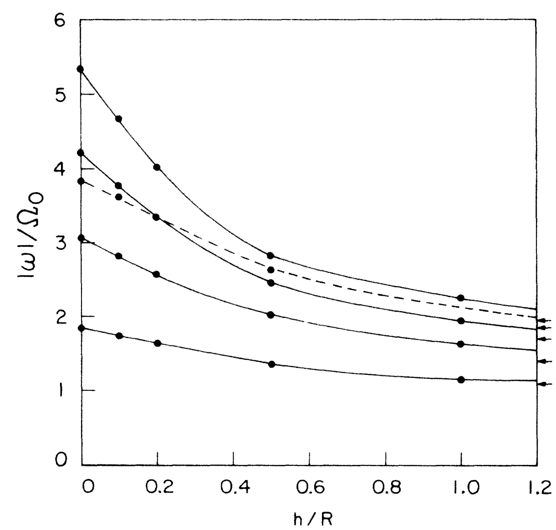

FIG. 2. Lowest zero-field plasma frequency for $l=0$ (dashed curve) and $L=|l|=1$, 2, 3, and 4 (solid curves) for different values of the screening parameter h/R. The dots are computed values and the lines are to guide the eye. Arrows at right denote unscreened ($h/R\gg 1$) values of these five modes, in order $L=1, 2, 3, 0, 4$.

V. AXISYMMETRIC MODES

The preceding sections considered only modes with angular dependence, whose azimuthal nodes ensure that there is no net induced charge density. The situation for axisymmetric modes is slightly different, since the condition that the perturbation in the electron density n have a zero integral over the surface of the disk now requires the presence of at least one radial node. Thus, it is necessary to reformulate the problem for $L=0$, as is clear from the apparent divergence in the Green's function [Eq. (19)].

The basic analysis leading to Eqs. (10) and (11) remains correct with $l=0$, and the integral of the corresponding Eq. (11a) over the surface of the disk leads to the condition ($l=0$)

$$\left[\frac{\partial}{\partial x}\left[\Omega_0^2\Phi-\frac{s^2}{R^2}N\right]\right]_{x=1^-}=(\omega^2-\omega_c^2)\int_0^1 dx'x'N(x')=0\ , \quad (35)$$

where the right-hand side vanishes because of charge conservation. Thus this relation provides the appropriate boundary condition for axisymmetric modes, which also agrees with Eq. (12) if l is set equal to 0. Note that neither the frequency nor the cyclotron frequency actually appears in the boundary condition (35). Thus they enter only in the combination $\omega^2-\omega_c^2$ [see Eq. (11a)], ensuring that all axisymmetric modes necessarily obey the usual rule that the squared magnetoplasma frequency varies linearly with the squared cyclotron frequency.[3] Consequently, axisymmetric modes have no anomalous behavior of the sort found for $l\neq 0$; as a corollary, all edge modes or perimeter waves have angular dependence.

To proceed, it is first necessary to construct the Green's function for $l=0$, analogous to that for Eq. (16). A straightforward calculation gives

$$G(x,x')=g(x,x')=-\ln x_>\ . \quad (36a)$$

In this case, however, it is not possible to impose a boundary condition at $x=1^-$, analogous to that in Eq. (17). For this reason, Green's theorem must be used explicitly to find the integral relation analogous to Eq. (18). For simplicity, I consider only the limit $(s/R\Omega_0)^2\ll 1$, and a straightforward calculation gives

$$\Omega_0^2[\Phi(x)-\Phi(1)]+(\omega^2-\omega_c^2)\int_0^1 dx'x'g(x,x')N(x')=0\ , \quad (36b)$$

where the additional constant $\Phi(1)$ arises from the altered boundary condition on $g(x,x')$ at the upper limit. A combination with Eq. (10a) yields the final integral equation

$$\Omega_0^2\int_0^1 dx'x'[K(x,x')-K(1,x')]N(x')-(\omega^2-\omega_c^2)\int_0^1 dx'x'g(x,x')N(x')=0 \quad (37)$$

that holds on the interval $0<x<1$.

The derivation of the matrix eigenvalue problem is similar to that in Sec. III, and the polynomials for $l=0$ are just the familiar Legendre polynomials. To ensure the conservation of charge, however, the expansion of the induced electron density [the analog of Eq. (26)] omits the constant term

$$N(x)=\sum_{j=1}^{\infty}c_jP_j(1-2x^2)\ . \quad (38a)$$

To find the linear algebraic equations that determine the coefficients of the truncated basis, Eq. (37) is orthogonalized to the same polynomials (a Galerkin procedure), which gives

$$\sum_{j=1}^{\infty}[\Omega_0^2K_{ij}-(\omega^2-\omega_c^2)g_{ij}]c_j=0\ . \quad (38b)$$

A detailed analysis shows that Eqs. (29), (30), and (33) remain correct for $l=0$, and the matrix elements are all bounded since i and j now start from 1. Furthermore, for any fixed value of h/R, the elements K_{ij} are simply related to those evaluated previously for $L=2$.

As noted above, the magnetic field dependence of the axisymmetric modes is trivial, and only the zero-field plasma frequency needs to be determined. In the fully screened case ($h/R=0$), a three-term approximation gives the lowest dimensionless zero-field frequency 3.831 72, which should be compared to the exact value $\alpha'_{02}=3.831\,71$. As an additional check on the convergence of the truncation scheme, the zero-field dimensionless frequency of the lowest mode in the unscreened limit ($h/R\to\infty$) was 1.8635 and 1.8610 for three and five terms, respectively. For comparison, Fig. 1 includes the field dependence of the lowest axisymmetric mode (shown as dashed curves), which differs from the split behavior of those for nonzero L. In addition, Fig. 2 includes the dependence on h/R of the lowest axisymmetric zero-field frequency (shown as a dashed curve).

VI. DISCUSSION

The present work considers the magnetoplasma modes of a 2D electron fluid on the surface of liquid He confined to a disk of radius R and screened by parallel grounded planes a distance h above and below. For any fixed h/R, it provides an exact reformulation in terms of a homogeneous integral equation that has solutions only for discrete eigenfrequencies. An expansion in a complete set of orthogonal polynomials yields an equivalent matrix problem, and successively larger truncations allow a systematic study of the accuracy of the solutions. This approach holds for any value of the screening parameter h/R; it differs from that of Ref. 3, which includes only the leading correction in an expansion for small h/R. On the other hand, Ref. 3 also considers the effect of spatial inhomogeneity in the static electron density, which can be important in obtaining quantitative fits to the measured eigenfrequencies. A systematic experimental study of the dependence on the ratio h/R would be of great interest.

As noted by Wu *et al.*,[16] the magnetoplasma modes of a layered array differ from those of a single 2D electron fluid only in the specific form of the screening function. Thus the present method also applies to a stack of identical disks, separated by a distance a with an insulating dielectric between adjacent layers. In this case, the hyperbolic functions in Eqs. (10b) and (29) are replaced by

$$\frac{\sinh(pa/R)}{\sinh(a/R)}\frac{\cosh(a/R)-\cos(q_z a)}{\cosh(pa/R)-\cos(q_z a)}\,, \tag{39}$$

where q_z is the wave number for propagation along the axis of the system. In particular, if $q_z=0$, this screening function reduces to $\tanh(a/2R)\coth(pa/2R)$, and the limit $a/R\rightarrow 0$ then yields the soluble behavior for three-dimensional (3D) modes in a continuous cylinder [because the integral kernel then is just the Green's function in Eq. (32)]. The solution for general values of a/R and q_z has not been examined, but it presents no fundamental difficulty.

A related geometry is a 2D electron fluid in a halfplane placed between a pair of grounded planes or a stack of such 2D electron fluids. References 4 and 16 construct an approximate solution to the associated edge magnetoplasmons with wave number q along the boundary, and the Wiener-Hopf technique in principle provides the corresponding exact dispersion relation. Since this latter method is cumbersome, however, it is important to note that these problems can also be reduced to a single integral equation of the sort studied in Secs. II–IV. In particular, an expansion in a complete set of polynomials again yields a matrix eigenvalue problem, but the convergence with successive truncations is slower than for the disk. This question is under investigation. It offers the appealing possibility of an effectively exact solution for any value of the screening parameter (which is now given for a single layer by qh). Comparison with the approximate solutions[4,16] will be most valuable.

ACKNOWLEDGMENTS

I am grateful to A. J. Dahm and D. B. Mast for valuable comments, and to J.-W. Wu for providing a copy of Ref. 16 prior to publication and for helpful correspondence. This research has been supported in part by the National Science Foundation (NSF) through Grants No. DMR-81-18386 and No. DMR-84-18865.

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[5]C. J. Tranter, *Integral Transforms in Mathematical Physics*, 3rd ed. (Methuen, London, 1966), Chap. IV.

[6]See, for example, F. Sauter, Z. Physik **203**, 488 (1967); A. R. Melnyk and M. J. Harrison, Phys. Rev. B **2**, 835 (1970); and A. D. Boardman, in *Electromagnetic Surface Modes*, edited by A. D. Boardman (Wiley, New York, 1982), pp. 49–53. An alternative approach is to take the view of macroscopic electrodynamics and require the usual boundary conditions on the horizontal components of **D** and **E** at the edge of the disk. For $r<R$, the conductivity tensor of the electrons is proportional to $\delta(z)$ and thus dominates the effective dielectric tensor $1+4\pi i\omega^{-1}\sigma$. The continuity of $D_r=(\epsilon\mathbf{E})_r$ at $r=R$ and the singularity in σ thus imply that $(\sigma\mathbf{E})_r=0$ at the edge of the disk, which is equivalent to the previous linearized condition that $v_r=0$ at $r=R$.

[7]A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continuua* (McGraw-Hill, New York, 1980), Sec. 43.

[8]With the following trick [R. B. Laughlin (private communication)], the problem can be reduced to a conventional but non-self-adjoint eigenvalue problem. Equation (22) is an integral equation of the form $AN+\omega BN=\omega^2 CN$, where A, B, and C are integral operators. Define $M=\omega CN-BN$ and introduce the two-component vector $\mathbf{V}$ with components M and N. Straightforward manipulation reduces the problem to the usual form $P\mathbf{V}=\omega Q\mathbf{V}$, where P and Q are real 2×2 matrices, with P explicitly not symmetric. (*Note added in proof.* R. Geller (private communication) has pointed out that P and Q can be made symmetric if A, B, and C are self-adjoint [P. Lancaster, *Lambda-Matrices and Vibrating Systems* (Pergamon, Oxford, 1968), Sec. 4.2].)

[9]The notation is that of M. Abramowitz and I. A. Stegun (editors) *Handbook of Mathematical Functions*, National Bureau of Standards AMS No. 55 (U.S. GPO, Washington, D.C., 1964), Chap. 22.

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