Solitary waves in elastic ferromagnets

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It is shown, on the basis of the continuum equations of the magnetoelasticity of ferromagnetic crystals, that Bloch walls in an infinite crystal and Néel walls in a thin elastic film can be represented by "magnetoelastic" solitary waves. In the first case the solitary waves are solutions of a simple sine-Gordon equation in which the only alteration as compared to Enz's case is a change in the reference length as a result of the presence of magnetostrictive internal strains. These solitary waves may therefore be true solitons. In the second case, in addition to the same effect resulting both from magnetostrictive internal strains and demagnetizing effects, the magnetic-spin orientation remains nonlinearly coupled to the elastic displacement polarized in the plane of the film. One therefore has to deal with a nonlinearly coupled system of a sine-Gordon or a double—sine-Gordon equation and two wave equations. Solitary-wave solutions are obtained in closed form for this system. These solitary waves, however, are not true solitons in that radiations always accompany the interactions of such two waves, as already shown in a parallel study concerning ferroelectric crystals.

I. INTRODUCTION

After the works of Seeger,[1] Enz,[2] Feldkeller,[3] and others[4] providing a dynamical generalization of the pioneering paper by Landau and Lifshitz,[5] it is well known that, in *rigid* ferromagnets, solitary waves, and solitons,[6] solutions of a sine-Gordon equation, may represent ferromagnetic domain walls in motion. Enz's derivation, in fact, nowadays is often used in reviews[7] and textbooks[6] on solitons as a physical introduction of what may be referred to as the *sine-Gordon model.* On the other hand, with a growing interest in the magnetoelasticity of crystals and magnon-phonon couplings,[8–10] we now have at our disposal a rather well-established theory of the nonlinear and linearized magnetoelasticity of ferromagnets,[11–15] so that all ingredients are available for a *magnetoelastic* generalization of, say, Enz's derivation. As a matter of fact, Motogi and Maugin[16,17] have already studied the small-amplitude *magnetoelastic* vibrations of Bloch and Néel walls in ferromagnetism. But this problem already involves a linearization, albeit about a spatially nonuniform, nonlinear solution. Here we directly tackle the *nonlinear* dynamical problem of the ferromagnetic domain-wall structure on the basis of the equations of magnetoelasticity. More precisely, by analogy with the recent works of Pouget and Maugin[18,19] on the "soliton" type of approach to the domain-wall structure in elastic ferroelectric crystals, we show, in two cases where ferromagnetic crystals find technological applications, that solitary waves, jointly in the magnetization orientation and the elastic-displacement components, provide an adequate modeling of the nonlinear motion of ferromagnetic domain walls. The derivation follows Enz's in the sense that a slight deviation of the magnetic spin out of the usual plane of rotation in Bloch and Néel walls must be envisaged so that, mathematically, *pure* Bloch and Néel walls cannot exist.[20] In particular, the existence of such solitary-wave solutions is established for 180° Bloch walls in an infinite cubic crystal on account of magnetostrictive internal strains and for 180° Néel walls in a thin cubic elastic film where both magnetostrictive and demagnetizing effects are taken into account. The second case is much more involved than the first one in that it generally involves a nonlinear dynamical coupling between the orientation of magnetic spins and the elastic displacement component in the plane of the film. Then the solitary waves obtained are not exactly solitons in the sense that radiations will accompany any interaction phenomena between such two waves—compare Ref. 19.

Section II is devoted to the statement of the basic system of field equations for the magnetoelastic motion in elastic ferromagnets. The magnetomechanical constitutive equations needed to close this system are given in Sec. III for cubic crystals presenting an easy axis of magnetization. Bloch and Néel walls are defined in Sec. IV. The dynamical nonlinear problem concerning the motion of a Bloch wall is examined and solved exactly in Sec. V, while that concerning the Néel wall is dealt with in Sec. VI, analytically. In particular, the branch of the pseudo-"magnetoacoustic"-dispersion relation which provides a stable magnetoelastic solitary-wave solution is determined, and a closed-form solution is given both for magnetic and mechanical entities. The work concludes in Sec. VII with possible generalizations (influence of an externally applied magnetic field, interaction of solitary waves, influence of coupled magnetomechanical dissipative effects) which are not studied in the present work.

II. EQUATIONS OF MOTION

In the magnetoelasticity of ferromagnets the balance equations which are needed in the magnetomechanical description of crystals at low temperature (much below the Curie temperature) are as follows (e.g., Ref. 15): (a)

Maxwell's equations of magnetostatics, (b) the conservation of mass, (c) the balance of linear momentum, and (d) the precession equation of magnetic spins or, at the long-wavelength limit, of the bulk magnetization. The *quasi*magnetostatic hypothesis implied at point (a) is sufficient since we are not interested in optical effects. Within the framework of infinitesimal strains envisaged here, (b) does not play any role. Equation (2.1) below, for infinitesimal strains, contains couplings with magnetic entities in two forms: (i) in the stress constitutive equation in the form of magnetostriction and (ii) if the magnetic field is not spatially uniform, in the form of a body force, the so-called ponderomotive force, acting on a magnetized crystal. Equation (d), referred to as the *Landau-Lifshitz equation,* describes the time evolution of magnetic spins in a phenomenological manner. It was shown in magnetoelasticity that this latter equation is entirely equivalent to the equation of balance of angular momentum for the crystal (see, e.g., Refs. 12 and 13). We therefore have the following equations: the balance of linear momentum,

$$\rho \frac{d\mathbf{v}}{dt} = \operatorname{div}\mathbf{t} + {}_M\mathbf{f} , \tag{2.1}$$

the spin-precession equation,

$$\frac{d\boldsymbol{\mu}}{dt} = \boldsymbol{\mu} \times \mathbf{C} , \tag{2.2}$$

and Maxwell's equations of magnetostatics in insulators (here Lorentz-Heaviside units are used so that neither factor 4π nor vacuum magnetic permeability appear in the formulation):

$$\nabla \times \mathbf{H} = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = 0 , \tag{2.3}$$

where (T = transpose)

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{e} = \tfrac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] , \tag{2.4}$$

$${}_M\mathbf{f} = (\mathbf{M}\cdot\nabla)\mathbf{H}, \quad \mathbf{M} = \rho\boldsymbol{\mu}, \quad \mathbf{H} = \mathbf{B} - \mathbf{M} , \tag{2.5}$$

$$\mathbf{C} = \gamma \mathbf{H}^{\text{eff}} , \tag{2.6}$$

$$\mathbf{H}^{\text{eff}} = \mathbf{H}^e + \mathbf{H}^d - \frac{\delta\Sigma}{\delta\mathbf{M}}, \quad \mathbf{t} = \frac{\partial\Sigma}{\partial\mathbf{e}} = \mathbf{t}^T , \tag{2.7}$$

and

$$\Sigma = \Sigma(\mathbf{e}, \mathbf{M}, \nabla\mathbf{M}) . \tag{2.8}$$

Here, ρ is the matter density at the natural state, $\mathbf{u}$ is the elastic displacement, $\mathbf{e}$ is the strain tensor of infinitesimal strain theory, $\mathbf{v}$ is the matter velocity, $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the magnetic induction, $\mathbf{M}$ and $\boldsymbol{\mu}$ are the magnetization per unit volume and mass, respectively, γ is the gyromagnetic ratio of the crystal, Σ is its internal energy per unit volume, $\mathbf{H}^e$ is the applied magnetic field, $\mathbf{H}^d$ is the demagnetizing field (which depends only on the magnetization and the geometrical shape of the sample), and $\delta/\delta\mathbf{M}$ is the functional derivative such that

$$\frac{\delta\Sigma}{\delta\mathbf{M}} = \frac{\partial\Sigma}{\partial\mathbf{M}} - \operatorname{div}\left[\frac{\partial\Sigma}{\partial(\nabla\mathbf{M})}\right] . \tag{2.9}$$

We use indifferently a direct intrinsic notation (in which ∇ is the nabla operator and "div" means the divergence—taken on the first index of second-order tensors) and a Cartesian tensorial index notation in rectangular coordinate frames.

Equation (2.6) means that it is an effective magnetic field $\mathbf{H}^{\text{eff}}$ which causes a couple $\mathbf{C}$ which, in turn, is responsible for the spin precession. The very form of Eq. (2.2) implies that

$$|\boldsymbol{\mu}| = \mu_S = \text{const} , \tag{2.10}$$

the saturation value. Instead of $\boldsymbol{\mu}$ or $\mathbf{M}$ one can introduce the director cosines such that

$$\alpha_i = \mu_i/\mu_S = M_i/M_S, \quad M_S = \rho\mu_S , \tag{2.11}$$

and Eqs. (2.7)—(2.9) are replaced by

$$\mathbf{H}^{\text{eff}} = \mathbf{H}^e + \mathbf{H}^d - \frac{1}{M_S}\frac{\delta\bar{\Sigma}}{\delta\boldsymbol{\alpha}} , \tag{2.12}$$

$$\Sigma = \bar{\Sigma}(e, \boldsymbol{\alpha}, \nabla\boldsymbol{\alpha}) , \tag{2.13}$$

$$\frac{\delta\bar{\Sigma}}{\delta\boldsymbol{\alpha}} = \frac{\partial\bar{\Sigma}}{\partial\boldsymbol{\alpha}} - \operatorname{div}\left[\frac{\partial\bar{\Sigma}}{\partial(\nabla\boldsymbol{\alpha})}\right] . \tag{2.14}$$

It must be noticed that in writing the second of Eqs. (2.7) we have somewhat simplified the theory in neglecting the coupled exerted by the field on the elastic continuum so that the stress tensor $\mathbf{t}$ is reduced to a symmetric contribution, which is generally not exact in a deformable ferromagnetic body (see Refs. 12—15 for the complete expressions).

III. INTERNAL ENERGY: CONSTITUTIVE EQUATIONS

The internal energy of the crystal at a given temperature much below the Curie temperature is the sum of four contributions:

(i) The *magnetic anisotropy energy* such that the crystal presents privileged directions of easy magnetization. We assume that the considered crystal is of the easy-axis type of direction of unit vector $\mathbf{d}$, hence the magnetic anisotropy energy

$$\Sigma_{\text{anis}} = -\tfrac{1}{2}KM_S^2(\boldsymbol{\alpha}\cdot\mathbf{d})^2, \quad K > 0 , \tag{3.1}$$

where K is the *dimensionless* magnetic anisotropy constant.

(ii) The *ferromagnetic exchange energy*. This is a continuum representation of Heisenberg's exchange interactions which contribute to the ferromagnetic ordering of spins. It does not depend on the direction of magnetization but it accounts for any deviation from a state of spatially uniform magnetization. Its continuum representation involves $\nabla\boldsymbol{\alpha}$ in the form

$$\Sigma_{\text{ex}} = \tfrac{1}{2}\lambda M_S^2 \alpha_{i,j}\alpha_{i,j}, \quad \lambda > 0 , \tag{3.2}$$

where the summation over repeated indices is understood and λ is the exchange constant with dimension L^2.

(iii) The *elastic strain energy*. For a cubic symmetry such as in iron, nickel, or YIG (yttrium iron granet) this energy reads[21,22] (also Ref. 14)

$$\Sigma_{el}=\frac{1}{2}c_{11}(e_{xx}^2+e_{yy}^2+e_{zz}^2)+c_{44}(e_{xy}^2+e_{yz}^2+e_{zx}^2)$$
$$+c_{12}(e_{xx}e_{yy}+e_{yy}e_{zz}+e_{zz}e_{xx}) , \quad (3.3)$$

where the axes x,y,z coincide with the edges of the cubic structure and c_{11}, c_{12}, and c_{44} are the elastic moduli.

(iv) The *magnetoelastic energy*. This is the interaction energy between magnetization and strains, which provides globally a *nonlinear* effect since, for *magnetostriction*, this energy is cubic simultaneously in α_i and e_{ij}, remaining linear in the strains. For a cubic crystal one has (Ref. 22)

$$\Sigma_{magel}=[B_1(\alpha_x^2e_{xx}+\alpha_y^2e_{yy}+\alpha_z^2e_{zz})$$
$$+2B_2(\alpha_x\alpha_ye_{xy}+\alpha_y\alpha_ze_{yz}+\alpha_z\alpha_xe_{zx})]M_S^2 , \quad (3.4)$$

where B_1 and B_2 are nondimensional coefficients of magnetostriction. Globally,

$$\overline{\Sigma}=\Sigma_{anis}+\Sigma_{ex}+\Sigma_{el}+\Sigma_{magel} . \quad (3.5)$$

IV. BLOCH AND NÉEL WALLS

A. Bloch wall

The following setting will be considered for the case of a Bloch wall (Fig. 1). The crystal is infinite in its three dimensions. The wall is pictured by a transition layer of which the plane is orthogonal to the x axis. Far from the wall, at infinity, we have the following limit conditions on the magnetization or its orientation:

$$\mathbf{M}(x\to\infty)=(0,-M_S,0), \quad \mathbf{M}(x\to-\infty)=(0,M_S,0) , \quad (4.1a)$$

or

$$\boldsymbol{\alpha}(x\to\infty)=(0,-1,0), \quad \boldsymbol{\alpha}(x\to-\infty)=(0,+1,0) . \quad (4.1b)$$

Within the wall, using the angles θ and φ in Fig. 1,

$$\mathbf{M}=M_S(\sin\varphi, \cos\varphi\cos\theta, \cos\varphi\sin\theta)=M_S\boldsymbol{\alpha} , \quad (4.2)$$

where $0<|\theta|<\pi$, while, following Enz[2] in his analysis of the rigid case, φ is considered as an infinitesimally small deviation out of the plane of the wall, i.e.,

$$|\varphi|\ll\pi . \quad (4.3)$$

Moreover, it is supposed that φ varies slowly with x so that

$$\left|\frac{\partial^2\varphi}{\partial x^2}\right|\ll\frac{1}{\delta^2}, \quad \left|\frac{\partial\varphi}{\partial x}\right|\ll\frac{1}{\delta} , \quad (4.4)$$

where

$$\delta=(\lambda/K)^{1/2} , \quad (4.5)$$

the thickness of a Bloch wall in the early Landau-Lifshitz theory (Ref. 5), is the only characteristic length which can be introduced so far. The above approximations allow one to replace (4.2) by

$$\mathbf{M}(\text{Bloch})\simeq M_S(\varphi,\cos\theta,\sin\theta)=M_S\boldsymbol{\alpha}(\text{Bloch}) . \quad (4.6)$$

Furthermore, the out-of-plane deviation of $\mathbf{M}$ related to the small angle φ implies the existence of a demagnetizing field $\mathbf{H}^d$ which opposes this deviation (see Winter[23]). Generally,

$$\mathbf{H}^d=-\mathbf{N}\cdot\mathbf{M} , \quad (4.7)$$

where $\mathbf{N}$ is the so-called demagnetizing tensor. In the present case

$$\mathbf{H}^d=(H_x^d,0,0) , \quad (4.8)$$

with

$$H_x^d=-N_{11}M_x=-M_S\sin\varphi\simeq-M_S\varphi, \quad N_{11}=1 . \quad (4.9)$$

B. Néel wall

The following setting is considered for the case of a Néel wall (Fig. 2). The crystal is a *thin film* of thickness T along the z axis, wide as compared to T along the y direction and of infinite extent along x. According to Néel[24] (also Soohoo[25] and Jones and Middleton[26]), insofar as demagnetizing effects are concerned, one can assimilate the wall to a uniformly magnetized cylinder of elliptic cross section with principal axes D_N and T. According to their analysis it follows that, along the x axis

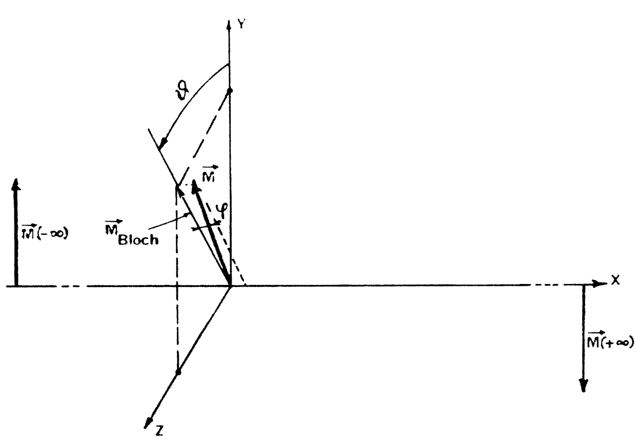

FIG. 1. Bloch wall in an infinite crystal.

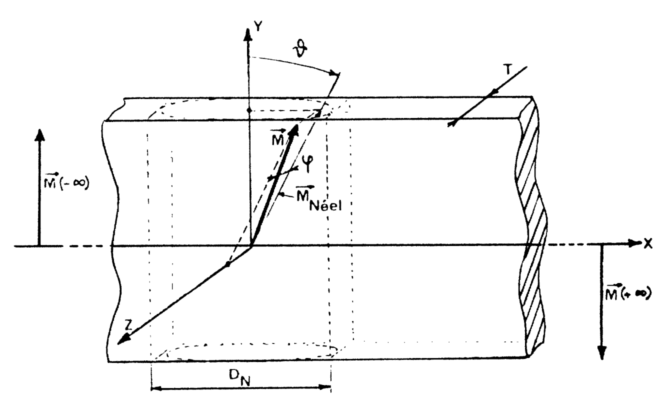

FIG. 2. Néel wall in a thin film.

$$H_x^d = -\frac{T}{D_N+T} M_x \,, \tag{4.10}$$

which reduces to $-M_S \sin\theta$ if $D_N \ll T$ and $\mathbf{M}$ is in the plane (x,y) of the wall. The out-of-plane deviation of the magnetization ($\varphi \neq 0$), by analogy with the Bloch case of the previous paragraph, yields a demagnetizing field component

$$H_z^d = -N_{33} M_S \alpha_z, \quad N_{33} = \frac{D_N}{D_N+T} \,, \tag{4.11}$$

with $H_z^d \simeq -M_S \alpha_z$ if $T \ll D_N$. However, if $D_N \ll T$, H_z^d will be much less important. Globally, on account of Eqs. (4.10) and (4.11), we set

$$\mathbf{H}^d(\text{Néel}) = (-N_{11} M_S \alpha_x, 0, -N_{33} M_S \alpha_z) \,, \tag{4.12}$$

with

$$N_{11} = \frac{T}{D_N+T}, \quad N_{33} = \frac{D_N}{D_N+T} \,. \tag{4.13}$$

Like for the Bloch wall, we have the following limit conditions

$$\mathbf{M}(x \to \pm\infty) = M_S(0, \mp 1, 0) \tag{4.14}$$

and [compare Eqs. (4.4)]

$$\left| \frac{\partial^2 \varphi}{\partial x^2} \right| \ll \frac{1}{T^2}, \quad \left| \frac{\partial \varphi}{\partial x} \right| \ll \frac{1}{T}, \quad |\varphi| \ll \pi \,, \tag{4.15}$$

for all x's, while in the region of the wall

$$\mathbf{M}(\text{Néel}) = M_S(\cos\varphi \sin\theta, \cos\varphi \cos\theta, \sin\varphi) = M_S \boldsymbol{\alpha}(\text{Néel}) \,, \tag{4.16}$$

which, on account of the third of Eqs. (4.15), can be simplified to

$$\mathbf{M}(\text{Néel}) = M_S \boldsymbol{\alpha}(\text{Néel}) \simeq M_S(\sin\theta, \cos\theta, \varphi) \,. \tag{4.17}$$

The walls described by the orientation fields (4.6) and (4.17) and the asymptotic conditions (4.1) and (4.14) are called 180° walls since the magnetization effects a rotation of 180° within the two limits. It must, however, be noticed that the orientation fields (4.6) and (4.17) are not *pure* Bloch and Néel walls since in both cases there is a slight out-of-plane deviation. The existence of this nonzero deviation is of utmost importance because the linearization of the theory with respect to the small variable φ allows for the forthcoming manipulations.

V. BLOCH WALL IN AN INFINITE ELASTIC MEDIUM

We consider a one-dimensional motion of a three-dimensional elastic body along the x axis. Let u_x, u_y, and u_z denote the three Cartesian components of the elastic displacement which, together with $\boldsymbol{\alpha}$ and $\mathbf{H}$, is assumed to depend only on x and t. With the asymptotic conditions (4.1) it is normal to assume that the crystal is practically *uniformly magnetized* far away from the wall where the dependence on x is markedly pronounced (the bulk of the 180° rotation occurs through the wall). In these distant regions ($x \to \pm\infty$) the cubic crystal is therefore subjected to the magnetostriction of uniformly magnetized bodies (see Ref. 22 and Kléman[27]). For a cubic crystal with elastic and magnetostrictive energies given by Eqs. (3.3) and (3.4) we set

$$\lambda_{100} = -\frac{2}{3} \frac{B_1 M_S^2}{c_{11} - c_{12}}, \quad \lambda_{111} = -\frac{B_2 M_S^2}{3 c_{44}} \,, \tag{5.1}$$

and it is shown (Ref. 22, pp. 272–273) that to a spatially uniform magnetization field of director cosines $\boldsymbol{\alpha}^0$ there correspond *internal strains* $\mathbf{e} = \mathbf{e}^0$ given by

$$e_{ij}^0 = \tfrac{3}{2} \lambda^{(ij)} \alpha_i^0 \alpha_j^0 \,, \tag{5.2}$$

where

$$\lambda^{(ij)} = \lambda_{100} \ \text{ if } i = j, \quad \lambda^{(ij)} = \lambda_{111}, \ \text{ if } i \neq j \,. \tag{5.3}$$

In our case, since the motion does not depend on y and z we can write

$$e_{yy} = e_{yy}^0 = \text{const}, \quad e_{yz} = e_{yz}^0 = \text{const} \,, \quad e_{zz} = e_{zz}^0 = \text{const} \,, \tag{5.4}$$

where the constants are determined by the values of internal strains at $\pm\infty$. For the 180° Bloch wall with asymptotic conditions (4.1), Eq. (5.2) yields

$$e_{yy}^0(\pm\infty) = -\frac{B_1 M_S^2}{(c_{11} - c_{12})} \quad \text{otherwise } e_{ij}^0(\pm\infty) = 0 \,. \tag{5.5}$$

On the other hand, Maxwell's equations (2.3), integrated along x, give

$$H_y = H_y^0, \quad H_z = H_z^0, \quad H_x = H_x^0 - M_x \,, \tag{5.6}$$

where $\mathbf{H}^0$ is a spatially uniform field which may possibly depend on time. On account of this and Eq. (4.6) the equations of motion (2.1) on the open x interval $(-\infty, +\infty)$ first read

$$\begin{aligned}
\frac{\partial^2 u_x}{\partial t^2} - c_L^2 \frac{\partial^2 u_x}{\partial x^2} &= M_S^2 (B_1 - \tfrac{1}{2}) \frac{\partial}{\partial x} (\varphi^2) \,, \\
\frac{\partial^2 u_y}{\partial t^2} - c_T^2 \frac{\partial^2 u_y}{\partial x^2} &= 2 (B_2 M_S^2 / \rho) \frac{\partial}{\partial x} (\varphi \cos\theta) \,, \\
\frac{\partial^2 u_z}{\partial t^2} - c_T^2 \frac{\partial^2 u_z}{\partial x^2} &= 2 (B_2 M_S^2 / \rho) \frac{\partial}{\partial x} (\varphi \sin\theta) \,,
\end{aligned} \tag{5.7}$$

where we have defined the speeds c_L and c_T of longitudinal and transverse elastic waves by

$$c_L^2 = c_{11} / \rho, \quad c_T^2 = c_{44} / \rho \,. \tag{5.8}$$

But, in view of the approximation (4.4), we can consider that the right-hand sides of Eqs. (5.7) are negligible, so that we obtain the following three uncoupled wave equations for the elastic-displacement components:

$$\frac{\partial^2 u_x}{\partial t^2} - c_L^2 \frac{\partial^2 u_x}{\partial x^2} = 0 \,, \tag{5.9}$$

$$\frac{\partial^2 u_y}{\partial t^2} - c_T^2 \frac{\partial^2 u_y}{\partial x^2} = 0, \quad \frac{\partial^2 u_z}{\partial t^2} - c_T^2 \frac{\partial^2 u_z}{\partial x^2} = 0 \,. \tag{5.10}$$

Now the spin-precession equation (2.2), on account of Eqs. (2.6), (2.12), (2.14), (3.1)–(3.5), (4.8), (5.6), and (5.4), yields the components:

$$\frac{1}{\gamma}\frac{\partial\varphi}{\partial t}=\lambda M_S\frac{\partial^2\theta}{\partial x^2}-\tfrac{1}{2}KM_S\sin2\theta-2B_2M_Se^0_{yz}\cos2\theta$$
$$+(H^0_z\cos\theta-H^0_y\sin\theta)+B_1M_S(e^0_{yy}-e^0_{zz})\sin2\theta\,, \tag{5.11}$$

$$-\frac{1}{\gamma}\frac{\partial\theta}{\partial t}\sin\theta=(H^0_x\sin\theta-\varphi M_S\sin\theta-\varphi H^0_z)$$
$$+2B_1M_S(e^0_{zz}-e_{xx})\varphi\sin\theta$$
$$-2B_2M_S(e_{zx}\sin^2\theta+\tfrac{1}{2}e_{yx}\sin2\theta)\,, \tag{5.12}$$

$$\frac{1}{\gamma}\frac{\partial\theta}{\partial t}\cos\theta=(\varphi H^0_y+\varphi KM_S\cos\theta-H^0_x\cos\theta+\varphi M_S\cos\theta)$$
$$+2B_1M_S(e_{xx}-e^0_{yy})\varphi\cos\theta$$
$$+2B_2M_S(e_{yx}\cos^2\theta+\tfrac{1}{2}e_{zx}\sin2\theta)\,. \tag{5.13}$$

Combining the last two equations while accounting for (4.4), we are left with the equations:

$$\frac{1}{\gamma}\frac{\partial\varphi}{\partial t}=\lambda M_S\frac{\partial^2\theta}{\partial x^2}+(H^0_z\cos\theta-H^0_y\sin\theta)$$
$$-\tfrac{1}{2}[K+B_1(e^0_{zz}-e^0_{yy})]M_S\sin2\theta$$
$$-2B_2M_Se^0_{yz}\cos2\theta\,, \tag{5.14}$$

$$\frac{1}{\gamma}\frac{\partial\theta}{\partial t}=H^0_x+\varphi[H^0_z\sin\theta+H^0_y\cos\theta+M_S(K-2B_1e^0_{yy})\cos^2\theta$$
$$-2B_1M_Se^0_{zz}\sin^2\theta+M_S]$$
$$+2B_2M_S(e_{yx}\cos\theta+e_{zx}\sin\theta)\,, \tag{5.15}$$

and

$$\varphi\{\tfrac{1}{2}[K+2B_1(e^0_{zz}-e^0_{yy})]M_S\sin2\theta$$
$$+(H^0_y\sin\theta-H^0_z\cos\theta)\}=0\,. \tag{5.16}$$

On account of Eqs. (5.5) and setting

$$\hat{K}=K-2B_1e^0_{yy}=K+\frac{2B_1^2M_S^2}{(c_{11}-c_{12})}\,, \tag{5.17}$$

we rewrite Eqs. (5.14) and (5.15) as

$$\frac{1}{\gamma}\frac{\partial\varphi}{\partial t}=\lambda M_S\frac{\partial^2\theta}{\partial x^2}-\tfrac{1}{2}\hat{K}M_S\sin2\theta+(H^0_z\cos\theta-H^0_y\sin\theta) \tag{5.18}$$

and

$$\frac{1}{\gamma}\frac{\partial\theta}{\partial t}=H^0_x+\varphi(H^0_z\sin\theta+H^0_y\cos\theta+\hat{K}M_S\cos^2\theta+M_S)$$
$$+2B_2M_S(e_{yx}\cos\theta+e_{zx}\sin\theta)\,. \tag{5.19}$$

We shall assume that

$$1\gg H^0_y/M_S,\;\; H^0_z/M_S,\;\; \hat{K}\,. \tag{5.20}$$

Taking then the time derivative of Eq. (5.19), eliminating φ between the resulting equation and Eq. (5.18), and introducing the notation

$$\omega_M=\gamma M_S,\quad \omega H_{x,y,z}=\gamma H^0_{x,y,z}\,, \tag{5.21}$$

we obtain the following equation for θ:

$$\frac{\partial^2\theta}{\partial t^2}=\omega_M^2\left[\lambda\frac{\partial^2\theta}{\partial x^2}-\tfrac{1}{2}\hat{K}\sin2\theta+(\omega_{H_z}\cos\theta-\omega_{H_y}\sin\theta)\right]$$
$$+2B_2\omega_M\left[\frac{\partial e_{yx}}{\partial t}\cos\theta+\frac{\partial e_{zx}}{\partial t}\sin\theta\right]$$
$$-2B_2\omega_M(e_{yx}\sin\theta-e_{zx}\cos\theta)\frac{\partial\theta}{\partial t}\,. \tag{5.22}$$

But, after Eq. (5.19)

$$\frac{\partial\theta}{\partial t}=\omega_{H_x}+2B_2\omega_M(e_{yx}\cos\theta+e_{zx}\sin\theta)+O(\varphi)\,, \tag{5.23}$$

so that we can reasonably rewrite Eq. (5.22) in the following final form

$$\frac{\partial^2\theta}{\partial t^2}-\frac{\partial\omega_{H_x}}{\partial t}-2B_2\omega_M\left[\left[\frac{\partial e_{yx}}{\partial t}+\omega_{H_x}e_{zx}\right]\cos\theta+\left[\frac{\partial e_{zx}}{\partial t}-\omega_{H_x}e_{yx}\right]\sin\theta\right]$$
$$+2B_2^2\omega_M^2[(e_{yx}^2-e_{zx}^2)\sin2\theta-2e_{zx}e_{yx}\cos2\theta]=\omega_M^2\left[\lambda\frac{\partial^2\theta}{\partial x^2}-\tfrac{1}{2}\hat{K}\sin2\theta\right]+(\omega_{H_z}\cos\theta-\omega_{H_y}\sin\theta)\,. \tag{5.24}$$

At this point we can remark that the vanishing of the first factor in the right-hand side of this equation defines, in statics, the orientational distribution obtained by Landau and Lifshitz within a Bloch wall.[5] Indeed, the vanishing of this term in statics, i.e.,

$$\lambda\frac{d^2\phi}{dx^2}-\hat{K}\sin\phi=0,\quad \phi=2\theta\,, \tag{5.25}$$

by integration over x on account of the asymptotic conditions (4.1), gives the distribution

$$\cos\theta=-\tanh(x/\delta_B),\quad \delta_B=(\lambda/\hat{K})^{1/2}\,, \tag{5.26}$$

but here we have a magnetostrictive effect which replaces the usual value K by $\hat{K}$, hence δ by δ_B. In fact, from Eq. (5.17)

$$\delta_B = \left[\frac{\lambda}{K + 2B_1^2 M_S^2/(c_{11} - c_{12})} \right]^{1/2}, \quad (5.27)$$

which is none other than the "magnetostrictively reduced" ($\delta_B < \delta$ since $\hat{K} > K$) Bloch-wall thickness introduced by Motogi and Maugin[16] in their study of the small-amplitude magnetoelastic vibrations of Bloch walls. Obviously, δ_B is the natural reference length for the nondimensionalization in the magnetoelastic case.

We now return to the fully dynamical case of Eqs. (5.9), (5.10), and (5.24) with the limit conditions

$$\theta(x \to -\infty) = 0, \quad \theta(x \to +\infty) = \pi . \quad (5.28)$$

To that purpose we consider propagative solutions for u_x, u_y, u_z and θ which are functions only of the phase variable

$$\xi = qx - \omega t + \xi_0, \quad \xi_0 = \text{const} . \quad (5.29)$$

If ω and q do *not* satisfy the usual "dispersion relation" for longitudinal and transverse elastic waves, i.e., the relations

$$\omega^2 = \omega_L^2 \equiv c_L^2 q^2, \quad \omega^2 = \omega_T^2 \equiv c_T^2 q^2 , \quad (5.30)$$

then Eqs. (5.9) and (5.10) imply that

$$\frac{d^2 u_x}{d\xi^2} = \frac{d^2 u_y}{d\xi^2} = \frac{d^2 u_z}{d\xi^2} = 0 \quad (5.31)$$

on the interval $(-\infty, +\infty)$, and thus

$$e_{xx} = \frac{\partial u_x}{\partial x} = q \frac{du_x}{d\xi} = q \left[\frac{du_x}{d\xi} \right]_0, \quad \frac{\partial e_{xx}}{\partial t} = 0 , \quad (5.32a)$$

$$e_{yx} = \frac{\partial u_y}{\partial x} = q \frac{du_y}{d\xi} = q \left[\frac{du_y}{d\xi} \right]_0, \quad \frac{\partial e_{yx}}{\partial t} = 0 , \quad (5.32b)$$

$$e_{zx} = \frac{\partial u_z}{\partial x} = q \frac{du_z}{d\xi} = q \left[\frac{du_z}{d\xi} \right]_0, \quad \frac{\partial e_{zx}}{\partial t} = 0 . \quad (5.32c)$$

But, on account of Eqs. (5.4) and (5.5),

$$e_{xx}^0 = e_{yx}^0 = e_{zx}^0 = 0 , \quad (5.33)$$

and thus, for asymptotic conditions such as (5.27), Eq. (5.24) reduces to

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{\partial \omega_{H_x}}{\partial t} = \omega_M^2 \left[\lambda \frac{\partial^2 \theta}{\partial x^2} - \tfrac{1}{2} \hat{K} \sin(2\theta) \right] + \omega_M (\omega_{H_z} \cos\theta - \omega_{H_y} \sin\theta) . \quad (5.34)$$

Setting

$$\phi = 2\theta, \quad X = x/\delta_B, \quad \tau = t\hat{\omega}_M, \quad \hat{\omega}_M^2 = \hat{K}_M \omega_M^2 , \quad (5.35)$$

we can rewrite this in nondimensional form as

$$\frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial X^2} + \sin\phi - 2\frac{\partial}{\partial \tau}\left[\frac{\omega_{H_x}}{\hat{\omega}_M} \right] - \frac{2}{(\hat{K})^{1/2}} \left[\frac{\omega_{H_z}}{\hat{\omega}_M} \cos\left[\frac{\phi}{2}\right] - \frac{\omega_{H_y}}{\hat{\omega}_M} \sin\left[\frac{\phi}{2}\right] \right] = 0 \quad (5.36)$$

with the asymptotic conditions

$$\phi(X \to -\infty) = 0, \quad \phi(X \to +\infty) = 2\pi . \quad (5.37)$$

Clearly, according to Eq. (5.36) which governs ϕ (or θ), magnetoelastic couplings contribute only by a redefinition (5.27) of the reference length of the problem. That is, Eq. (5.36) would be exactly the same in *rigid* crystals but with δ and K replacing δ_B and $\hat{K}$. This manifestation of magnetostriction in the spin-precession equation can be traced back to the introduction of internal strains at the asymptotic, uniformly magnetized states at $\pm\infty$.

Whenever $\mathbf{H}^0$ vanishes, $\omega_{H_x} = \omega_{H_y} = \omega_{H_z} = 0$ and Eq. (5.36) reduces to the celebrated *sine-Gordon equation* as established by Enz and others

$$\frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial X^2} + \sin\phi = 0 , \quad (5.38)$$

with an exact stable *solitary-wave* solution, satisfying Eqs. (5.36), in the form

$$\phi = 4 \tan^{-1} \left[C \exp \left[\frac{X - c\tau}{(1 - c^2)^{1/2}} \right] \right] , \quad C = \text{const}, \quad c = \text{const} < 1 . \quad (5.39)$$

Equations (5.38), (5.37), and (5.39) are exactly the same as those obtained by Wesołowski[28] in his study of the mixed torsion flexure of an elastic ribbon with rectangular cross section subjected to a longitudinal tension. As is well known (e.g., Ref. 7), the sine-Gordon equation (5.38) admits not only solitary-wave solutions but also *solitons* of various types depending on the asymptotic conditions. Note that the stable solution (5.39) is obtained for a subsonic state ($c < 1$, where 1 is the characteristic speed of the linearized equation).

If $\mathbf{H}^0$ is directed along the easy axis of magnetization, then only $H_y^0 \neq 0$ and Eq. (5.36) can be written as

$$\frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial^2 \psi}{\partial X^2} - \sin\psi = \eta(\tau) \sin\left[\frac{\psi}{2}\right] , \quad (5.40)$$

wherein

$$\psi = \pi - \phi , \quad (5.41)$$

and

$$\eta(\tau) = \frac{2}{(\hat{K})^{1/2}} H_y^0(\tau)/M_S . \quad (5.42)$$

If H_y^0 is a constant, then $\eta = \text{const}$ and Eq. (5.40) is a *double-sine-Gordon* equation, an equation which also admits solutions of the soliton type.[29] If H_y^0 is not constant in time but its magnitude remains small in the sense that

$|\eta(\tau)|$ remains small for all times, then Eq. (5.40) is a *nonlinearly perturbed* (simple) *sine-Gordon* equation. This situation resembles the one obtained by Pouget and Maugin[18] in ferroelectric crystals of the molecular-group type. For $H_y^0(t)$ represented by a step function—a mathematical assumption which supposedly models the starting motion of a Bloch wall from rest under the switching on of a constant, spatially uniform, magnetic field which favors the growth of one of the magnetic domains to the expenses of the other one—, the perturbation in Eq. (5.40) can be treated by a simple energy method[30] which provides the time modulation of the solitary-wave speed only or, more sophistically, by using Whitham's "averaged Lagrangian" method,[31] in which case a time modulation of both the speed and phase of the solitary wave can be obtained analytically.[32] The evolution of the amplitude requires a numerical study.[33] We do not consider such processes that would directly duplicate what has already been achieved for elastic ferroelectric crystals.

Finally, if $\mathbf{H}^0$ depends on time but is oriented along x, then Eq. (5.36) reduces to

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial X^2}+\sin\phi=f(\tau)\,, \tag{5.43}$$

where

$$f(\tau)=2\frac{\partial}{\partial\tau}[\omega_{H_x}(\tau)/\hat{\omega}_M]\,. \tag{5.44}$$

Equation (5.43) is a sine-Gordon equation which is *forced* by a temporal term. For $|f(\tau)|$ sufficiently small, standard perturbational methods can be used.[34]

VI. NÉEL WALL IN A THIN ELASTIC FILM

A. Governing equations

The case of a Néel wall proves to be much more involved than that of a Bloch wall. Indeed, we have to account for Eqs. (4.12) and (4.17) along with Eq. (4.16). The external field $\mathbf{H}^0$ is spatially uniform, but it may vary in time. The expressions (5.3)–(5.5) are still valid. The equations of motion (2.1) first give

$$\frac{\partial^2 u_x}{\partial t^2}-c_L^2\frac{\partial^2 u_x}{\partial x^2}=\frac{1}{\rho}(B_1-\tfrac{1}{2}N_{11})M_S^2\frac{\partial}{\partial x}(\sin^2\theta)\,, \tag{6.1}$$

$$\frac{\partial^2 u_y}{\partial t^2}-c_T^2\frac{\partial^2 u_y}{\partial x^2}=\frac{1}{\rho}B_2M_S^2\frac{\partial}{\partial x}[\sin(2\theta)]\,, \tag{6.2}$$

$$\frac{\partial^2 u_z}{\partial t^2}-c_T^2\frac{\partial^2 u_z}{\partial x^2}=\frac{1}{\rho}2B_2M_S^2\frac{\partial}{\partial x}(\varphi\sin\theta)\,. \tag{6.3}$$

But, on account of the hypotheses (4.15) and setting

$$\bar{B}_1=B_1-\tfrac{1}{2}N_{11}\simeq B_1\,, \tag{6.4}$$

these equations can be rewritten as

$$\frac{\partial^2 u_x}{\partial t^2}-c_L^2\frac{\partial^2 u_x}{\partial x^2}=-\frac{1}{2\rho}\bar{B}_1M_S^2\frac{\partial}{\partial x}[\cos(2\theta)]\,, \tag{6.5}$$

$$\frac{\partial^2 u_y}{\partial t^2}-c_T^2\frac{\partial^2 u_y}{\partial x^2}=\frac{1}{\rho}B_2M_S^2\frac{\partial}{\partial x}[\sin(2\theta)]\,, \tag{6.6}$$

$$\frac{\partial^2 u_z}{\partial t^2}-c_T^2\frac{\partial^2 u_z}{\partial x^2}=0\,. \tag{6.7}$$

That is, in contrast to the Bloch case [see Eqs. (5.9)–(5.10)] the x- and y-components of the equation of elastic motion remain coupled to the rotation θ through magnetostriction.

The precession equation (2.2) is now shown to have the following components:

$$\begin{aligned}\frac{1}{\gamma}\frac{\partial\theta}{\partial t}\cos\theta=&-\varphi[H_y^0+M_S(K+N_{33})\\&-2B_1M_Se_{yy}^0\cos\theta-2B_2M_Se_{xy}\sin\theta]\\&+H_z^0\cos\theta-B_2M_Se_{yx}\sin(2\theta)\,,\end{aligned} \tag{6.8}$$

$$\begin{aligned}-\frac{1}{\gamma}\frac{\partial\theta}{\partial t}\sin\theta=&+\varphi[H_x^0-(N_{11}-N_{33})M_S\sin\theta\\&-2B_1M_Se_{yx}\cos\theta-2B_2M_Se_{xx}\sin\theta]\\&-H_y^0\sin\theta+2B_2M_Se_{zx}\sin^2\theta\,,\end{aligned} \tag{6.9}$$

$$\begin{aligned}\frac{1}{\gamma}\frac{\partial\varphi}{\partial t}=&-(H_x^0\cos\theta-H_y^0\sin\theta)+\tfrac{1}{2}M_S(K+N_{11})\sin(2\theta)\\&-\lambda M_S\frac{\partial^2\theta}{\partial x^2}+B_1M_S(e_{xx}-e_{yy}^0)\sin(2\theta)\\&+2B_2M_Se_{xy}\cos(2\theta)+2B_2M_Se_{zx}\varphi\cos\theta\,,\end{aligned} \tag{6.10}$$

where we used (4.12) and neglected terms including contributions of the type $\varphi(\partial^2\theta/\partial x^2)$. Combining now Eqs. (6.8) and (6.9) and assuming that

$$N_{33}\gg\frac{\omega_{H_y}}{\omega_M},\ \frac{\omega_{H_x}}{\omega_M},\ K,\ N_{11}\,, \tag{6.11}$$

where the notation (5.21) is used, we obtain

$$\frac{\partial\theta}{\partial t}\simeq-N_{33}\omega_M\varphi+\omega_{H_z}-2B_2\omega_Me_{zx}\sin\theta\,. \tag{6.12}$$

Eliminating φ between this equation and Eq. (6.10) we get

$$\begin{aligned}\frac{\partial^2\theta}{\partial t^2}=&(N_{33}\omega_M^2)\left[\lambda\frac{\partial^2\theta}{\partial x^2}-\tfrac{1}{2}[\tilde{K}-2B_1(e_{yy}^0-e_{xx})]\sin(2\theta)\right.\\&+\frac{1}{\omega_M}(\omega_{H_x}\cos\theta-\omega_{H_y}\sin\theta)\\&\left.-2B_2e_{xy}\cos(2\theta)-2B_2e_{zx}\varphi\cos\theta\right]\\&+2B_2\omega_M\left[\frac{\partial e_{zx}}{\partial t}\sin\theta+e_{zx}\cos\theta\frac{\partial\theta}{\partial t}\right]-\frac{\partial\omega_{H_z}}{\partial t}\,,\end{aligned} \tag{6.13}$$

where

$$\tilde{K}=K+N_{11}\,. \tag{6.14}$$

But, in view of Eq. (6.12),

$$\frac{\partial\theta}{\partial t}=\omega_{H_z}-2B_2\omega_M e_{zx}\sin\theta+\omega_M O(\varphi) \ . \tag{6.15}$$

Therefore, noting that contributions in e_{zx} can be discarded in the precession equation by virtue of Eqs. (6.7) and (5.5), we can rewrite Eq. (6.13) in the following form:

$$(N_{33}\omega_M^2)^{-1}\frac{\partial^2\theta}{\partial t^2}=\left[\lambda\frac{\partial^2\theta}{\partial x^2}-\tfrac{1}{2}\hat{K}\sin(2\theta)\right]+\left[\frac{\omega_{H_x}}{\omega_M}\cos\theta-\frac{\omega_{H_z}}{\omega_M}\sin\theta\right]-[B_1e_{xx}\sin(2\theta)+2B_2e_{xy}\cos(2\theta)]-(N_{33}\omega_M^2)^{-1}\frac{\partial\omega_{H_z}}{\partial t} \ , \tag{6.16}$$

where we have set

$$\hat{K}=\tilde{K}-2B_1e_{yy}^0=K+N_{11}-2B_1e_{yy}^0 \ . \tag{6.17}$$

This last quantity, together with λ, allows one to introduce the characteristic thickness of a Néel wall in a thin elastic film, $\hat{\delta}_N$, in agreement with the definition introduced by Motogi and Maugin[16] in their study of small-amplitude magnetoelastic vibrations of such walls:

$$\hat{\delta}_N=(\lambda/\hat{K})^{1/2}=\left[\frac{\lambda}{K+2B_1^2M_S^2/(c_{11}-c_{12})+T/(D_N+T)}\right]^{1/2} . \tag{6.18}$$

Here both the magnetostrictive internal strains and demagnetizing effects diminish the value of the characteristic thickness as compared to the Landau-Lifshitz value. If $T \ll D_N$, which is usually the case, then $N_{11}=0$ and Eq. (6.18) reduces to the Bloch expression (5.27). The following space and time change of scale is natural:

$$X=x/\hat{\delta}_N, \quad \tau=t\hat{\omega}_M, \quad \hat{\omega}_M^2=N_{33}\hat{K}\omega_M^2 \ . \tag{6.19}$$

Then, with $\phi=2\theta$, Eq. (6.16) takes on the following form:

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial X^2}+\sin\phi+2\hat{a}\left[\frac{\omega_{H_y}}{\hat{\omega}_M}\sin\left[\frac{\phi}{2}\right]-\frac{\omega_{H_x}}{\hat{\omega}_M}\cos\left[\frac{\phi}{2}\right]\right]+2\frac{\partial}{\partial\tau}\left[\frac{\omega_{H_z}}{\hat{\omega}_M}\right]-\left[\frac{B_1}{\hat{\delta}_N\hat{K}}\frac{\partial u_x}{\partial X}\sin\phi+\frac{2B_2}{\hat{\delta}_N\hat{K}}\frac{\partial u_y}{\partial X}\cos\phi\right]=0 \ , \tag{6.20}$$

where we have set

$$\hat{a}=(N_{33}/\hat{K})^{1/2}=\left[\frac{D_N}{[K+2B_1^2M_S^2/(c_{11}-c_{12})](D_N+T)+T}\right]^{1/2} . \tag{6.21}$$

In view of Eq. (6.4) and the fact that $D_N \gg T$, hence $N_{11}\simeq 0$, we shall take $\bar{B}_1=B_1$ in Eq. (6.5). Setting

$$c_M^2=\hat{\delta}_N^2\hat{\omega}_M^2, \quad V_L^2=c_L^2/c_M^2, \quad V_T^2=c_T^2/c_M^2 \ , \tag{6.22}$$

and introducing nondimensional elastic displacements U and V and nondimensional magnetostriction coefficients by

$$u_x=UL_x, \quad u_y=VL_y \ , \tag{6.23}$$

$$L_x=L_y=\frac{M_S}{\hat{\omega}_M}(\hat{K}/2\rho)^{1/2}=(\gamma N_{33})^{-1}(2\rho\hat{K})^{-1/2} \ , \tag{6.24}$$

and

$$\alpha=\frac{B_1M_S^2}{2\rho\hat{\delta}_N\hat{\omega}_M^2}\frac{1}{L_x}=(2\rho\hat{K})^{-1/2}\frac{B_1M_S}{\hat{\omega}_M\hat{\delta}_M} \ , \tag{6.25a}$$

$$\beta=\frac{B_2M_S^2}{\rho\hat{\delta}_N\hat{\omega}_N^2}\frac{1}{L_y}=(2/\rho\hat{K})^{1/2}\frac{B_2M_S}{\hat{\omega}_M\hat{\delta}_N} \ , \tag{6.25b}$$

we can rewrite the system of coupled equations (6.5), (6.6), and (6.20) in the following nondimensional form:

$$\frac{\partial^2U}{\partial\tau^2}-V_L^2\frac{\partial^2U}{\partial X^2}=-\alpha\frac{\partial}{\partial X}(\cos\phi) \ , \tag{6.26a}$$

$$\frac{\partial^2V}{\partial\tau^2}-V_T^2\frac{\partial^2V}{\partial X^2}=\beta\frac{\partial}{\partial X}(\sin\phi) \ , \tag{6.26b}$$

and

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial X^2}+\sin\phi=-\left[\alpha\frac{\partial U}{\partial X}\sin\phi+\beta\frac{\partial V}{\partial X}\cos\phi\right]-2\frac{\partial}{\partial\tau}(\omega_{H_z}/\hat{\omega}_M)+2\hat{a}\left[\frac{\omega_{H_x}}{\hat{\omega}_M}\cos\left[\frac{\phi}{2}\right]-\frac{\omega_{H_y}}{\hat{\omega}_M}\sin\left[\frac{\phi}{2}\right]\right] , \tag{6.27}$$

where $\mathbf{H}^0(\tau)$, or ω_{H_x}, ω_{H_y}, and ω_{H_z}, is a prescribed function of time. Often $\mathbf{H}^0=\mathbf{0}$ for all times and the most complicated case we can envisage is the one where $\mathbf{H}^0(\tau)\neq\mathbf{0}$ is directed along the easy axis of magnetization, in which case Eq. (6.27) reduces to

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial X^2}+\sin\phi=-\left[\alpha\frac{\partial U}{\partial X}\sin\phi+\beta\frac{\partial V}{\partial X}\cos\phi\right]+\eta(\tau)\sin\frac{\phi}{2} \ , \tag{6.28}$$

wherein

$$\eta(\tau) = -2\hat{a}\omega_{H_y}(\tau)/\hat{\omega}_M \ . \tag{6.29}$$

The system formed by Eqs. (6.26) and (6.28) is quite remarkable, first by reason of its symmetry. Indeed, apart from the applied field contribution $\eta(\tau)$, this system consists in a sine-Gordon equation for $\phi=2\theta$, which is nonlinearly coupled in a symmetric manner with two d'Alembert wave equations by the intermediate of the two "magnetostriction coefficients" α and β. If $\eta=0$ and the "longitudinal" magnetostriction coefficient α is practically zero ($\alpha \ll 1$ and, in practice, much smaller than β), then U uncouples from ϕ, and there remain the coupled equations (6.26b) and (6.28) as

$$\begin{aligned} &\frac{\partial^2 V}{\partial\tau^2} - V_T^2\frac{\partial^2 V}{\partial X^2} = \beta\frac{\partial}{\partial X}(\sin\phi) \ , \\ &\frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial^2\phi}{\partial X^2} + \sin\phi = -\beta\frac{\partial V}{\partial X}\cos\phi \ , \end{aligned} \tag{6.30}$$

which is formally the same system as the one obtained from a lattice-dynamics approach in elastic ferroelectric crystals of the molecular-group type (such as sodium nitrite) by Pouget and Maugin.[18] The system (6.30) was shown to possess stable, propagative, solitary-wave solutions. The same result will be proven here for the more general system (6.26) and (6.28) for $\eta=0$.

B. Magnetoelastic Néel wall and solitary waves

With $\eta=0$ we seek propagative solutions of Eq. (6.26) and (6.28), (U,V,ϕ), which depend on X and τ via the phase variable

$$\xi = QX - \Omega\tau + \xi_0, \quad \xi_0 = \text{const} \ , \tag{6.31}$$

where Q and Ω are the pseudo-wave-number and the circular frequency. We thus deduce the following system of *ordinary* nonlinear differential equations:

$$(\Omega^2-\Omega_L^2)\frac{d^2U}{d\xi^2} = -\alpha Q\frac{d}{d\xi}(\cos\phi) \ , \tag{6.32a}$$

$$(\Omega^2-\Omega_T^2)\frac{d^2V}{d\xi^2} = \beta Q\frac{d}{d\xi}(\sin\phi) \ , \tag{6.32b}$$

$$(\Omega^2-Q^2)\frac{d^2\phi}{d\xi^2} + \sin\phi = -Q\left[\alpha\frac{dU}{dX}\sin\phi + \beta\frac{dV}{d\xi}\cos\phi\right] , \tag{6.32c}$$

where we have set

$$\Omega_L^2 = V_L^2Q^2, \quad \Omega_T^2 = V_T^2Q^2 \ . \tag{6.33}$$

The integration of Eqs. (6.32) with respect to ξ gives

$$(\Omega^2-\Omega_L^2)\frac{dU}{d\xi} = -\alpha Q\cos\phi + C_1 \ , \tag{6.34a}$$

$$(\Omega^2-\Omega_T^2)\frac{dV}{d\xi} = \beta Q\sin\phi + C_2 \ . \tag{6.34b}$$

The constants of integration C_1 and C_2 are determined by the following asymptotic conditions:

$$\begin{aligned} &\phi \to 0 \ (\text{mod } 2\pi) \ , \\ &\frac{dU}{d\xi}\to 0, \quad \frac{dV}{d\xi}\to 0 \ , \end{aligned} \tag{6.35}$$

as $|\xi|$ goes to infinity. Hence,

$$C_1 = \alpha Q, \quad C_2 = 0 \ . \tag{6.36}$$

In the following development it is assumed that the nonlinear wave motion studied does *not* satisfy the dispersion relations (6.33) of linear longitudinal and transverse acoustic waves. Then $\Omega\neq\Omega_L$ and $\Omega\neq\Omega_T$. This allows one to eliminate $dU/d\xi$ and $dV/d\xi$ between Eqs. (6.34) and (6.32c) which, on account of Eqs. (6.36), yields the following *unique ordinary* nonlinear differential equation for the angle variable ϕ:

$$(\hat{Q}^2-\hat{\Omega}^2)\frac{d^2\phi}{d\xi^2} - \sin\phi + \zeta(\Omega,Q)\sin(2\phi) = 0 \ , \tag{6.37}$$

where we have set

$$\hat{\Omega}_L^2 = \hat{V}_L^2Q^2, \quad \hat{V}_L^2 = V_L^2(1-\epsilon_{ML}), \quad \epsilon_{ML} = \alpha^2/V_L^2 \ , \tag{6.38}$$

$$\mathscr{A} = \left[\frac{\Omega^2-\hat{\Omega}_L^2}{\Omega^2-\Omega_L^2}\right]^{1/2}, \quad \hat{\Omega} = \Omega/\mathscr{A}, \quad \hat{Q} = Q/\mathscr{A} \ , \tag{6.39}$$

and

$$\zeta(\Omega,Q) = \tfrac{1}{2}\hat{Q}^2\left[\frac{\alpha^2}{\Omega^2-\Omega_L^2} - \frac{\beta^2}{\Omega^2-\Omega_T^2}\right] . \tag{6.40}$$

Here ϵ_{ML} is a typical small parameter of magnetoelastic wave propagation (see, e.g., Ref. 15) while $\hat{V}_L$ is the "magnetostrictively" reduced speed of longitudinal elastic waves (see, for instance, Ristic[35] or Maugin and Hakmi[36]).

For a functional dependence of the type (6.31) and fixed parameters Ω and Q, Eq. (6.37) is an ordinary differential equation equivalent to a *double* sine-Gordon equation. Therefore, we have reached the same result as in the simpler case envisaged by Pouget and Maugin[18] in ferroelectrics, but this time in elastic ferromagnetic crystals. We can state, thus, the motion of a Néel wall in a thin elastic ferromagnetic film can be represented by a solitary wave in ϕ and the *longitudinal* and *transverse* elastic displacements, the transverse component being that contained in the plane of rotation of the magnetization.

Equation (6.37) admits a first integral

$$\tfrac{1}{2}\lambda\dot{\phi}^2 + \mathscr{V}(\phi) = \text{const} = E_0 \ , \tag{6.41}$$

with

$$\lambda = \hat{Q}^2 - \hat{\Omega}^2, \quad \dot{A} \equiv dA/d\xi \tag{6.42}$$

and

$$\mathscr{V}(\phi) = \cos\phi - \frac{\zeta}{2}\cos(2\phi) \ . \tag{6.43}$$

Equation (6.41) is the total-energy conservation equation for a particle of "mass" λ (which would be negative for $\Omega > Q$) in a potential $\mathscr{V}$ of period 2π. Formally, the integration of Eq. (6.41) yields

$$\xi=\int^{\phi}\frac{\hat{Q}(1-c^2)^{1/2}}{\{2[E_0-\mathscr{V}(\phi)]\}^{1/2}}d\phi \tag{6.44}$$

for $c=\Omega/Q<1$, and the type of solution depends on the value of the constant E_0. However, here we can directly integrate Eqs. (6.37) and (6.41). With the limit conditions

$$\frac{d\phi}{d\xi}\rightarrow 0,\ \ \phi\rightarrow 0\ (\mathrm{mod}\,2\pi)\ \ \text{as}\ |\xi|\rightarrow\infty\ , \tag{6.45}$$

we have $E_0=(2-\zeta)/2$. To avoid any problem of determination of trigonometric arguments we set

$$\psi=\phi-\pi,\ \ \tilde{\theta}=\psi/2\rightarrow\pm\pi/2\ \ \text{as}\ |\xi|\rightarrow\infty \tag{6.46}$$

so that Eqs. (6.37) and (6.41) transform to

$$\lambda\frac{d^2\psi}{d\xi^2}+\sin\psi+\zeta\sin(2\psi)=0\ , \tag{6.47a}$$

$$\lambda\left[\frac{d\psi}{d\xi}\right]^2=2(1+\cos\psi)+\zeta[\cos(2\psi)-1]\ . \tag{6.47b}$$

Setting

$$\psi=2\tan^{-1}\varphi,\ \ \varphi(\xi)=\tan\tilde{\theta}\ , \tag{6.48}$$

from Eq. (6.47a) we obtain the well-behaved solution

$$\varphi=a\sinh\xi\ , \tag{6.49}$$

with

$$a=\pm|1-2\zeta|^{-1/2},\ \ \lambda=1-2\zeta \tag{6.50}$$

if $\varphi\rightarrow\mp\infty$ as $\xi\rightarrow\mp\infty$. The second of Eqs. (6.50), on account of the definitions (6.42) and (6.40), is nothing but a *pseudodispersion relation* to be satisfied by the couple (Ω,Q), i.e.,

$$\mathscr{D}_{\mathrm{sol}}(\Omega,Q)\equiv[1-2\zeta(\Omega,Q)]-(\hat{Q}^2-\hat{\Omega}^2)=0\ . \tag{6.51}$$

Therefore, the solutions for the real rotation angle θ of the magnetization are given by

$$\theta(\xi)=\tan^{-1}\left[\pm\frac{\sinh\xi}{|1-2\zeta|^{1/2}}\right]\pm\frac{\pi}{2}\ , \tag{6.52}$$

where the $+$ and $-$ signs correspond to two possibilities of rotation [clockwise and anticlockwise rotation in the (x,y) plane, respectively]. In fact, these possibilities are restricted by stability considerations. To examine this question a closer look at the dispersion relation (6.51) is necessary and a digression to the linearized case is enlightening.

C. Digression-linearized case

A linearization about a spatially nonuniform static state is to be found in Motogi and Maugin.[16] Here we linearize Eqs. (6.26) and (6.28)—in the absence of applied field—about a fully ordered ferromagnetic state, i.e., a spatially uniform state of magnetization as it can exist sufficiently far away from a wall, hence at $x=\pm\infty$, so that $\phi_0=0$ $(\mathrm{mod}\,2\pi)$ and $U_0=V_0=0$. The linearization renders the following system:

$$\frac{\partial^2 U}{\partial\tau^2}-V_L^2\frac{\partial U}{\partial X^2}=0\ , \tag{6.53}$$

and

$$\frac{\partial^2 V}{\partial\tau^2}-V_T^2\frac{\partial^2 V}{\partial X^2}=\beta\frac{\partial\phi}{\partial X}\ , \tag{6.54a}$$

$$\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial^2\phi}{\partial X^2}+\phi=-\beta\frac{\partial V}{\partial X}\ . \tag{6.54b}$$

In this linearization the longitudinal elastic mode can propagate independently of other components—see Eq. (6.53). The remaining coupled system (6.54) consists in a Klein-Gordon equation for ϕ, thus providing a *dispersive mode,* which is coupled by *resonance*[37] with the nondispersive transverse-acoustic mode V. Plane harmonic wave solutions of *true* wave number Q and circular frequency Ω of Eqs. (6.54) must satisfy the following *true* dispersion relation:

$$\mathscr{D}_L(\Omega,Q)\equiv(\Omega^2-\Omega_S^2)(\Omega^2-\Omega_T^2)-\beta^2Q^2=0\ , \tag{6.55}$$

where we have set

$$\Omega_S^2=Q^2+1\ . \tag{6.56}$$

Equation (6.55), in the positive (Ω,Q) quadrant admits the classical representation by branches Ω_+ and Ω_- in Fig. 3 with a characteristic repulsion at the magnetoacoustic resonance point (Ω^*,Q^*) defined by[36]

$$\Omega^*=\Omega_T(Q^*)=\Omega_S(Q^*)\ . \tag{6.57}$$

With branches

$$\Omega_\pm^2=\tfrac{1}{2}(\Omega_T^2+\Omega_S^2)\pm[\tfrac{1}{4}(\Omega_T^2-\Omega_S^2)^2+\beta^2Q^2]^{1/2}\ , \tag{6.58}$$

the vertical repulsion between (c) and (d) is classically

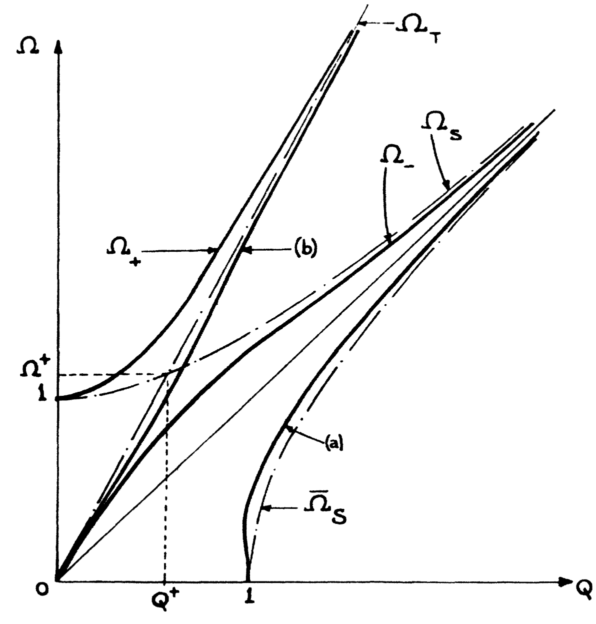

FIG. 3. Dispersion relation for "magnetoelastic" solitary waves in thin films.

given by[36]

$$|\Omega_+ - \Omega_-|_{Q=Q^*} \simeq \beta Q^* . \tag{6.59}$$

One point is to be emphasized. The uncoupled dispersion relation (6.56) is *not* exactly the nondimensional relation of spin waves or magnons in a ferromagnetic crystal. As a matter of fact, the latter is usually parabolic, $\Omega_S \simeq 1+Q^2$ (see Ref. 9) while (6.56) is hyperbolic. This difference is not significant in practice because the magnetoacoustic resonance point Q^* usually occurs in the rather flat part of the spin spectrum, with a frequency not very different from $\Omega_S(0)$; but this discrepancy is ultimately related to the fact that a real *spin wave* corresponds to the propagation of a true precession about the easy axis of magnetization directed along the y axis, while in the present linearized analysis only the angle ϕ about the z axis has been retained, the other angle φ having been eliminated.

D. Stable solitary-wave solution

The *pseudo*dispersion relation (6.51) can be written as

$$\mathscr{D}_{\rm sol}(\Omega,Q)=(\Omega^2-\tilde{\Omega}_S^2)(\Omega^2-\Omega_T^2)+\beta^2Q^2=0 , \tag{6.60}$$

where

$$\tilde{\Omega}_S^2=Q^2-1 . \tag{6.61}$$

Both Eqs. (6.60) and (6.61) are to be directly compared to Eqs. (6.55) and (6.56). It can be noticed that while Ω_L still appeared in the expression (6.40), in fact it does not contribute in Eq. (6.60) and the situation is quite similar to that of the linearized case in spite of the coupling present in Eq. (6.26a).

For small β, the solution branches (a) and (b) (see Fig. 3) of Eq. (6.60) are given by

$$\Omega(Q)\simeq\tilde{\Omega}_S\left[1+\beta^2\frac{Q^2}{2(Q^2-1)[(V_T^2-1)Q^2+1]}\right] \quad [\text{branch (a)}] , \tag{6.62a}$$

$$\Omega(Q)\simeq\Omega_T\left[1-\beta^2\frac{1}{2V_T^2[(V_T^2-1)Q^2+1]}\right] \quad [\text{branch (b)}] . \tag{6.62b}$$

They are thus very close to the uncoupled solutions Ω_T and $\tilde{\Omega}_S$. With $V_T>1$, the expressions (6.62) indicates the position of the coupled branches vis-à-vis the uncoupled solutions. A stability criterion decides on which of the branches (a) and (b) corresponds to a stable solitary-wave solution (6.52). The linear stability criterion which requests that the solution be always increasing (kink solution) for all ξ's,

$$\frac{d\psi}{d\xi}>0, \quad c\equiv\Omega/Q<1 , \tag{6.63}$$

i.e., a subsonic propagation with respect to the characteristic speed, here equal to one (compare Callegar and Reiss[38]) imposes that

$$\lambda>0, \quad a>0 . \tag{6.64}$$

Hence, only the plus signs must be considered in Eqs. (6.52) and (6.50), since only the points (Ω,Q) of the pseudodispersion relation situated below the bisectrix of the positive quadrant are admissible; hence branch (a), for which the only stable solution (6.52) is the one corresponding to a clockwise rotation of the magnetization in the (x,y) plane, i.e.,

$$\theta(\xi)=\frac{\pi}{2}+\tan^{-1}\left[\frac{\sinh\xi}{\hat{Q}(1-c^2)^{1/2}}\right] , \tag{6.65}$$

where, in fact, $\hat{Q}\simeq Q$ if the longitudinal magnetostriction coefficient is vanishingly small—see Eqs. (6.38) and (6.39).

E. Acoustic field generated by the Néel wall

Equations (6.34) can be integrated on account of (6.36) and the solution (6.65). After a somewhat lengthy calculation it is found that the longitudinal and transverse elastic displacements which accompany the rotation (6.65) of magnetization, are given by

$$U(\xi)=U_0+\frac{\alpha Q}{(\Omega^2-\Omega_L^2)}\left[\frac{1-2\zeta}{-2\zeta}\right]^{1/2}\ln\left|\frac{e^{-2\xi}-a^+}{e^{-2\xi}-a^-}\right| , \tag{6.66a}$$

$$V(\xi)=V_0-\frac{2\beta Q}{(\Omega^2-\Omega_T^2)}\left[\frac{1-2\zeta}{-2\zeta}\right]^{1/2}\tan^{-1}\left[\frac{\cosh\xi}{\sqrt{-2\zeta}}\right] , \tag{6.66b}$$

wherein

$$a^{\pm}=(4\zeta-1)\pm2[2\zeta(2\zeta-1)]^{1/2} , \tag{6.67}$$

with $a^+>0$ and $a^-<0$. Simultaneously, the nondimensional stress components Σ_{ij} developed by the displacements $U(\xi)$ and $V(\xi)$ in the (x,y) plane are found as

$$\Sigma_{xx}(\xi)=\frac{dU}{d\xi}=\frac{2\alpha Q(1-2\zeta)}{[(1-2\zeta)+\sinh^2\xi](\Omega^2-\Omega_L^2)} , \tag{6.68a}$$

$$\Sigma_{xy}(\xi)=\frac{dV}{d\xi}=-\frac{2\beta Q(1-2\zeta)^{1/2}}{(\Omega^2-\Omega_T^2)}\frac{\sinh\xi}{(1-2\zeta)+\sinh^2\xi} , \tag{6.68b}$$

to which can be added the internal stresses Σ_{yy}^0 associated with the internal strain e_{yy}^0 via the elasticity constitutive relation.

Equations (6.65), (6.66), and (6.68) provide a complete, *exact*, stable nonlinear dynamical solution (solitary wave) of the magnetoelastic equations that govern a thin elastic ferromagnetic film in the plane of the film in the absence of an applied magnetic field $\mathbf{H}^0$. Numerical simulations of this solution would be practically identical to those carried out and reproduced by Pouget and Maugin[18] by using a Lax-Wendroff leap-frog numerical scheme for nonlinear hyperbolic systems rewritten in the form of time-evolution systems in the case of elastic ferroelectrics.

Whenever H_y^0 is not zero but constant, and hence $\eta=$const, Eq. (6.68) becomes a *double*-sine-Gordon equation which is nonlinearly coupled to the elastic displacements U and V. If H_y^0 depends on time—e.g., through a step function—then with $|\eta(\tau)|$ small for all times some perturbation procedure has to be used to study the influence of the applied field. For this mathematical point we refer the reader to previous works on ferroelectrics (Refs. 29, 32, and 33) which provide the adequate treatment that needs no duplication.

VII. CONCLUSION

Solitary-wave solutions representing the nonlinear motion of a Bloch wall in an infinite, cubic elastic ferromagnetic crystal and of a Néel wall in a thin ferromagnetic film made of a cubic elastic crystal have been obtained in closed form. In the first case (Bloch wall) the whole problem reduces to a *sine-Gordon equation*, the influence of magnetoelastic couplings being felt only through a slight change in the reference length (wall thickness) of the problem. Therefore, this problem admits true *soliton* solutions. In the second case (Néel wall), in the absence of external field, the problem consists of solving simultaneously two wave equations and a sine-Gordon equation, the three equations being nonlinearly coupled via magnetostriction. This also admits solitary-wave solutions which consist of a solitary wave for the magnetic-spin orientation within the wall and accompanying solitary waves in the longitudinal elastic displacement and the transverse elastic displacement polarized in the plane of rotation of spins. This rather complex mathematical structure, already encountered in elastic ferroelectrics[18]—where only one elastic (transverse) displacement was involved—appears to be a common feature of the problem of propagating walls in elastic crystals with a microstructure.[39] It is this microstructure (in the present case, the magnetic-spin field) which brings the dispersive and nonlinear features in the wave problem. From the already thoroughly examined case of elastic ferroelectrics,[19] we know that the coupling of the sine-Gordon equation with one or two wave equations makes that pure solitons (i.e., solitary waves interacting as solitons do, recovering their full identity after interaction) are not possible, being always accompanied by radiation wavelets that propagate preceding and tail trailing the solitons. We refer the reader to the ferroelectric case for the corresponding theoretical and numerical analysis and to a review paper[40] for the general mathematical features of this problem. Another possible generalization of the present problem consists of accounting for coupled dissipative processes (viscosity and spin-lattice relaxation) as formulated in a good invariant framework by one of the authors.[41] This is left for further works.

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