Criticality and superfluidity in a dilute Bose fluid

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The crossover from ideal-Bose-gas behavior to nonclassical, *XY*-like criticality in a dilute, interacting Bose fluid is considered in detail with emphasis on the superfluid density, $\rho_s(T)$, below the transition. A general discussion of crossover scaling in constrained systems shows that ρ_s and other temperature-dependent thermodynamic functions at controlled overall density can be expressed in terms of universal scaling functions. The finite-temperature Green's function formalism is used to calculate the associated linear scaling fields and the crossover exponent *exactly* for all dimensionalities $d>2$, scaling predictions, including logarithmic singularities just at $d=3$, being checked via second-order computations. A mapping of the Bose problem onto the classical s^4 spin model is achieved which is asymptotically exact in the critical region. In this way, previously known renormalization-group results are used to calculate the scaling functions for ρ_s, for the off-diagonal susceptibility, etc., to first order in $\epsilon=4-d$. Good agreement is found with the experimental measurements by Reppy and co-workers of $\rho_s(T)$ for helium absorbed at low coverages in Vycor glass (of pore size 50—80 Å). An effective-mass ratio $m^*/m\simeq1.5$, and an effective-scattering-length ratio $a^*/a\simeq1.3$ are indicated for mobile helium in the Vycor (although the irregular character of Vycor has not, at this point, been fully allowed for in the theory). Other approaches in the literature to computing the critical behavior of quantum-mechanical systems are reviewed and related to the present analysis.

I. INTRODUCTION

Recent experiments on superfluid helium in a dilute limit[1,2] and on spin-polarized hydrogen[3] have renewed interest in the theory of the interacting, three-dimensional Bose fluid in the regime of low densities. In particular, it has become important to understand how, as the density increases from zero, the critical behavior changes over from that for an ideal, noninteracting Bose gas to that observed at the lambda point of a bulk, fully interacting Bose superfluid. Indeed, the effect of critical fluctuations on the superfluid order parameter, $\Psi_s(T)$, as evidenced by the behavior of the superfluid density, $\rho_s(T)$, has been studied recently in experiments by Crooker, Hebral, Smith, Takano, and Reppy.[1,2] Helium four was condensed in Vycor, which is a highly connected, semiregular spongelike glass, at a carefully controlled overall filling density $\bar{\rho}$. The overall superfluid density was measured by a delicate torsional oscillator method down to temperatures of around 5 mK.[1,2] When the Vycor was completely full the superfluid density was found to vary when $T\to T_c(\bar{\rho})-$ as

$$\rho_s(T)/\rho_s(0)\approx D\,|\,t\,|^{\zeta},\quad t=(T-T_c)/T_c\ , \tag{1.1}$$

with critical exponent $\zeta\simeq0.64\pm0.05$, which corresponds closely to the observed value, $\zeta\simeq0.674$, in pure bulk helium. Indeed a plot of ρ_s versus T closely resembles the bulk behavior. As $\bar{\rho}$ is reduced, T_c falls from $T_c\simeq1.96$ K,[2] as does $\rho_s(0)$; however, down to densities at which $T_c(\bar{\rho})\simeq60$ mK the shape of the curves and the exponent ζ, remain unchanged. One concludes that the Vycor glass, despite its somewhat random character, is not playing a significant role in the nature of the critical singularities or the superfluid ordered state. [Nevertheless, a small "tail" in $\rho_s(T)$ is observed, which smears the transition slightly:[1,2] this might be attributable to semimacroscopic inhomogeneities in the Vycor.]

On the other hand, when the density $\bar{\rho}$ is increased from zero one finds, initially, that there is no superfluidity even at $T\simeq0$. One must presume that all the helium is strongly adsorbed on the surface of the Vycor in "localized," "crystalline" or "solid" form. At a coverage $\bar{\rho}=\rho_0$ corresponding to about 1.5 atomic monolayers, however, one discovers the onset of superfluidity. Apart from some slight initial curvature, one then observes an almost linear increase of $\rho_s(0)$ with density. This can be interpreted naturally as revealing that the excess helium density, $\rho=\bar{\rho}-\rho_0$, corresponds to that of a dilute Bose gas moving in the Vycor pores. There is some residual attraction to the walls of the pores, which, in the experiments of Crooker *et al.*, had diameters in the range 40–80 Å. However, the thermal de Broglie wavelength,

$$\Lambda_T=h/(2\pi m k_B T)^{1/2}, \tag{1.2}$$

of a helium atom of mass $m=m_{\rm He}$ at 10 mK reaches 90 Å which exceeds the pore dimensions: thus it is reasonable to regard the Vycor as merely providing some sort of uniform background medium. Furthermore, because of the full three-dimensional interconnectivity of all accessible pores, superfluid or off-diagonal long-range order should be rapidly propagated in all three spatial dimensions. (This contrasts with the experiments in a Mylar "jelly roll" where the helium is confined to a two-dimensional film resulting in completely different transition behavior[4] of Kosterlitz-Thouless type.)

This picture of a three-dimensional dilute Bose fluid is supported by the observed variation of ρ_s at very low excess densities, ρ. For an ideal Bose gas in spatial dimensionality d, one has[5]

$$\rho_s(T)/\rho_s(0)=1-(T/T_c)^{d/2}\approx D_0\,|t|\,, \tag{1.3}$$

with $D_0=\frac{1}{2}d$, provided that $d>2$. Thus the exponent changes from $\zeta\simeq\frac{2}{3}$ to $\zeta_0=1$. In fact, as ρ increases so does T_c and the plots of $\rho_s(T)$ for $T_c\lesssim 15$ mK quite closely resemble the more-or-less linear variation predicted by (1.3). In this regime the mean interparticle spacing of the excess helium of density ρ is 40 Å or greater:[1,2] this greatly exceeds the atomic dimensions and, thus, the interparticle interactions should indeed be playing a relatively small role here. However, close to the transition for $T_c\gtrsim 15$ mK the data reveal a sharper decrease approximating the bulk, interacting behavior with $\zeta\simeq\frac{2}{3}$. As ρ and T_c increase further this "nonclassical" or nonideal critical region expands in size until the characteristic bulk variation completely takes over.

Even if one does not accept the physical picture presented, these experiments undoubtedly demonstrate a *crossover* in critical behavior from the bulk interacting form (1.1) to some new form, described near T_c by (1.3). In order not to beg the question of "ideality" in this limit it may be termed "dilute." Now the crossover can be regarded as controlled by $T_c(\rho)$ which decreases to zero as the dilute limit is approached. Rather generally, the theory of critical phenomena predicts that such a change in behavior should be described more fully by a *crossover scaling formulation.* Specifically, one should anticipate[6]

$$\rho_s(T;T_c)/\rho_s(0;T_c)\approx D_0\,|t|\,Y_T(T_c/E\,|t|^{\phi_T})\,, \tag{1.4}$$

in which ϕ_T is the crossover exponent and $Y_T(z)$ is the crossover scaling function while E is a metrical factor. (The subscripts T indicate that T_c is being used as the control variable: this is appropriate at the phenomenological level but, as will be seen, another form of variable is more natural if the theoretical picture of a dilute Bose fluid is accepted.[6]) The dilute limit is now reproduced, provided that $Y(z)\to Y(0)\neq 0$ as $z\to 0$. Conversely, to recover (1.1) for nonzero T_c as $t\to 0$, one must have $Y_T(z)\approx Y_T^\infty z^{(1-\zeta)/\phi_T}$ when $z\to\infty$. The scale of the critical region is then set by $z\gtrsim 1$ which shows that it corresponds to $t\lesssim[T_c(\rho)/E]^{1/\phi_T}$. Evidently ϕ_T determines the rate at which the critical region shrinks when $T_c\to 0$, and this reflects a dominant qualitative feature of the data.[1,2]

The scaling prediction (1.4) can be tested experimentally by plotting $\rho_s(T)/\rho_s(0)\,|t|$ versus $T_c/|t|^{\phi_T}$ using trial values of ϕ_T: all the points should lie on a common locus which then represents $Y_T(z)$. A tentative analysis[7] with early data suggested $\phi_T\simeq 1.5$ to 2. The theory[6] to be presented in detail here predicts $\phi_T=2$ (for $d=3$) and prefers somewhat different forms for the scaling variable. It is found to yield very satisfactory scaling fits to the data,[2,6] and the theoretical scaling function calculated to leading order in $\epsilon=4-d$ also gives a reasonable description of the observations.

Our task here then, is as follows. First, (a) accepting the picture of a dilute Bose fluid, we must check that a crossover scaling form is correct and determine the appropriate forms in which the physical parameters enter the scaling variables. Second, (b) we must determine the crossover exponent, ϕ_T, which should control *all* features of the changeover in criticality: in fact, we will find

$$\phi_T=2(4-d)/(d-2)^2\,, \tag{1.5}$$

for $2<d<4$, which range of dimensionality, d, will be adopted as standard. Then, (c) we must calculate the scaling function for the superfluid density, $Y_T(z)$ or its equivalent. (The scaling function, of course, depends on the property considered: various other scaling functions will be encountered and presented.) Finally, (d) we must check the predictions against the data of Crooker *et al.*[1] and examine the reasonableness of the metrical fitting parameters (for the effective mass and interaction strength). We will also discuss a little further the reasonableness of the physical picture, which certainly seems to be well vindicated by our calculations. Nevertheless, our analysis takes no direct account of the residual random character of Vycor: a more-detailed discussion of the effects of randomness on these calculations will be reserved for presentation elsewhere.[8]

This paper is arranged as follows. In Sec. II A the theoretical model is presented; aspects of its validity are discussed in Sec. II B. The scaling ansatz (1.4) and its generalizations are discussed in Sec. III. An aim is to show how the crossover exponent and scaling fields can be calculated and how the validity of scaling can be checked by doing perturbation theory about the ideal-Bose-gas limit. The appropriate quantum-mechanical perturbation calculations are explained in Sec. IV. In order to obtain scaling functions, more elaborate calculations are necessary. Other authors, particularly Nicoll and Chang[9] and Rudnick and Jasnow,[10] have undertaken appropriate renormalization-group analyses within the ϵ expansion, but they consider only the nonquantal, n-vector classical-spin models of ferromagnetism.[11,12] To take advantage of this body of work, we show, in Sec. V, how the quantal-

Bose-fluid model may be mapped onto the ($n=2$)-vector or classical XY model in a way that is asymptotically *exact* in the critical region. In Sec. VI this mapping is used to derive scaling functions to first order in $\epsilon=4-d$. The final scaling-function expressions for the superfluid density are displayed in Eqs. (6.58)–(6.67) and, in Sec. VI F, the fitting to the experimental data is undertaken. Finally, in Sec. VII some previous approaches to the problem of criticality in an interacting Bose fluid are reviewed briefly but critically and related to our analysis. (Previous authors have considered neither the equation of state in the crossover regime below T_c nor the superfluid density.) An appendix shows how logarithmic singularities which appear at second order in the quantal perturbation theories when $d=3$ are fully consistent with nonlinear renormalization-group recursion relations although apparently violating the simplest scaling formulation.

II. A MODEL FOR HELIUM IN VYCOR

A. Hamiltonian for a Bose system

We will consider spinless bosons confined to a region Ω of volume V_Ω in the presence of an external single-particle potential $w(\mathbf{r})$ which may represent the Vycor. In terms of Bose field operators $\psi^\dagger(\mathbf{r})$ and $\psi(\mathbf{r})$, the second-quantized Hamiltonian will then be taken as

$$\mathcal{H}_B=\mathcal{H}_0+\mathcal{H}_{1/2}-\mu\mathcal{N}+\mathcal{H}_1+\mathcal{H}_2 , \tag{2.1}$$

with kinetic energy

$$\mathcal{H}_0=\int_\Omega d^dr\,\psi^\dagger(\mathbf{r})\left[-\frac{\hbar^2}{2m}\right]\nabla^2\psi(\mathbf{r}) , \tag{2.2}$$

number-operator

$$\mathcal{N}=\int_\Omega d^dr\,\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) , \tag{2.3a}$$

external-potential term

$$\mathcal{H}_1=\int_\Omega d^dr\,w(\mathbf{r})\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) , \tag{2.3b}$$

and, if $v(r)$ is the translationally invariant pair-potential, interaction energy

$$\mathcal{H}_2=\tfrac{1}{2}\int_\Omega d^dr\int_\Omega d^dr'\,\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r})v(\mathbf{r}'-\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r}') . \tag{2.4}$$

Finally, for theoretical purposes in understanding the ordering we include the physically "anomalous," off-diagonal term

$$\mathcal{H}_{1/2}=-\tfrac{1}{2}\int_\Omega d^dr[\nu^*\psi(\mathbf{r})+\nu\psi^\dagger(\mathbf{r})] , \tag{2.5}$$

where ν, which may be complex, is the off-diagonal or Bose field. As usual μ denotes the chemical potential.

Thermodynamic properties follow from the free-energy density

$$f(\beta,\mu,\nu)=F/V_\Omega=-(k_BT/V_\Omega)\ln Z , \tag{2.6}$$

where $\beta=1/k_BT$ and, as always, the partition function is given by

$$Z=\mathrm{Tr}\{\exp(-\beta\mathcal{H}_B)\} , \tag{2.7}$$

the trace being taken over a full set of symmetric states for all N. To make comparisons with experiment, the chemical potential needs to be adjusted to keep the overall density,

$$\rho\equiv\frac{N}{V_\Omega}\equiv\frac{\langle\mathcal{N}\rangle}{V_\Omega}=-\left[\frac{\partial f}{\partial\mu}\right]_{\beta,\nu} , \tag{2.8}$$

fixed. The order parameter (sometimes called the "condensate wave function") is given by

$$\Psi_0=\langle\psi\rangle_{\nu\to0}=-2\left[\frac{\partial f}{\partial\nu^*}\right]_{\beta,\nu} . \tag{2.9}$$

Even above any Bose-condensation critical point there is a corresponding reduced *off-diagonal susceptibility* defined by

$$\chi\equiv\chi_\nu/\beta\equiv\left[\frac{\partial\Psi_0}{\partial\beta\nu}\right]_{\beta,\mu} , \tag{2.10}$$

which diverges at the transition and is, thus, theoretically significant.

Most of the analysis below will be performed under the assumption that, for the critical behavior of interest, the external potential $w(\mathbf{r})$ can be replaced by a spatially uniform effective medium which gives rise to an effective mass, m^*, for the bosons (in place of a bare mass m) and which likewise modifies the interaction potential $v(\mathbf{r})$. This assumption will be discussed further in Sec. II B. It has the technical merit of restoring translational invariance so that a momentum space formulation is advantageous. Accordingly, we put

$$v_\mathbf{q}=\int_\Omega d^dr\,e^{i\mathbf{q}\cdot\mathbf{r}}v(\mathbf{r}) , \tag{2.11}$$

$$a_\mathbf{k}=V_\Omega^{-1/2}\int_\Omega d^dr\,e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r}) , \tag{2.12}$$

and then have

$$\mathcal{H}_0-\mu\mathcal{N}=\sum_\mathbf{k}(\varepsilon_\mathbf{k}-\mu)a_\mathbf{k}^\dagger a_\mathbf{k} , \tag{2.13}$$

$$\mathcal{H}_{1/2}=\tfrac{1}{2}V_\Omega^{1/2}(\nu^*a_0+\nu a_0^\dagger) , \tag{2.14}$$

$$\mathcal{H}_2=\tfrac{1}{2}V_\Omega^{-1}\sum_\mathbf{k}\sum_{\mathbf{k}'}\sum_\mathbf{q}v_\mathbf{q}a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}'-\mathbf{q}}^\dagger a_{\mathbf{k}'}a_\mathbf{k} , \tag{2.15}$$

where, as usual, the single-particle energies are

$$\varepsilon_\mathbf{k}=\hbar^2k^2/2m^*+O(k^4) , \tag{2.16}$$

the higher-order term allowing for the effects of the medium.

It is common practice in the literature[13] to take $v(\mathbf{r})=v_0\delta(\mathbf{r})$ where, generalizing from $d=3$, one then has

$$v_0=4\pi^{d/2}\hbar^2a^{d-2}/\Gamma(\tfrac{1}{2}d-1)m^* , \tag{2.17}$$

in which a is the s-wave scattering length or effective hard-core diameter, while $\Gamma(z)$ is the standard gamma function. However, such an assignment causes divergences in perturbation theory in second and higher order which we wish to investigate in order to check the consistency of scaling theory. Accordingly, we will retain a smooth, bounded form for $v(\mathbf{r})$ and, correspondingly, for $v_{\mathbf{k}}$. Not surprisingly, we will see that only $v_{\mathbf{q}\to 0}$ plays a major role at low T and, hence, when necessary we may suppose that $v_{\mathbf{q}}$ equals v_0 for $|\mathbf{q}|$ up to some *cutoff momentum* q_Λ beyond which it vanishes.

For reference below we recall here the thermodynamic properties of an ideal Bose gas[14] corresponding to $v_{\mathbf{q}}\equiv 0$. By performing shifts on $a_0^\dagger$ and a_0 the Hamiltonian is readily diagonalized and yields a free energy

$$f(\beta,\mu,\nu)=|\nu|^2/4\mu+\beta^{-1}\int_{\mathbf{k}}\ln[1-e^{-\beta(\varepsilon_{\mathbf{k}}-\mu)}] \ . \quad (2.18)$$

where we adopt the convenient notation

$$\int_{\mathbf{k}}\equiv\int d^dk/(2\pi)^d \ , \quad (2.19)$$

in which $\mathbf{k}$ ranges over all values or up to a cutoff at $|\mathbf{k}|=q_\Lambda$ if one is to be understood. The density is thus

$$\rho=|\nu|^2/4\mu^2+\int_{\mathbf{k}}n_B(\varepsilon_{\mathbf{k}}-\mu) \ , \quad (2.20)$$

in which the standard Bose occupation number factor is

$$n_B(\varepsilon)=1/(e^{\beta\varepsilon}-1) \ . \quad (2.21)$$

By (2.9) one has $\Psi_0=-\nu/2\mu$ whence the "condensate density" is

$$n_0(T)=|\Psi_0(T)|^2=\rho-\int_{\mathbf{k}}n_B(\varepsilon_{\mathbf{k}}-\mu) \ . \quad (2.22)$$

For T above the transition point, $n_0(T)$ vanishes identically in zero off-diagonal field because $\mu\neq 0$ and thus the first term in (2.20) vanishes when $|\nu|\to 0$. However, for $d>2$ the integral on n_B is bounded above as $\mu\to 0$. Thus below $T_c^0(\rho)$, which is determined by

$$\rho(h^2/2\pi m^* k_B T_c^0)^{d/2}=\zeta(\tfrac{1}{2}d) \ , \quad (2.23)$$

the ratio ν/μ cannot vanish as $\nu\to 0$ even when $\mu\to 0$: then (2.22) yields the standard result[5,14]

$$n^0(T)/\rho=1-[T/T_c^0(\rho)]^{d/2} \ . \quad (2.24)$$

B. Validity of the model

The critical behavior of an interacting but dilute Bose gas and, especially, the behavior of the superfluid density as the crossover to ideal behavior occurs, is a problem of interest in its own right. Any reader concerned only with the theory of this crossover may wish to skip this subsection in which we discuss briefly the degree to which such a model may be valid as a description of the Crooker *et al.*[1] experiments on helium absorbed in Vycor.

The manufacture of Vycor involves the leaching out of one component of a finely divided two-component glass mixture. The process begins at the surface, and hence ensures the presence of an infinite cluster of open and accessible pores. Porosity measurements[15] yield values of about 40%, which is well above the percolation threshold and is consistent with a highly-connected three-dimensional structure. Furthermore, the pore sizes have a definite physical scale (in the range 50 to 80 Å in the Crooker *et al.* experiments) and an apparently well-behaved distribution. It therefore seems unlikely that Vycor should be viewed in any way as a fractal structure. Nevertheless, we may note that the effect of a fractal structure on the transition in an ideal Bose gas should be to give a superfluid density and, in this case,[5] a condensate fraction varying like $[1-(T/T_c)^{\tilde{d}/2}]$. Here $\tilde{d}$, which should exceed 2 for a transition, is the "fracton" or "acoustic" dimension,[16] which describes the density of states at low energies according to $N(\varepsilon)\sim\varepsilon^{(\tilde{d}/2)-1}$.[17] (The states are presumed to be extended.) Near T_c such behavior clearly cannot be distinguished from ideal Bose behavior. The effect of such a fractal structure on the *interacting* Bose fluid critical behavior is difficult to assess definitively, but one might hope that it corresponds reasonably closely to the corresponding uniform system of intermediate dimensionality $\tilde{d}$ as suggested by studies on various hierarchical fractal structures.[18]

However, we believe that the excess helium atoms responsible for the observed superfluidity may be regarded, in the range of most interest where the thermal wavelength, Λ_T, and mean interparticle spacing are larger or comparable to the typical pore dimensions, as moving in a normal three-dimensional random potential, $w(r)$. In this respect the problem of helium-atom motion in Vycor is analogous to an electron in a disordered lattice where the Fermi momentum sets the scale of the potential variation. In that context it is certainly customary to take account of the background potential via an effective-mass approximation.

The modification of the bare helium-helium van der Waals attractions and the repulsive core interactions by the Vycor seems very difficult to assess *a priori*. It is certain that the attractive tail of the interaction potential is much reduced since nothing like a bulk liquid-vapor condensation is seen in Vycor at any level of filling. Indeed, from the experimental viewpoint the great merit of Vycor in these experiments is that the interatomic attractions are effectively suppressed so that helium can be examined as a low-density but still three-dimensional superfluid phase, something that is impossible with pure bulk helium. On the other hand, the strong hard-core part of the interaction, having a scale of 3 Å which is much less than the pore diameters, would seem unlikely to be greatly altered. Furthermore, for excess interparticle spacings much larger than 3 Å only the low-momentum part of the interaction should matter so that an effective $v_{\mathbf{q}}$ should be adequate.[13]

As the critical temperature increases the corresponding thermal de Broglie wavelength, Λ_T, drops and becomes smaller than the pore size. Likewise the correlation length amplitude, ξ_0, which can be estimated[2] from the observed superfluid density via

$$\xi_0\simeq(m/h)^2 k_B T|t|^{\zeta}/\rho_s(T) \quad \text{for } d=3 \ ,$$

also falls as T_c increases and eventually becomes comparable with the pore size (although one finds $\xi_0>200$ Å for $T_c<100$ mK). One might then recall that the helium atoms will predominantly stay relatively close to the walls of the pores and might hence ask if one should not see some characteristics of critical behavior in a *reduced* dimensionality $d'=2$. Indeed, some sound attenuation experiments in Vycor (but with appreciably larger pores) have been interpreted as revealing two-dimensional effects.[19] Furthermore, studies of helium films condensed on particles of an Al_2O_3 powder show behavior reflecting two-dimensional criticality:[20] the particle sizes, of order 500 Å, were much larger than the pore dimensions in the Crooker *et al.* experiments. These observations suggest that the interconnections between films on different particles and between different pores are relatively much weaker in systems with large particles and pore sizes. On the other hand, it seems likely that as the pore size is reduced the structure becomes more strongly connected with neighboring pores increasingly well coupled so that the order extends essentially in the full three dimensions. (Note that this must happen close enough to the transition, even in the powders and systems with large pores, i.e., dimensional crossover from $d=2$ to $d=3$ must eventually occur as $T \rightarrow T_c$ in any system that is actually three-dimensionally connected even if only weakly.)

We conclude that an interesting and reasonable model for helium in Vycor is that of a low-density, mobile, three-dimensionally connected gas above a substrate of localized helium atoms. Various studies of heat capacity,[15] general thermodynamics,[21] NMR on ^{3}He films,[22] as well as the Crooker *et al.* observations of superfluid density seem quite consistent with this picture.

Undoubtedly, however, the residual external potential, represented by $w(\mathbf{r})$ in the Hamiltonian (2.1), has a spatially nonuniform or random component which should be allowed for theoretically. Now at the bulk superfluid transition, which is expected to be in the XY or $n=2$ universality class, as our analysis confirms, the specific-heat exponent is slightly negative: $\alpha=-0.02\pm0.02$. The Harris criterion,[23] which should be applicable,[24] then indicates that the randomness represented by $w(\mathbf{r})$ should be *irrelevant*. Hence the nature of the interacting critical behavior should not be changed by the randomness (although one might be worried, in view of the smallness of α, about rather slowly varying corrections to scaling). This fact supports the view that the transition in Vycor for T_c exceeding say 50 mK should, ideally, be sharp despite the *micro*scopic randomness of Vycor. Macroscopic inhomogeneities or superflow paths between the outside of the Vycor block and the container will, however, cause some smearing. Thus it is reasonable to suppose this is the cause of the small "tails" on the experimental $\rho_s(T)$ plots above the expected transition point.

On the other hand, the crossover behavior of interest is from the ideal Bose gas. Here the interactions described by $v(\mathbf{r})$ represent a relevant renormalization-group perturbation as we will see in detail. However, it is not difficult to see by the perturbation methods we explain or by renormalization calculations, that a random external potential represents *another, independent* relevant perturbation about the Gaussian fixed point which, as we show, controls the ideal Bose gas criticality. As such the strength of the randomness should enter separately in a full crossover scaling calculation. Such calculations are not reported here: indeed they seem to present considerable difficulties. However, the effects of the randomness can be studied further[8] and, in the light of special properties of Vycor and of the mapping to spin systems, plausible if not compelling arguments can be developed which suggest why the residual randomness of the helium-coated Vycor glass does not seem to play a strong role. For the present purposes we will accept this conclusion and proceed by ignoring any randomness in $w(\mathbf{r})$. It is then reasonable to expect that the effective-mass approximation and a modified pair-interaction potential will suffice to account for the regular part of $w(\mathbf{r})$.

III. SCALING FORMULATION

In this section we set out the expectations of scaling theory for a crossover in critical behavior of the sort which is anticipated to occur in a dilute Bose fluid. The main aim is to show how straightforward perturbation theory suffices, if scaling is accepted, to determine many features *exactly*, including the crossover exponent. Furthermore, perturbation theory carried to higher order can be used to check the consistency of the scaling hypothesis and, finally, it provides a route to a precise, unambiguous means of mapping the dilute Bose fluid in its critical region onto an n-vector spin model. An important feature, especially in the mappings, relates to the fact that a full scaling formulation must pay attention to the existence and nature of the *linear* and *nonlinear scaling fields* which, in general, serve to mix and recombine the various physical parameters which naturally enter the original problem. Another relevant issue addressed is the influence of a *constraint* (such as observation at constant density) on the crossover scaling formulation.

A. General considerations

Let us for concreteness and also, as it transpires, for computational simplicity, consider the reduced off-diagonal susceptibility $\chi(\beta,\mu,\nu)$, defined in (2.10). However, we will *not* here use many specific properties of a Bose fluid. When $\nu=0$, to which we restrict attention, the susceptibility diverges when μ approaches a critical value which in the ideal, noninteracting case we call $\mu_0(\beta)$. If we set

$$\dot{\mu}=\mu-\mu_0(\beta)\ , \tag{3.1}$$

one anticipates $\chi \approx C/|\dot{\mu}|^{\gamma_0}$, where γ_0 is the ideal critical exponent and C is the corresponding ideal amplitude. But deviations from ideal behavior are caused by the presence of the interaction potential $v(\mathbf{r})$. One expects the universal critical features to be controlled, for weak interactions, only by

$$v_0=\int d^d r\, v(\mathbf{r}) \equiv v_{\mathbf{q}=0}\ . \tag{3.2}$$

Then the simplest scaling ansatz is

$$\chi(\mu;v_0) \approx \frac{C}{|\dot{\mu}|^{\gamma_0}} X\left[\frac{Bv_0}{|\dot{\mu}|^{\phi}}\right]\ , \tag{3.3}$$

in which ϕ is the *crossover exponent* (for the variable μ) while B is a nonuniversal *metrical factor* which can be chosen so that the scaling function has an expansion

$$X(x)=1+x+X_2x^2+X_3x^3+\cdots\ , \tag{3.4}$$

with derivative $X_1 \equiv (dX/dx)_0$ normalized to unity. Note that $X(0)=1$ is needed so that (3.3) reproduces the ideal result when $v_0 \to 0$. By the same token, the existence of a perturbation expansion for χ in powers of v_0 indicates that $X(x)$ should have a power-series expansion (but see also the Appendix).

The nonideal ($v_0>0$) critical behavior of χ will be controlled by a singularity of the scaling function $X(x)$. Specifically, if

$$X(x) \approx X_c/(x_c-x)^{\gamma} \quad \text{as } x \to x_c - \ , \tag{3.5}$$

then for $v_0 \neq 0$ the susceptibility diverges as $\widetilde{C}(v_0)/|\mu-\mu_c(v_0;\beta)|^{\gamma}$ where the nonideal critical locus must depend on v_0 as

$$\mu_c(v_0;\beta) \approx \mu_0(\beta) + |Bv_0/x_c|^{1/\phi} \ , \tag{3.6}$$

when $v_0 \to 0$. An expression for the amplitude $\widetilde{C}(v_0)$ as $v_0 \to 0$ also follows. It may happen that $x_c=\infty$; then (3.6) is still valid but (3.5) is no longer adequate: for this case, see (3.22) below.

Note that the relation (3.6) for the nonideal critical locus must, physically, be *independent* of the particular property, susceptibility, specific heat, etc., studied. This is ensured by the *universality* of the crossover exponent ϕ, which is the *same* for all properties both above *and* below the ideal transition: conversely, if ϕ is known for one property it is known for all. Similarly, after a single normalization fixing B, as in (3.4), the critical value x_c of the scaling combination $x=Bv_0/|\dot{\mu}|^{\phi}$ is universal.

Now the utility of the scaling hypothesis for our present purposes is seen by combining (3.3) and (3.4) to yield the expansion

$$\chi \approx \frac{C}{|\dot{\mu}|^{\gamma_0}} + \frac{BC}{|\dot{\mu}|^{\gamma_0+\phi}} v_0 + \frac{X_2 B^2 C}{|\dot{\mu}|^{\gamma_0+2\phi}} v_0^2 + O(v_0^3) \ . \tag{3.7}$$

This predicts the divergences to be found in the perturbation expansion of $\chi(\dot{\mu};v_0)$ in powers of v_0. Conversely, the first-order perturbation term should yield (i) the universal crossover exponent ϕ and (ii) the metrical factor B; the second-order term should (iii) *check* scaling by revealing a divergence with exponent $\gamma_0+2\phi$, and (iv) yield the universal susceptibility scaling function expansion coefficient X_2 in (3.4), and so on!

We shall, in fact, implement this matching strategy for the Bose fluid. However, before doing so one must recognize that the scaling postulate (3.3) is somewhat too naive: specifically, it makes no allowance for linear or, more generally, *nonlinear scaling fields* nor for *irrelevant variables*.[25,26] At special values of the dimensionality, d, one must also be prepared to find *logarithmic singularities* in the perturbation expansion and critical behavior. We shall sidestep this technical complication by working always with *general d*. However, $d=3$ *is* a special value for which logarithms *do* appear. To see how scaling nonetheless works when properly formulated via the renormalization group,[25] the extra features arising in this case are explained in the Appendix.

In light of general renormalization-group theory, therefore,[25,26] we introduce the more complete scaling form

$$\chi \approx \frac{C}{|g_\mu|^{\gamma_0}} X\left[\frac{Bg_v}{|g_\mu|^{\phi}}, \frac{B_2 g_2}{|g_\mu|^{\phi_2}}, \ldots\right] + \chi_0(\mu;v_0) \ , \tag{3.8}$$

in which χ_0 represents an analytic "background" contribution. The g_i ($i=2,3,\ldots$) represent nonlinear scaling fields for the further variables which have scaling exponents ϕ_i and metrical factors B_i [which can again be chosen so that $(\partial X/\partial x_i)_0=1$]. We expect these extra scaling fields to depend on further details of the interaction potential $v(\mathbf{r})$, etc., and to be *irrelevant* so that their scaling exponents are negative, i.e., $-\phi_i=\theta_i>0$. Then they give rise merely to asymptotic *corrections to scaling* varying as $|g_\mu|^{\theta_i}$.[25,26] If, however, one of these fields turns out to have $\phi_i>0$ it also is relevant and must then enter the discussion of the crossover: this, in fact, is what happens if *random* perturbations, controlled by $w(\mathbf{r})$ in (2.3), are included.[8]

Let us, initially, neglect the irrelevant variables: then the nonlinear scaling fields for the chemical potential, μ, and for the interaction should have expansions in integral powers as

$$g_\mu = a_0\dot{\mu} + a_1 v_0 + a_2\dot{\mu}^2 + a_3\dot{\mu}v_0 + a_4 v_0^2 + \cdots \ , \tag{3.9}$$

$$g_v = v_0(b_0 + b_1\dot{\mu} + b_2 v_0 + \cdots) \ , \tag{3.10}$$

where the terms neglected are of cubic order in $\dot{\mu}$ and v_0. In this expansion the leading coefficients a_0 and b_0 may be normalized to unity by absorbing factors into the amplitudes B and C. However, it proves more convenient to retain them so that g_μ and g_v may be assumed dimensionless. The form for g_v has been restricted here by knowledge of the physics, namely, that ideal behavior must be recaptured to all orders when $v_0 \to 0$ so that $g_v(v_0=0)\equiv 0$.

The perturbation expansion (3.7) now becomes

$$\chi(\mu;v_0) = \chi_{(0)}(\mu) + \chi_{(1)}(\mu)v_0 + \tfrac{1}{2}\chi_{(2)}(\mu)v_0^2 + \cdots \ , \tag{3.11}$$

where, putting $g_\mu(v_0=0)\equiv g_\mu^0 = a_0\dot{\mu} + a_2\dot{\mu}^2 + \cdots$, we have

$$\chi_{(0)} \approx C/|g_\mu^0|^{\gamma_0} + \chi_0(\mu;0) \ , \tag{3.12}$$

$$\chi_{(1)} \approx D_0/|g_\mu^0|^{\gamma_0+1} + D_1/|g_\mu^0|^{\gamma_0+\phi} + \chi_0'(\mu,0) \ , \tag{3.13}$$

in which the prime denotes differentiation with respect to v_0, and

$$\chi_{(2)} \approx \frac{E_0}{|g_\mu^0|^{\gamma_0+2}} + \frac{E_1}{|g_\mu^0|^{\gamma_0+1}} + \frac{E_2}{|g_\mu^0|^{\gamma_0+1+\phi}} + \frac{E_3}{|g_\mu^0|^{\gamma_0+2\phi}} + \frac{E_4}{|g_\mu^0|^{\gamma_0+\phi}} + \chi_0''(\mu,0) \ . \tag{3.14}$$

The main feature to notice, by comparison with (3.7), is the appearance of new singularities associated, in fact, with the mixing coefficient $g_\mu'(0)\approx a_1$ in (3.9). If $\phi<1$, as will transpire in our case, some of these new singularities actually *dominate* the perturbation coefficients. Thus in any matching program they must clearly be allowed for. A nonzero value of a_1 means that v_0 enters linearly in g_μ and so there is a nontrivial *linear* scaling field. As follows by equating $g_v/|g_\mu|^\phi$ to x_c/B, this corresponds to a shift in the critical locus μ_c analytic in v_0 *in addition* to the nonanalytic term displayed in (3.6). Conversely, the appearance of a first-order perturbation term diverging like $1/|g_\mu^0|^{\gamma_0+1} \sim 1/|\mathring{\mu}|^{\gamma_0+1}$ implies the presence of such linear mixing.

The various amplitudes in (3.13) and (3.14) are easily found to be

$$D_0=-C\gamma_0 g'_\mu, \quad D_1=BCg'_v \tag{3.15}$$

where, neglecting $O(\mathring{\mu}^2)$ terms,

$$g'_\mu \approx a_1+a_3\mathring{\mu}, \quad g'_v \approx b_0+b_1\mathring{\mu} , \tag{3.16}$$

and

$$\begin{aligned} &E_0=C\gamma_0(\gamma_0+1)g'^2_\mu , \\ &E_1=C\gamma_0 g''_\mu , \\ &E_2=-2BC(\gamma_0+\phi)g'_\mu g'_v , \\ &E_3=2X_2B^2Cg'^2_v , \text{ and } E_4=BCg''_v , \end{aligned} \tag{3.17}$$

where $g''_\mu \approx 2a_4+O(\mathring{\mu})$ and $g''_v=2b_2+O(\mathring{\mu})$.

Including the irrelevant variables complicates the spectrum of singularities in the perturbation expansion still further. Each field g_i has an expansion analogous to (3.9) and (3.10) except that the g_i themselves enter, as they should also in (3.9) and (3.10). In first-order perturbation theory the extra terms,

$$D_i/|g_\mu^0|^{\gamma_0+\phi_i} \quad (i=2,3,4,\dots) , \tag{3.18}$$

appear with $D_i=B_iCg'_i$. In second and higher orders, cross terms involving ϕ and the ϕ_i also enter. However, if all the extra fields are, indeed, irrelevant the $\phi_i=-|\theta_i|$ are negative, and so the secondary singularities appearing beyond the primary ones displayed in (3.12) and (3.13) are relatively weaker.

We are thus in a position to compare (3.11)–(3.18) with the actual perturbation expansions about the ideal Bose gas. This is carried out in the next section to which readers may wish to jump at this stage. For our problem, however, there is a further complication which it is convenient to treat here. The most appropriate theoretical variables for scaling and calculations are the chemical potential, μ, and potential strength, v_0, since these enter linearly as "coupling constants" in the Hamiltonian. The inverse temperature β is best regarded as a parameter combined into the more basic variables $\beta\mu$ and βv_0. However, the experiments (as well as the normal view of an ideal Bose fluid) entail the *constraint* of *constant density*,[27] with temperature T acting as primary variable while v_0 is held fixed. Such constraints can be dealt with purely by thermodynamics but they may lead to significant changes in critical behavior, in particular, to a *renormalization of critical exponents*.[28] Constraints also complicate the question of the full crossover scaling form. Since the issues arising are general, although some special features enter for a dilute Bose fluid, they will be explored at this point; but, as indicated, readers may prefer to return here for reference when the results are alluded to in Secs. IV and VI.

B. Crossover scaling in a constrained system

Consider a thermodynamic property, $p(\beta,\mu;v_0)$, which obeys the crossover scaling form

$$p \approx |g_\mu|^{\pi_0}P(Bg_v/|g_\mu|^\phi)+p_0(\beta,\mu;v_0) , \tag{3.19}$$

where π_0 is the ideal exponent, the amplitude prefactor has been absorbed in $P(x)$, and p_0 is a smooth background. We will assume, following (3.10), that $g_v(\beta,\mu;v_0) \propto v_0$. Our concern is to study the transition under the constraint

$$r(\beta,\mu;v_0)=r_c , \tag{3.20}$$

where r_c is fixed while r is related analytically to some thermodynamic property and so satisfies a corresponding scaling expression

$$r \approx A\ |g_\mu|^{\psi_0}R(Bg_v/|g_\mu|^\phi)+r_0(\beta,\mu;v_0) , \tag{3.21}$$

in which we will assume that both the amplitude A and the background r_0 have Taylor expansions in β, μ, and v_0. The nonideal ($v_0>0$) critical locus is given by $Bg_v/|g_\mu|^\phi \approx x_c$ where x_c might be ∞ [as, in fact, happens to $O(\epsilon)$ in the Gaussian to n-vector crossover we study in Sec. VI]. To allow for $x_c=\infty$ we may write the scaling function for $x \to x_c-$ as[29]

$$R(x) \approx R_c x^{\psi_0/\phi}+R_\psi x^{(\psi_0-\psi)/\phi}[1-(x/x_c)]^\psi+\cdots , \tag{3.22}$$

where ψ is the nonideal exponent for r: if $\psi>1$ terms regular in $[1-(x/x_c)]$ must also enter. Note that we necessarily have ψ_0, $\psi>0$ if the constrained system actually achieves criticality. Most frequently r will couple directly to the energy (as the most relevant "even" or "thermal"[26] operator) and one then has $\psi_0=1-\alpha_0$ and $\psi=1-\alpha$, with α_0 and α the ideal ($v_0=0$) and nonideal specific-heat exponents, respectively. This is, indeed, the situation for the dilute Bose gas at constant density, $r=\rho$, for which $\alpha_0=\frac{1}{2}\epsilon$ for $\epsilon=4-d>0$ while $\alpha \simeq -0.02$ for $d=3$.

Now in the ideal limit the critical locus is given by $g_\mu(\beta,\mu;v_0=0)=0$. The constraint $r(\beta,\mu;0)=0$ then specifies a particular ideal critical point, say (β_c^0,μ_c^0), on this locus. (For simplicity we ignore the possibility of multiple or coincident roots.) The scaling fields g_μ and g_v may be expanded about this ideal constrained critical point in powers of

$$t_0=\beta-\beta_c^0, \quad \mu_0=\mu-\mu_c^0, \quad \text{and } v_0 . \tag{3.23}$$

Our aim is to use the constraint to eliminate μ or, equivalently, g_μ in favor of t_0 or some corresponding scaling field, say, $g_\beta(\beta,v_0)$, and to study the consequent variation of p. We wish to discover, if possible, a *constrained scaling form* for p parallel to (3.19). Under appropriate conditions, in particular when $\psi_0<1$, we will show that one does, in fact, obtain the constrained scaling behavior

$$p(\beta,\mu;v_0)\,|_{r=r_c}\approx |g_\beta|^{\tilde{\pi}_0}\tilde{P}(\tilde{B}\tilde{g}_v/|g_\beta|^{\tilde{\phi}}) + p_0^c+\tilde{p}_1t_0+\tilde{p}_2v_0+\cdots , \quad (3.24)$$

where

$$g_\beta\approx c_1t_0+c_2v_0 \text{ and } \tilde{g}_v\approx v_0(b_0+\tilde{b}_1t_0+\tilde{b}_2v_0) , \quad (3.25)$$

while the *"renormalized" critical exponents*[28] are

$$\tilde{\pi}_0=\pi_0/\psi_0 \text{ and } \tilde{\phi}=\phi/\psi_0 . \quad (3.26)$$

It will be found, in contrast to (2.9) and (2.10), that the higher-order corrections omitted in (3.25) *cannot*, in general, be expressed as a power series in t_0 and v_0: the same goes for the omitted background terms in (3.24). It will be shown, however, that the missing terms vanish faster than those shown by factors $g_\beta^{(1-\psi_0)/\psi_0}$. When $\psi_0>1$ a similar result holds except that the exponents in (3.24) are not renormalized: in other words (3.19) still holds in form but with more singular corrections.

To establish these results note that the constraint may be written

$$r_c-r_0\approx A|g_\mu|^{\psi_0}R(x) \text{ with } x=Bg_v/|g_\mu|^{\phi} . \quad (3.27)$$

On dividing by $A|Bg_v|^{\psi_0/\phi}$, and rearranging we get

$$y\equiv\bar{B}g_v/|r_c-r_0|^{\bar{\phi}}=x/|R(x)|^{\bar{\phi}} , \quad (3.28)$$

with $\bar{B}=B|A|^{\bar{\phi}}$. This equation can be inverted to yield $x=\Xi(y)$ (see further below) which may be substituted into (3.19). Then, if one could express (r_c-r_0) and g_μ in terms of t_0 and v_0, the job would be done.

Accordingly we must analyze the constraint (3.27) further. To this end, note that since r_0 and g_μ are both expandable in powers of t_0, μ_0, and v_0, we can write

$$r_0(\beta,\mu;v_0)=k_0(\beta,v_0)+g_\mu k_1(\beta,v_0)+O(g_\mu^2) . \quad (3.29)$$

To linear order this yields

$$r_c-r_0\approx c_1t_0+c_2v_0-c_3g_\mu\equiv\tilde{t}-c_3g_\mu , \quad (3.30)$$

with, in an obvious notation, $c_1=-(\partial k_0/\partial\beta)_c^0$, $c_2=-(\partial k_0/\partial v_0)_c^0$, and $c_3=k_1(0,0)$. Similarly we can write

$$g_v=v_0(b_0+b_1t_0+b_2v_0+b_3g_\mu+\cdots) , \quad (3.31)$$

$$p_0=p_0^c+p_1t_0+p_2v_0+p_3g_\mu+\cdots , \quad (3.32)$$

$$A=A_c+A_1t_0+A_2v_0+A_3g_\mu+\cdots . \quad (3.33)$$

To leading order the constraint then becomes

$$c_3g_\mu+A_c|g_\mu|^{\psi_0}R(x)=\tilde{t} . \quad (3.34)$$

This is an equation for g_μ in terms of $\tilde{t}$ and v_0.

There are two principal cases to consider: (i) $\psi_0>1$ and (ii) $\psi_0<1$. In the *first case*, c_3g_μ dominates the left-hand side of (3.34) and solving by iteration yields

$$g_\mu=c_3^{-1}\tilde{t}[1+\mathscr{E}_>(t_0,v_0)] , \quad (3.35)$$

where, as $t_0,v_0\to 0$, the error term is bounded as

$$|\mathscr{E}_>(t_0,v_0)|\le|A_c|\,|c_3|^{-\psi_0}R_{\max}|\tilde{t}|^{\psi_0-1} , \quad (3.36)$$

in which $R_{\max}=\max_{0\le x\le x_c}\{|R(x)|\}$ which we suppose is *bounded.* The expression (3.35) for g_μ can now be substituted directly into the original scaling form (3.19) for p. One obtains (3.24) and (3.25) but (3.26) is replaced by $\tilde{\pi}_0=\pi_0$ and $\tilde{\phi}=\phi$. In addition, via (3.30)–(3.32), one finds

$$\tilde{B}=Bc_3^{\phi} ,$$

$$\tilde{b}_1=b_1+(b_3c_1/c_3) ,$$

$$\tilde{b}_2=b_2+(b_3c_2/c_3) ,$$

$$\tilde{p}_1=p_1+(p_3c_1/c_3) ,$$

$$\tilde{p}_2=p_2+(p_3c_2/c_3) ,$$

and, finally, sees that singular corrections of the form

$$\tilde{t}\mathscr{E}_>(t_0,v_0)\sim|\tilde{t}|^{\psi_0}\sim|g_\beta|^{\psi_0}$$

enter, in higher order, into g_β, $\tilde{g}_v$, and the constrained background in (3.24).

In the *second case,* $\psi_0<1$, the first term on the left-hand side of (3.24) is subdominant and one then finds

$$|g_\mu|=|\tilde{t}/A_cR(x)|^{1/\psi_0}[1+\mathscr{E}_<(t_0,v_0)] . \quad (3.37)$$

where, as $t_0,v_0\to 0$, the error term is bounded by

$$|\mathscr{E}_<(t_0,v_0)|\le|c_3/\psi_0|\,|A_cR_{\min}|^{-1/\psi_0}|\tilde{t}|^{(1/\psi_0)-1} . \quad (3.38)$$

Here $R_{\min}=\min_{0\le x\le x_c}\{|R(x)|\}$ is assumed to be *strictly positive;* in the event that $R_{\min}$ vanishes, further analysis is required: see below. Note that $\mathscr{E}_<$ diverges when $A_c\to 0$ since the system then experiences a crossover back to unrenormalized behavior as described by (3.35).

One can now use (3.37), first in (3.30) for r_c-r_0 and then with (3.31) and (3.33) in (3.28) to obtain the scaling variable y. On substituting with $x=\Xi(y)$ from (3.28) in (3.19), one finds the scaling argument in (3.24) with $\tilde{B}=\bar{B}=B|A|^{\bar{\phi}}$, $\tilde{b}_1=b_1$, and $\tilde{b}_2=b_2$. Finally, using (3.37) for g_μ in (3.32) and in the prefactor in (3.19) completes the derivation of (3.24) and yields $\tilde{p}_1=p_1$, $\tilde{p}_2=p_2$, and the scaling function

$$\tilde{P}(y)=|A_c|^{-\tilde{\pi}_0}|y/\Xi(y)|^{\pi_0/\phi}P(\Xi(y)) . \quad (3.39)$$

Note that this expression diverges when $A_c\to 0$, another indication of crossover back to unrenormalized behavior. In addition, we see that the g_μ terms in (3.30)–(3.33) introduce singular corrections proportional to $\tilde{t}\mathscr{E}_<(t_0,v_0)\sim|\tilde{t}|^{1/\psi_0}$ in the fields $g_\beta,\tilde{g}_v$ and in the background in (3.24), as well as in the amplitude in (3.39).

It is of some interest to know the nature of the function $\Xi(y)$ which solves (3.28). For small x we may suppose the scaling function for r varies as

$$R(x)=x^{-\Gamma}R_0(1+r_1x+r_2x^2+\cdots) . \quad (3.40)$$

Normally one would have $\Gamma\equiv 0$ but we will allow for

$\Gamma \geq 0$ since in the dilute Bose gas beneath T_c one finds a density scaling function with $\Gamma=1$: see Sec. VI. Solving (3.28) for small y then yields

$$\Xi(y)=\Xi_1 y^{\Delta}+\Xi_2 y^{2\Delta}+\cdots \quad (3.41)$$

with an exponent $\Delta=1/(1+\Gamma\tilde{\phi})$, which takes the value $1-\frac{1}{2}\epsilon$ for the dilute Bose gas; in addition one has $\Xi_1=R_0^{\tilde{\phi}\Delta}$, $\Xi_2=\tilde{\phi}\Delta r_1 R_0^{2\tilde{\phi}\Delta}$, etc..

As regards the behavior for larger y, we find from (3.22) the critical value $y_c=|R_c|^{-\phi}$ which is independent of x_c but becomes infinite if $R_c=0$. *For bounded x_c and $\psi>1$* we then obtain

$$\Xi(y)/x_c \approx 1-\tilde{\Xi}_1\dot{y}+\Xi_\psi \dot{y}^{\psi}, \text{ with } \dot{y}=1-(y/y_c), \quad (3.42)$$

where $\tilde{\Xi}_1$ and Ξ_ψ depend also on the coefficient, say R_1, of the term linear in $1-(x/x_c)$ not shown in (3.22). The leading linear variation of $\Xi(y)$ near y_c means that the singularity of p *on* the nonideal but constrainted critical line is described by an *unrenormalized* exponent, say π.[28] Conversely, *for $\psi<1$ and bounded x_c* we find instead

$$\Xi(y)/x_c \approx 1-\Xi_0 \dot{y}^{\,1/\psi}+O(\dot{y}^{\,(2/\psi)-1}), \quad (3.43)$$

with $\Xi_0=(R_c/\tilde{\phi}R_\psi)^{1/\psi}x_c^{1/\phi}$. It follows now that p displays a *renormalized* singularity for $\tilde{g}_v, v_0>0$ with exponent $\tilde{\pi}=\pi/\psi$, independent of the value of ψ_0.[28]

As mentioned, *the case $x_c=\infty$* is actually of interest in connection with the $O(\epsilon)$ calculations of Sec. VI. *Provided $R_c\neq 0$*, the first term in the scaling function $R(x)$ in (3.22) dominates as $x\to\infty$ and we find

$$\Xi(y)\approx(|\tilde{\phi}R_\psi|/R_c\dot{y})^{\phi/\psi} \text{ as } \dot{y}\to 0+ \,. \quad (3.44)$$

On the other hand, if R_c *vanishes* one obtains, first, $y_c=\infty$ and, then,

$$\Xi(y)\approx|R_\psi|^{\phi/\psi}y^{\psi_0/\psi} \text{ as } y\to\infty \,. \quad (3.45)$$

In fact, for the dilute Bose crossover in order ϵ we find, in Sec. VI, that $R_c=0$ so this case is relevant. It also transpires, however, that $\lambda=(\psi_0-\psi)/\phi$ is negative in $O(\epsilon)$ so that the scaling function decays to zero as

$$R(x)\approx R_\psi x^{-|\lambda|} \text{ when } x\to x_c=\infty \,. \quad (3.46)$$

As a result, the assumption $R_{\min}>0$, used in deriving (3.37) and (3.38), is violated! To complete the analysis as far as our present purposes go, therefore, we will reexamine the constraint equation (3.34) for this special situation.

As a first step let us accept the solution (3.37) and drop the error term $\mathscr{E}_<$. On using (3.46) for large x and solving for g_μ in this limit, we find

$$|g_\mu|=|(|Bg_v|^{|\lambda|}/A_c R_\psi)\tilde{t}|^{1/\psi}, \quad (3.47)$$

where we have used $|\lambda|\phi+\psi_0=\psi$ so that ψ_0 effectively drops out and is replaced by ψ. We will *suppose* $\psi<1$, as is appropriate in $O(\epsilon)$ for $n<4$, which will fulfill our needs: see Sec. VI. Then g_μ still vanishes faster than $\tilde{t}$ and, further, has a factor vanishing as $g_v^{|\lambda|}$. Thus (r_c-r_0) in (3.30) is still given, up to higher-order corrections, by g_β in (3.25). The correction term in (3.37) is, in leading order, just $c_3|g_\mu|/\psi_0\tilde{t}$ and so as $t_0,v_0\to 0$ it is bounded for, say, $x\lesssim 1$ by (3.38) with $R_{\min}$ replaced by $|R_\psi|$ or, for $x\gtrsim 1$, where we suppose (3.46) applies, by

$$|\mathscr{E}_<(t_0,v_0)|\leq|c_3/\psi_0|\,|(|Bg_v|^{|\lambda|}/A_c R_\psi)|^{1/\psi} \times\tilde{t}^{(1/\psi)-1}\,. \quad (3.48)$$

Thus, provided $\psi<1$, our conclusions remain essentially unchanged.

If $\psi>1$ [and all lower order terms in (3.22) still vanish] the bound (3.38) applies on any scaling trajectory at fixed y provided $R_{\min}$ is replaced by $|R(\Xi(y))|$. However, the amplitude of the corrections diverges as $y\to\infty$. This means that there is a small region close to the nonideal critical line, given by $t\lesssim v_0^{(\psi-\psi_0)/\phi(\psi-1)}$, in which deviations from the constrained scaling form (3.24) arise. In the present case, the exponent of v_0 here is $1/|\alpha|+2/\epsilon$; this is very large ($\gtrsim 20$) for $d=3$ so that (even if x_c remains infinite and R_c, etc., still vanishes at $d=3$) the region of breakdown becomes extremely small when $v_0\to 0$.

A few further remarks are in order. It was noted that the amplitude A in (3.21) might depend on β. If this dependence is strong, as transpires for a dilute Bose gas, it is preferable numerically to absorb the factor $|A|^{\tilde{\phi}}$ into the scaling field $\tilde{g}_v$ in (3.24) to leave the amplitude B in place of $\tilde{B}$. It must also be recognized that there may be deviations from the scaling form (3.21) for r which go beyond changes in the form of A and r_0. This can be seen, for example, by postulating a perfect scaling form for the free energy $f(\beta,\mu,v_0)$ in terms of nonlinear scaling fields g_μ and g_v and supposing $r=\rho=-(\partial f/\partial\mu)$. The expression for ρ then contains a term proportional to $g_{v,\mu}|g_\mu|^{2-\alpha-\phi}F_1(x)$ where $g_{v,\mu}=(\partial g_v/\partial\mu)$ while $F_1(x)$ derives from the scaling function for f and $x=g_v/|g_\mu|^{\phi}$. Unless $g_{v,\mu}\equiv 0$ this term cannot be represented in the expected scaling form $|g_\mu|^{1-\alpha}R(x)$. However, by (3.10) we have $g_{v,\mu}\approx b_1 v_0\approx b_1 g_v=b_1 x|g_\mu|^{\phi}$ and so it can be written $b_1|g_\mu|^{1-\alpha}xF_1(x)|g_\mu|$. In this version it can be recognized as correcting the scaling function $R(x)$ to $[R(x)+b_1|g_\mu|xF(x)]$. Then it can clearly be neglected when $g_\mu\to 0$. Indeed, it gives rise only to corrections of higher order than those normally appearing in (3.37). Finally, *irrelevant variables* have been neglected in the scaling of p and r. It is clear, however, that they may be included with little effort, at least in leading order, and do not affect the validity of the constrained scaling result (3.24).

IV. QUANTAL PERTURBATION THEORY

In this section we consider quantum-mechanical perturbation theory about the ideal-Bose-gas limit and use the results, in conjunction with the scaling analysis of Sec. III, to obtain, for the unconstrained ideal Bose fluid, the crossover exponent ϕ and the corresponding linear scaling fields. Thence we will derive the crossover exponent ϕ_T entering the phenomenological scaling form (1.4) for the superfluid density.

A. Zeroth order

As mentioned in Sec. III, the most convenient and basic property to examine is the off-diagonal susceptibility,

$\chi(\beta,\mu;v_0)$, *above* the superfluid transition. The finite-temperature Green's function formalism[30] allows one to calculate thermodynamic functions perturbatively in the interaction $v(\mathbf{r})$: see (2.1)—(2.5). In terms of the Matsubara Green's function,[30] $\mathscr{G}(\mathbf{p},ip_n)$, one has

$$\chi(\beta,\mu;v_0)=-(2\beta)^{-1}\mathscr{G}(0,0) \ . \tag{4.1}$$

As usual $\mathbf{p}$ is the momentum or wave number and the $p_n=2\pi n/\beta$, with $n=0,\pm 1,\pm 2,\ldots$, are the Matsubara frequencies. For the ideal Bose gas [indicated by a superscript (0)] one has

$$\mathscr{G}^{(0)}(\mathbf{p},ip_n)=1/[ip_n-\varepsilon_{\mathbf{p}}+\mu]$$

whence

$$\chi^{(0)}(\beta,\mu)=\tfrac{1}{2}(-\beta\mu)^{-1} \ , \tag{4.2}$$

which, of course, also follows directly from the exact solution (2.18). Comparison with the scaling result (3.12) yields the identifications

$$C=\tfrac{1}{2}, \ \ g_\mu^0\equiv g_\mu(v_0=0)=-\beta\mu, \ \ \gamma_0=1 \ , \tag{4.3}$$

and $\chi_0(\mu,0)=0$. Here, and below, $g_\mu^0=-\beta\mu$ must be kept *positive* to remain above the transition.

B. First order: Principle results

Corrections to (4.2) may be expressed, as usual, in terms of Feynman diagrams.[30] In first order, only the two single-loop diagrams shown in Fig. 1 are needed for the susceptibility. On using the definition (2.11) and the convention (2.19), the resulting term is

$$\chi^{(1)}(\beta,\mu;v)=\frac{-1}{2(-\beta\mu)^2}\int_{\mathbf{k}}\beta(v_0+v_{\mathbf{k}})n_B(\varepsilon_{\mathbf{k}}-\mu) \ . \tag{4.4}$$

Note that we now use a superscript (n) to denote the *total* nth-order contribution to χ; previously in (3.11)—(3.14) we used a subscript and wrote the term as $\chi_{(n)}v_0^n/n!$. In Sec. IV C we will discuss the evaluation of the integral here in a systematic way which reveals the nature of the dependence on the *shape* of the potential, $v(\mathbf{r})$, as expressed via $v_{\mathbf{k}}$. Let us first, however, demonstrate how the main results needed follow very simply if we suppose, as is, in fact, fully justifiable, that only the behavior of the integrand at low momenta matters as regards the singular dependence on μ.

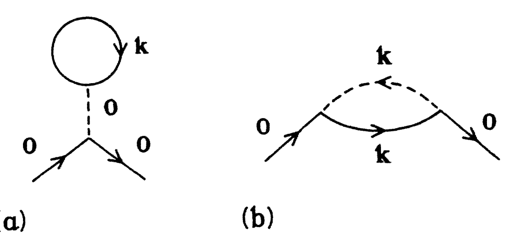

FIG. 1. First-order diagrams in the graphical expansion for the off-diagonal susceptibility χ. Dashed lines carry a factor $v_{\mathbf{k}}$, solid lines a factor $\mathscr{G}^0(\mathbf{k},ik_n)$; momentum $\mathbf{k}$, and frequency k_n, are conserved at each vertex. The two external lines carry zero momentum and frequency.

Thus let us separate the μ-independent part by introducing

$$\Delta n_B(\varepsilon-\mu)\equiv n_B(\varepsilon)-n_B(\varepsilon-\mu) = \frac{(-\beta\mu)[1+O(\beta\mu,\beta\varepsilon)]}{\beta\varepsilon(\beta\varepsilon-\beta\mu)} \ , \tag{4.5}$$

where the definition (2.21) has been used for small $\beta\mu$ and $\beta\varepsilon$. Then we can write

$$\chi^{(1)}=\frac{D_{0,0}v_0}{(-\beta\mu)^2}+\frac{1}{2(-\beta\mu)^2}\int_{\mathbf{k}}\beta(v_0+v_{\mathbf{k}})\Delta n_B(\varepsilon_{\mathbf{k}}-\mu) \ , \tag{4.6}$$

where

$$D_{0,0}=-\tfrac{1}{2}\int_{\mathbf{k}}\beta(1+\varphi_{\mathbf{k}})n_B(\varepsilon_{\mathbf{k}}) \ , \tag{4.7}$$

in which

$$\varphi_{\mathbf{k}}=v_{\mathbf{k}}/v_0 \tag{4.8}$$

measures the "shape" of the potential. The dispersion relation (2.16) yields

$$\beta\varepsilon_{\mathbf{k}}\approx\Lambda_T^2k^2/4\pi \ \text{ as } k\to 0 \ , \tag{4.9}$$

where the *thermal de Broglie wavelength* is

$$\Lambda_T=h/(2\pi m^*k_BT)^{1/2}=(4\pi\beta\hbar^2/2m^*)^{1/2} \ . \tag{4.10}$$

Focusing now only on small k, the behavior of the second term in (4.6) is thus given by

$$\Delta\chi^{(1)}\approx\frac{\beta v_0}{g_\mu^0}\int_{\mathbf{k}}\frac{16\pi^2}{\Lambda_T^2k^2(\Lambda_T^2k^2+4\pi g_\mu^0)} \ . \tag{4.11}$$

This integral is well known: an upper cutoff $|\mathbf{k}|\le q_\Lambda$, should be understood. Putting $w=\Lambda_Tk/(4\pi g_\mu^0)^{1/2}$ and

$$K_d=C_d/(2\pi)^d=2/(4\pi)^{d/2}\Gamma(\tfrac{1}{2}d) \ , \tag{4.12}$$

where C_d is the surface area of a unit sphere in d dimensions, leads to

$$\Delta\chi^{(1)}\approx\left[\frac{\beta v_0}{\Lambda_T^d}\right]\frac{(4\pi)^{d/2}K_d}{(g_\mu^0)^{1+\epsilon/2}}\int_0\frac{w^{d-3}}{1+w^2}dw \ , \tag{4.13}$$

in which

$$\epsilon=4-d \ . \tag{4.14}$$

The upper cutoff in (4.13), namely $q_\Lambda\Lambda_T/(4\pi g_\mu^0)^{1/2}$, diverges as $g_\mu^0\to 0$ but for $d<4$, *which will be assumed henceforth*, the integral remains bounded. This is the route by which the upper borderline dimensionality, $d_>=4$, enters our analysis. Finally, on performing the integral on w, we obtain the first-order correction to χ in leading orders

$$\chi^{(1)}\approx\frac{D_{0,0}v_0}{(g_\mu^0)^2}-\frac{\Gamma(1-\frac{1}{2}d)}{(g_\mu^0)^{1+\epsilon/2}}\left[\frac{\beta v_0}{\Lambda_T^d}\right] \ . \tag{4.15}$$

We may now compare with the scaling expectation (3.13). Since $\gamma_0=1$, by (4.3), the first term is just as anticipated. The amplitude $D_{0,0}$ is given by (4.7) and, via

(3.15), this yields the *leading analytic shift* in critical temperature due to the interactions. Specifically, the scaling field is

$$g_\mu(\beta,\mu;v_0)=-\beta\mu-2D_{0,0}v_0+O(\beta^2\mu^2,\beta\mu v_0,v_0^2)\ . \qquad (4.16)$$

Notice that $D_{0,0}$ carries a factor β and, essentially, $1/\Lambda_T^d$ which arises from the Bose factor which decays as $\exp(-k^2\Lambda_T^2/4\pi)$ for large k. A more explicit expression for $D_{0,0}$ will be obtained below.

The second term in (4.15) corresponds with scaling expectations if we make the identification

$$\phi=\tfrac{1}{2}\epsilon=\tfrac{1}{2}(4-d)\quad (d<4)\ . \qquad (4.17)$$

Thus we have found the crossover exponent quite generally. As regards the scaling formulation in the Introduction, however, we must recall first that that implicitly entailed the *constraint* of constant overall density whereas here we may consider varying T at *constant chemical potential* μ. As explained in Sec. III B, especially near (3.26), this leads to a *renormalization*[28] of the crossover exponent ϕ according to

$$\phi\Longrightarrow\tilde{\phi}=\frac{\phi}{1-\alpha_0}=\frac{2\phi}{d-2}=\frac{4-d}{d-2}\ , \qquad (4.18)$$

where α_0 is the specific-heat exponent of the ideal Bose gas at constant μ. The value of α_0 used here follows from (2.18) as[14] $\alpha_0=\frac{1}{2}\epsilon$. The renormalized scaling field g_β which replaces $g_\mu\approx-\beta\mu$ becomes proportional to $t_{(0)}=(T-T_c^0)/T_c^0$ where $T_c^0(\rho)$ is the critical temperature of the ideal Bose gas at constant density. However, as regards crossover scaling, it follows from general arguments (see Sec. III and Ref. 29) that we may, asymptotically, equally well use the standard reduced variable $t=(T-T_c)/T_c$, in which T_c is the actual critical temperature.

To compare with the Introduction, however, we also need to remember that T_c was there used as the control parameter in place of v_0 (which is essentially fixed in the experiments). How can T_c be brought into play? The answer lies in examining the *amplitude* of the second term in $\chi^{(1)}$ and comparing with (3.11) and (3.15) to obtain the identifications

$$D_1v_0=BCg_v'v_0=-\Gamma(1-\tfrac{1}{2}d)(\beta v_0/\Lambda_T^d)\ .$$

There is now some freedom but, clearly, the most sensible assignment is to take the amplitude B in the basic scaling ansatz (3.8) as the pure number

$$B=-2\Gamma(1-\tfrac{1}{2}d)\ . \qquad (4.19)$$

Then the *full interaction scaling field* is identified as

$$g_v(\beta,\mu;v_0)=(\beta v_0/\Lambda_T^d)[1+O(\beta\mu,v_0)]\ . \qquad (4.20)$$

The dimensionless combination of interaction strength, temperature, and de Broglie wavelength appearing here is most natural and, indeed, might well have been guessed *a priori* as the appropriate dimensionless "coupling constant." Note that by (2.17) it can be written in the appealing form[6]

$$\beta v_0/\Lambda_T^d\equiv k_d(a/\Lambda_T)^{d-2}\ , \qquad (4.21)$$

where a is the scattering length and

$$k_d=2\pi^{(d-2)/2}/\Gamma(\tfrac{1}{2}d-1)$$

is a purely numerical constant. The significance of the lower critical dimensionality, $d_<=2$, at which the ideal transition ceases to exist, emerges here.

We see now that the basic scaled combination entering in the dilute-Bose-gas crossover at *constant density* must be

$$y\propto\frac{g_v}{(g_\beta)^{\tilde{\phi}}}\propto\frac{(a/\Lambda_T)^{d-2}}{|t|^{(4-d)/(d-2)}}\propto\left[\frac{m^*a^2k_BT_c}{\hbar^2|t|^{\phi_T}}\right]^{(d-2)/2}\ , \qquad (4.22)$$

where one finds

$$\phi_T=2\tilde{\phi}/(d-2)=2(4-d)/(d-2)^2\ . \qquad (4.23)$$

It is evident from (4.22) that ϕ_T is the crossover exponent when T_c is regarded as a variable while m^* and a are fixed (or slowly varying). Our result for ϕ_T thus confirms (1.5) and completes the derivation of the first step beyond the phenomenological scaling theory developed in the Introduction.

The next central task is the calculation of the *scaling functions* for the susceptibility, superfluid density, etc. This is taken up in Sec. V to which the reader may wish to proceed directly. However, it is worthwhile to investigate more closely the role of the detailed form of the potential $v(\mathbf{r})$ in first-order perturbation theory and also to check that consistency with the scaling ansatz (3.8) extends to second order. These issues are addressed in the balance of this section.

C. Influence of the potential shape

In order to detect the effects of the changing details of the interaction potential and to verify that the dominant singularities of the first-order perturbation term, $\chi^{(1)}$, have been properly obtained, a more systematic method of calculation is needed. Let us, more concretely, suppose the Fourier transformed potential may be written

$$v_{\mathbf{k}}=v_0\varphi(a_0^2k^2)\ . \qquad (4.24)$$

Then φ specifies the shape of the potential while a_0 measures its range. With reasonable generality we may suppose that φ has the representation

$$\varphi(x)=\sum_{i=0}x^{\sigma_i}\varphi_i(x)\quad\text{with } \sigma_0=0\ , \qquad (4.25)$$

where the σ_i for $i>0$ are *not* positive integers while the form factors $\varphi_i(x)$ are analytic at $x=0$ so that

$$\varphi_i(x)=\sum_{l=0}^{\infty}\varphi_{i,l}x^l,\quad \varphi_{i,0}\neq0\ . \qquad (4.26)$$

In the simplest, short-range case only the term $i=0$ is required and $\varphi_0(0)=1$. Long-range pieces of $v(\mathbf{r})$ decaying as $1/|\mathbf{r}|^{d+\sigma_i}$ yield contributions with $\sigma_i\neq0$.

Now to calculate $\chi^{(1)}$ we evidently need integrals of the form

$$I_{n+\sigma}(\beta,\mu)=\Lambda_T^d \int_{\mathbf{k}} (\Lambda_T k)^{2(n+\sigma)} n_B(\varepsilon_{\mathbf{k}}-\mu) . \tag{4.27}$$

These are most readily calculated as nth derivatives with respect to λ of the generating function

$$G_{\lambda,\sigma}(\beta,\mu)=\Lambda_T^d \int_{\mathbf{k}} (\Lambda_T k)^{\sigma} \exp(-\lambda\Lambda_T^2 k^2/4\pi) n_B(\varepsilon_{\mathbf{k}}-\mu) . \tag{4.28}$$

To evaluate this we may expand $n_B(\varepsilon)$ in powers of $e^{-\beta\varepsilon}$ and, for simplicitly, suppose that the *dispersion relation* (4.9) holds *without corrections.* The resulting Gaussian integrals are readily performed and yield

$$G_{\lambda,\sigma}(\beta,\mu)=[(4\pi)^{\sigma}\Gamma(\tfrac{1}{2}d+\sigma)/\Gamma(\tfrac{1}{2}d)] \times e^{\beta\mu} f_{(d/2)+\sigma,1+\lambda}(e^{\beta\mu}) , \tag{4.29}$$

where the extended generalized zeta function[31] is defined by

$$f_{s,u}(z)=\sum_{n=0}^{\infty} z^n/(n+u)^s . \tag{4.30}$$

Then the required integrals are

$$I_{n+\sigma}(\beta,\mu)=[(4\pi)^{n+\sigma}\Gamma(\tfrac{1}{2}d+n+\sigma)/\Gamma(\tfrac{1}{2}d)] \times e^{\beta\mu} f_{(d/2)+n+\sigma,1}(e^{\beta\mu}) . \tag{4.31}$$

Our primary interest is in the singular behavior which arises when $-\beta\mu\to 0+$ or $z\to 1-$. This follows from[31]

$$f_{s,u}(e^{-t})=e^{ut}\Big[\Gamma(1-s)t^{s-1}+\sum_{n=0}^{\infty}\zeta(s-n,u)(-t)^n/n!\Big] , \tag{4.32}$$

valid for $s\neq 1,2,3,\ldots,\ u\neq 0,-1,-2,\ldots$. Here $\zeta(s,u)$ is the generalized zeta function.[31] When s is a positive integer, say $s=m+1$, the mth term in the sum combines with the previously singular term $\Gamma(1-s)t^{s-1}$ to generate the new singularity

$$[\ln t+\psi(m+1)-\psi(u)](-t)^m/m! , \tag{4.33}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function.[31,32] Thus logarithmic singularities appear whenever $s=\frac{1}{2}d+n+\sigma$ is an integer; in particular, for the short-range case $\sigma=0$, they arise for $d=2$ and 4. However, we will not normally explicitly indicate the presence of logarithmic factors.

If we use these results to evaluate the first-order integral in (4.4) the *singular parts* of $(-\beta\mu)^2\Delta\chi^{(1)}$ are found to be

$$\tfrac{1}{2}\Gamma(1-\tfrac{1}{2}d)\left[\frac{\beta v_0}{\Lambda_T^d}\right](-\beta\mu)^{(d-2)/2}\left[1+\sum_{i=0} y_\varphi^{\sigma_i}\sum_{l=0}^{\infty}\varphi_{i,l}\gamma_{i,l}y_\varphi^l\right] \tag{4.34}$$

in which the coefficients

$$\gamma_{i,l}=\Gamma(\tfrac{1}{2}d+\sigma_i+l)\Gamma(1-\tfrac{1}{2}d-\sigma_i-l)/\Gamma(\tfrac{1}{2}d)\Gamma(1-\tfrac{1}{2}d)$$

reduce to

$$\gamma_{i,l}=(-1)^l\gamma_i\equiv(-1)^l\Gamma(\tfrac{1}{2}d+\sigma_i) \times\Gamma(1-\tfrac{1}{2}d-\sigma_i)/\Gamma(\tfrac{1}{2}d)\Gamma(1-\tfrac{1}{2}d) . \tag{4.35}$$

The new variable, y_φ, appearing in (4.34) can be written

$$y_\varphi(\beta,\mu;a_0)=4\pi(a_0/\Lambda_T)^2(-\beta\mu)=[a_0/\xi^0(\beta,\mu)]^2 , \tag{4.36}$$

where, ξ^0, the correlation length of the ideal Bose gas, is given by[14]

$$\xi^0(\beta,\mu)=(\Lambda_T/\sqrt{4\pi})/(-\beta\mu)^{1/2} . \tag{4.37}$$

This expression, embodying the exponent $\nu_0=\frac{1}{2}$, is easily derived[14] and can also be read off from the denominator in (4.11). Evidently y_φ represents the properly *scaled range* of the potential $v(\mathbf{r})$ which could have been anticipated as one of the correction-to-scaling contributions in (3.8).

One can clearly resum the series in (4.34) to obtain the surprisingly explicit result

$$(-\beta\mu)^2\Delta\chi^{(1)}_{\rm sing}=\tfrac{1}{2}\Gamma(1-\tfrac{1}{2}d)\left[\frac{\beta v_0}{\Lambda_T^d}\right](-\beta\mu)^{(d-2)/2} \times\left[1+\varphi_0(-y_\varphi)+\sum_{i>0}\gamma_i y_\varphi^{\sigma_i}\varphi_i(-y_\varphi)\right] . \tag{4.38}$$

Now suppose that the potential $v(\mathbf{r})$ is sufficiently short range that the integral $\int v(r)d^dr=v_0$ is bounded or, equivalently, that $\sigma_i>0$ for $i>0$. Then one has $\varphi_0(0)\equiv\varphi(0)=1$ and, since $y_\varphi\to 0$ as $g_\mu^0=-\beta\mu\to 0$, one sees that the leading singularity in (4.38) agrees precisely with the previous result in (4.15). Conversely, if the forces are of such *long range* that $\sigma_-=\min_i\sigma_i$ is *negative,* the leading singularity *changes* and one obtains, instead of (4.17), the new crossover exponent

$$\phi_-=\tfrac{1}{2}(4-d)-\sigma_- . \tag{4.39}$$

The fact that long-range forces can change the nature of the crossover behavior is not surprising; note, however, that since $v(\mathbf{r})$ enters the Hamiltonian in the $|\psi|^4$ term the situation here differs from that usually considered in critical phenomena,[33] where the long-range interactions change the $|\nabla\psi|^2$ term. We will not discuss such long-range forces further.

Returning to the case of sufficiently short-range interactions with $\sigma_i > 0$ for $i > 0$, we see that $\Delta\chi^{(1)}$ contains an infinite series of higher-order singularities determined by the powers of $-\beta\mu a_0^2/\Lambda_T^2$. Since a prefactor $(\beta v_0/\Lambda_T^d)$ is always present these terms correspond to singularities of the form $(g_\mu^0)^{-\gamma_0-\phi-\phi_i}$ in the perturbation expansions (3.11)–(3.14) of the scaling ansatz. For each i, therefore, one finds a distinct scaling exponent

$$\phi_0 = -1 \equiv -2\nu_0, \quad \phi_i = -\sigma_i \equiv -2\sigma_i \nu_0 \quad (i > 0) , \tag{4.40}$$

associated with the leading potential moments $\varphi_{0,1}$ and $\varphi_{i,0}$ $(i>0)$, respectively. As expected, these crossover exponents are all negative and so the detailed shape of the potential is *irrelevant*. If the higher moments, $\varphi_{0,l+1}$ and $\varphi_{i,l}$ $(i>0)$ are also regarded as independent variables they would enter scaling with the still more negative exponents $\phi_{0,l+1} = -(l+1)$ and $\phi_{i,l} = -\sigma_i - l$. However, since the $\phi_{0,l+1}$ are integers in the present case it must be recognized that the corresponding terms in $\chi^{(1)}$ will appear as if analytic and will therefore be mixed with contributions arising from nonlinear terms in the full scaling fields $g_\mu(\beta,\mu;v)$ and $g_v(\beta,\mu;v)$. By formally taking $\sigma_0 \neq 0$ and then letting $\sigma_0 \to 0$ one can, however, distinguish the various sources of such terms.

Finally, we can use the analytic pieces of the $f_{s,u}(z)$ in (4.32) to study the leading divergence of $\chi^{(1)}$ associated with the amplitude

$$D_0 = D_{0,0} + D_{0,1}(-\beta\mu) + O(\beta^2\mu^2) . \tag{4.41}$$

For simplicity we consider only the fully short-range case with $\varphi \equiv \varphi_0$. Then we can write

$$D_{0,j} v_0 = -(\beta v_0/\Lambda_T^d)\zeta(\tfrac{1}{2}d)[1 + \overline{\varphi}_j(4\pi a_0^2/\Lambda_T^2)] , \tag{4.42}$$

for $j = 0, 1$, where the modified form factors are defined by

$$\overline{\varphi}_j(x) = \tfrac{1}{2} \sum_{l=1}^{\infty} \varphi_{0,l} \overline{\gamma}_{j,l} x^l \quad (j = 0, 1) , \tag{4.43}$$

with coefficients

$$\overline{\gamma}_{j,l} = \zeta(\tfrac{1}{2}d + l - j)\Gamma(\tfrac{1}{2}d + l)/\Gamma(\tfrac{1}{2}d)\zeta(\tfrac{1}{2}d - j) . \tag{4.44}$$

Note, therefore, that the $\overline{\varphi}_j$ terms in (4.42) vanish linearly when $T_c \to 0$. The scaling field for μ can then be given to higher order as

$$g_\mu(\beta,\mu;v) = -\beta\mu - 2D_{0,0}v_0 - 2D_{0,1}(-\beta\mu)v_0 + O(\beta^2\mu^2 v_0, v_0^2) . \tag{4.45}$$

Notice that since $\gamma_0 = 1$ is an integer it is impossible at this stage to decide whether the further powers of $\beta\mu$ arising from the analytic parts of $\chi^{(1)}$ contribute to g_μ or to the background term $\chi_0'(\mu,0)$ in (3.13). In order to obtain further terms in the scaling fields and in the backgrounds, χ_0, it is necessary to carry perturbation theory to higher order. More importantly, this also serves as a check on scaling; e.g., by (3.14) and (3.17) the second-order amplitude E_0 should be given by quantities already found in first order.

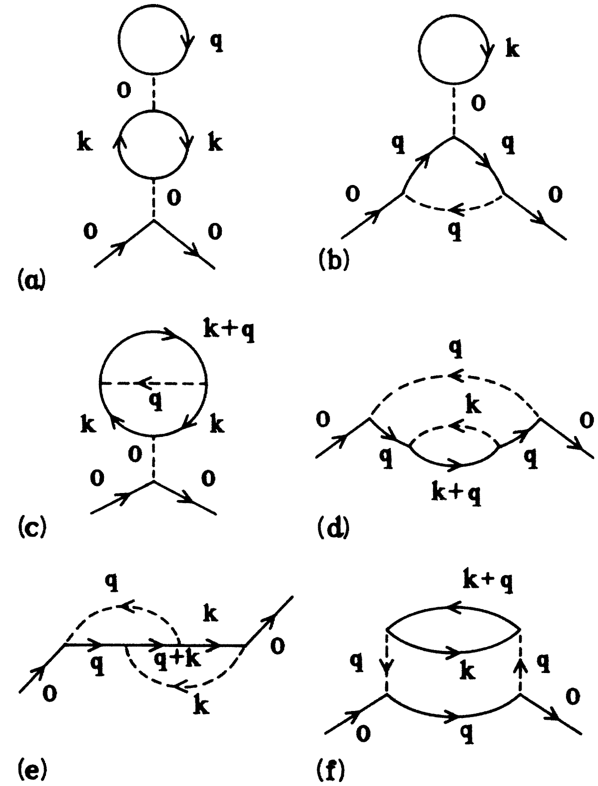

FIG. 2. Irreducible second-order diagrams contributing to the off-diagonal susceptibility χ.

D. Second-order calculations

The Feynman diagrams relevant to the calculation of $\chi^{(2)}$, the second-order correction to the susceptibility, are shown in Fig. 2. If we write the first-order correction term as

$$\chi^{(1)}(\beta,\mu;v) = -(1/2\beta)\mu^{-2}\Sigma^{(1)}(\beta,\mu,v) , \tag{4.46}$$

the total second-order term may be written

$$\chi^{(2)} = -(1/2\beta)[\mu^{-3}(\Sigma^{(1)})^2 + \mu^{-2}(\Sigma^{(2)})] . \tag{4.47}$$

Suppose, first, that $\Sigma^{(2)}(\beta,\mu,v)$ diverges less rapidly than μ^{-1} when $\mu \to 0$: in fact, this turns out to be so. It follows from (4.6) and its extension, (4.41), that the leading singularity is

$$\chi^{(2)} \approx \tfrac{1}{2}E_0 v_0^2/(-\beta\mu)^3 \text{ with } E_0 = 4D_0^2 . \tag{4.48}$$

Since $\gamma_0 = 1$ and $\phi < 1$ for $d > 2$ this is just the type of divergence expected by scaling: see (3.14). Furthermore, the *amplitudes* E_0 and D_0 are related just as scaling predicts: see (3.15) and (3.17) and use $C=\frac{1}{2}$ from (4.3). This is our first *check* on the validity of the scaling properties of the perturbation expansion.

To proceed further we need the two contributions to $\Sigma^{(2)}$, namely,

$$\Sigma_a^{(2)}=\int_{\mathbf{k}}\int_{\mathbf{q}}(v_0+v_{\mathbf{k}})n_B'(\varepsilon_{\mathbf{k}}-\mu)(v_0+v_{\mathbf{k}-\mathbf{q}})n_B(\varepsilon_{\mathbf{q}}-\mu)\ , \tag{4.49}$$

which arises from the diagrams (a)–(d) in Fig. 2, which have the topology of a figure eight, and

$$\Sigma_e^{(2)}=\int_{\mathbf{k}}\int_{\mathbf{q}}v_{\mathbf{q}}(v_{\mathbf{k}}+v_{\mathbf{q}})\frac{[n_B(\varepsilon_{\mathbf{k}+\mathbf{q}}-\mu)-n_B(\varepsilon_{\mathbf{k}}-\mu)][n_B(\varepsilon_{\mathbf{q}}-\mu)-n_B(\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_{\mathbf{k}})]}{\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_{\mathbf{q}}-\varepsilon_{\mathbf{k}}+\mu}\ , \tag{4.50}$$

which comes from diagrams (e) and (f) in Fig. 2, which have the topology of a theta. Note that $n_B'(\varepsilon)=dn_B/d\varepsilon$.

1. Figure-eight contribution

The evaluation of $\Sigma_a^{(2)}$ requires little more beyond the techniques used in Sec. IV C. For simplicity we restrict attention to the case where $\varphi(x)\equiv\varphi_0(x)$ in (4.24) and (4.25), i.e., short-range interactions. Then the necessary integrals can be written

$$J_{l,m}'(\mu)=\frac{\Lambda_T^{2(d+l+m)}}{\beta(-4\pi)^{l+m}}\int_{\mathbf{k}}\int_{\mathbf{q}}|\mathbf{k}-\mathbf{q}|^{2l}|\mathbf{k}|^{2m}n_B'(\varepsilon_{\mathbf{k}}-\mu)n_B(\varepsilon_{\mathbf{q}}-\mu)$$

$$=\left[-\frac{d}{d\beta\nu}J_{l,m}(\mu,\nu)\right]_{\mu=\nu}. \tag{4.51}$$

The auxiliary integrals, $J_{l,m}$ follow from a generating function analogous to (4.28) as

$$J_{l,m}(\mu,\nu)=\left[\left[\frac{d}{d\kappa}\right]^l\left[\frac{d}{d\lambda}\right]^m\sum_{r,s=1}^{\infty}e^{r\beta\mu+s\beta\nu}/[(r+\kappa)(s+\lambda)+\kappa r]^{d/2}\right]_{\kappa=\lambda=0}. \tag{4.52}$$

Thence the $J_{l,m}'(\mu)$ can be found in terms of the Bose functions

$$f_s(z)=zf_{s,1}(z)=\sum_{n=1}^{\infty}z^n/n^s\ , \tag{4.53}$$

whose singularities as $z=e^{\beta\mu}\rightarrow 1$ follow from (4.32). The general integral is of the form

$$J_{l,m}'(\mu)=-\sum_{j=m}^{l+m}e_{l,m;j}(d)f_{(d/2)-1+j}(e^{\beta\mu})\times f_{(d/2)+l+m-j}(e^{\beta\mu})\ , \tag{4.54}$$

where the coefficients for the first few cases are

$$e_{0,0;0}=1,\ \ e_{0,1;1}=e_{1,0;0}=e_{1,0;1}=-\tfrac{1}{2}d\ ,$$
$$e_{1,1;1}=e_{1,1;2}=\tfrac{1}{4}d(d+1)\ , \tag{4.55}$$
$$e_{0,2;2}=e_{2,0;0}=\tfrac{2}{3}e_{2,0;1}=\tfrac{2}{3}e_{2,0;2}=\tfrac{1}{4}d(d+2)\ .$$

Finally one finds

$$\beta\Sigma_a^{(2)}=\left[\frac{\beta v_0}{\Lambda_T^d}\right]^2\sum_{l,m=0}^{\infty}\varphi_{0,l}^{+}\varphi_{0,m}^{+}\left[\frac{-4\pi a_0^2}{\Lambda_T^2}\right]^{l+m}J_{l,m}'(\mu)\ , \tag{4.56}$$

where for compactness we have taken

$$\varphi_{0,l}^{+}=\varphi_{0,l}+\delta_{0,l}\ . \tag{4.57}$$

This may be compared with (4.34).

Now the strongest singularities in $\Sigma_a^{(2)}$ arise from the most singular Bose functions entering via (4.54): by (4.32) we see that $f_s(e^{\beta\mu})$ remains bounded when $\beta\mu\rightarrow 0-$ unless $s\leq 1$. For $d>2$ the only divergent terms thus arise from the terms with $j=m=0$. The corresponding coefficients $e_{l,0;0}$ can be found from (4.51) and (4.52) and so, as $\beta\mu\rightarrow 0-$, one finds

$$J_{l,0}'(\beta\mu)\approx(-1)^l[\Gamma(\tfrac{1}{2}d+l)/\Gamma(\tfrac{1}{2}d)]\times f_{(d/2)-1}(e^{\beta\mu})f_{(d/2)+l}(e^{\beta\mu})\ ,$$
$$\approx(-1)^l[\Gamma(\tfrac{1}{2}d+l)/\Gamma(\tfrac{1}{2}d)]\times\Gamma(\tfrac{1}{2}\epsilon)\zeta(\tfrac{1}{2}d+l)(-\beta\mu)^{-\epsilon/2}\ , \tag{4.58}$$

with, as before, $\epsilon=4-d$. Correspondingly, the leading singularity of $\Sigma_a^{(2)}$ is given by

$$\beta\Sigma_a^{(2)}\approx\left[\frac{\beta v_0}{\Lambda_T^d}\right]^2 4\zeta(\tfrac{1}{2}d)\Gamma(\tfrac{1}{2}\epsilon)\times[1+\overline{\varphi}_0(4\pi a_0^2/\Lambda_T^2)](-\beta\mu)^{-\epsilon/2}\ , \tag{4.59}$$

while the remaining pieces remain bounded as $\beta\mu\rightarrow 0-$. [The function $\overline{\varphi}_0(x)$ was defined in (4.43).]

Now, since $\phi=\frac{1}{2}\epsilon$ the singularity of $\Sigma_a^{(2)}$ contributes a term to $\chi^{(2)}$ diverging as $(-\beta\mu)^{-2-\phi}$. This evidently corresponds to the expected scaling term $E_2/|g_\mu^0|^{1+\gamma_0+\phi}$ in

(3.14). However, the term $\mu^{-3}(\Sigma^{(1)})^2$ also contributes to this singularity in $\chi^{(2)}$. On the other hand, we will show below that $\Sigma_e^{(2)}$ does not diverge more rapidly than $(-\beta\mu)^{d-3}$ and so cannot contribute. From (4.59), (4.56), (4.15), and (4.42) we thus find the correspondence

$$\tfrac{1}{2}v_0^2 E_2=\left[\frac{\beta v_0}{\Lambda_T^d}\right]^2\Gamma(1-\tfrac{1}{2}d)(6-d)\zeta(\tfrac{1}{2}d)\times[1+\bar{\varphi}_0(4\pi a_0^2/\Lambda_T^2)]\ . \quad (4.60)$$

If one recognizes that $\frac{1}{2}(6-d)=1+\frac{1}{2}\epsilon=\gamma_0+\phi$ and recalls (4.3), (4.19), (4.20), and (4.45) one sees that, to leading order in μ, this result for E_2 agrees precisely with the scaling prediction (3.17). The agreement is fairly spectacular in that it holds for general $v_{\mathbf{k}}$!

2. *Theta contribution*

We turn now to $\Sigma_e^{(2)}$: in justification of (4.60) we show that it cannot diverge as strongly as $(-\beta\mu)^{-\epsilon/2}$; in addition, we will identify a divergence corresponding to the scaling term $E_3/|g_\mu^0|^{\gamma_0+2\phi}$ and thence learn something new, namely, the universal amplitude X_2 of the scaling function $X(x)$ in (3.4). The integrand in (4.50) looks potentially rather singular but, in fact, remains regular except when $\mu\to 0$. To show this, note

$$n_B(\varepsilon)=\tfrac{1}{2}\coth(\tfrac{1}{2}\varepsilon)-\tfrac{1}{2}$$

and use the addition formula for $\coth x$ to obtain

$$\Sigma_e^{(2)}=\tfrac{1}{4}\int_{\mathbf{q}}\int_{\mathbf{k}}\mathscr{S}^{-1}\tanh(\tfrac{1}{2}\beta\mathscr{S})v_{\mathbf{q}}(v_{\mathbf{k}}+v_{\mathbf{q}})\times(c_{\mathbf{q}}+c_{\mathbf{k}}-c_{\mathbf{k}+\mathbf{q}}-c_{\mathbf{k}}c_{\mathbf{q}}c_{\mathbf{k}+\mathbf{q}})\ , \quad (4.61)$$

in which for brevity we have written

$$\mathscr{S}=\varepsilon_{\mathbf{k}+\mathbf{q}}-\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{q}}+\mu,\quad c_{\mathbf{p}}=\coth[\tfrac{1}{2}\beta(\varepsilon_{\mathbf{p}}-\mu)]\ . \quad (4.62)$$

The factor $\tanh(\frac{1}{2}\beta\mathscr{S})/\mathscr{S}$ remains bounded for all real $\mathscr{S}$; singularities can arise only from the coth terms when $\mu\to 0$.

It is at this point that one encounters the difficulty mentioned in Sec. II A if one supposes $v(\mathbf{r})=v_0\delta(\mathbf{r})$ so that $v_{\mathbf{k}}\equiv v_0$. Indeed, whenever $d>2$ the integral over the unbounded region $|\mathbf{k}+\mathbf{q}|<\delta$ with, say, $\delta=|\boldsymbol{\delta}|$ chosen so that $\frac{1}{2}\beta\varepsilon_{\boldsymbol{\delta}}=O(1)$, then diverges. However, if $v_{\mathbf{q}}$ decays sufficiently rapidly as $|\mathbf{q}|\to\infty$ the integral remains bounded: we shall assume this is the case.

Now the single coth terms in (4.61) behave like $(\Lambda_T^2p^2+4\pi g_\mu^0)^{-1}$ as $g_\mu^0=-\beta\mu\to 0$ where $p=k$, q, or $|\mathbf{k}+\mathbf{q}|$. They can thus be analyzed (after making a rotation to coordinates $\mathbf{p}_\pm=\mathbf{k}\pm\mathbf{q}$ in the third case) along the lines used above to handle (4.4). Each generates contributions to $\Sigma^{(2)}$ of the form $F_0\mu^{-2}+F_1\mu^{-1-\epsilon/2}$ which correspond to the expected scaling terms $E_1/|g_\mu^0|^{\gamma_0+1}$ and $E_4/|g_\mu^0|^{\gamma_0+\phi}$. Together with contributions of the same form which arise from $\Sigma_a^{(2)}$ and $(\Sigma^{(1)})^2$ they would serve, via (3.17), to determine the scaling fields to second order in v_0. We shall not, however, attempt to compute these amplitudes.

The dominant singularity of $\Sigma_e^{(2)}$ as $\mu\to 0$ arises from the triple coth product in (4.61). If we neglect the momentum dependence of the other factors, which is justifiable if appropriate upper cutoffs on $\mathbf{k}$ and $\mathbf{q}$ are understood, the integral becomes simply a convolution and can thus be written

$$\Delta\Sigma_e^{(2)}(\beta,\mu)=-\tfrac{1}{4}\beta v_0^2\int d^dr[G(\beta,\mu;\mathbf{r})]^3\ , \quad (4.63)$$

in which a *lower* cutoff on r may be needed, while

$$G(\beta,\mu;\mathbf{r})=\int_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}\coth[\tfrac{1}{2}\beta(\varepsilon_{\mathbf{k}}-\mu)]\ . \quad (4.64)$$

The singular behavior is determined by the variation of the integrand as $8\pi/(\Lambda_T^2k^2-4\pi\beta\mu)$ for small k. This leads to[32]

$$G(\mathbf{r})\approx 2^{(d+2)/2}\Lambda_T^{-d}(-\beta\mu)^{(d-2)/2}\times x^{-(d-2)/2}K_{(d-2)/2}(x)\ , \quad (4.65)$$

where large values of

$$x=(-4\pi\beta\mu)^{1/2}r/\Lambda_T \quad (4.66)$$

are relevant, while $K_\nu(x)$ is the standard modified Bessel function. When $\beta\mu\to 0-$ one finds $G(\mathbf{r})\approx G_0(\beta)/r^{d-2}$.

Three cases now arise: (a) For $d<3$ the integral in (4.63) evidently diverges for large r as $\int^\infty r^{5-2d}dr$, and by changing to x as a variable one finds the required result

$$\beta\Sigma_e^{(2)}(\beta,\mu)\approx\mathscr{C}_d(\beta v_0/\Lambda_T^d)^2(-\beta\mu)^{d-3},\quad (d<3)\ , \quad (4.67)$$

where

$$\mathscr{C}_d=-2^{(d+4)/2}[\Gamma(\tfrac{1}{2}d)]^{-1}\times\int_0^\infty x^{2-(d/2)}[K_{(d-2)/2}(x)]^3dx\ , \quad (4.68)$$

and we have recognized that the other contributions to $\Sigma_e^{(2)}$ are bounded. Next, (b) for $d=3$ one has

$$K_{1/2}(x)=(\pi/2x)^{1/2}e^{-x}$$

and thence finds explicitly

$$\beta\Sigma_e^{(2)}(\beta,\mu)=-2\pi(\beta v_0/\Lambda_T^d)^2[\ln(-\beta\mu)^{-1}+O(1)]\ . \quad (4.69)$$

Finally, (c) for $d>3$ the integral on r in (4.63) converges even for $\beta\mu=0$. In that case one must subtract the limiting behavior in order to find the singular term: this leads to

$$\beta\Sigma_e^{(2)}(\beta,\mu)=\beta\Sigma_e^{(2)}(\beta,0)+\mathscr{C}_d(\beta v_0/\Lambda_T^d)^2(\beta\mu)^{d-3}+\cdots\ , \quad (4.70)$$

for $3<d<4$, where the constant is given by the convergent integral

$$\mathscr{C}_d = -\frac{2^{(d+4)/2}}{\Gamma(\frac{1}{2}d)} \int_0^\infty dx \frac{1}{x^{(d-4)/2}} \left[[K_{(d-2)/2}(x)]^3 - \left[\frac{2^{(d-2)/2}\Gamma(\frac{1}{2}d)}{(d-2)x^{(d-2)/2}} \right]^3 \right] . \tag{4.71}$$

Note that $\Sigma_e^{(2)}(\beta,0)$ is proportional to $(\beta v_0/\Lambda_T^d)^2$ even though it also contains contributions from the single coth terms in (4.61). These also yield correction terms varying as $(\beta\mu)^{(d-2)/2}$ which correspond, as explained, to the $E_4/|g_\mu^0|^{\gamma_0+\phi}$ scaling pieces.

Now we may compare with the scaling expectations. Since

$$2-(d-3)=1+(4-d)=\gamma_0+2\phi$$

we have found the last predicted term in (3.14): for $d\neq 3$ its amplitude is

$$E_3=(\beta/\Lambda_T^d)^2\mathscr{C}_d , \tag{4.72}$$

which, via (3.17), gives the universal result

$$X_2=\mathscr{C}_d/4[\Gamma(1-\tfrac{1}{2}d)]^2 , \tag{4.73}$$

for the second Taylor coefficient in the expansion, (3.4) of the susceptibility scaling function, $X(x)$, about the origin.

But what about the case of most interest, $d=3$? The result (4.69) suggests that the scaling function no longer has a Taylor expansion but, rather, varies as

$$X(x)=1+x+X_2'x^2\ln x+X_2''x^2+\cdots , \tag{4.74}$$

for $d=3$ with $X_2'=-\pi/[\Gamma(1-\frac{1}{2}d)]^2$. Why this should, in fact, be the case is explained in the Appendix on the basis of renormalization-group arguments.[25,34] The appearance of the logarithms in $X(x)$ and in the second-order perturbation expansions hinges on the fact that $\phi=\frac{1}{2}(4-d)$ is a *rational fraction* when $d=3$. More generally, logarithms will appear in nth order if $\phi=m/n$ where m is an integer relatively prime to n.

3. *Summary of second-order calculations*

To conclude this section, note that we have now identified all the five scaling singularities predicted in (3.14) and, furthermore, have shown that no other singularities occur other than higher-order singularities associated with irrelevant variables: in particular, the scaled combination y_φ defined in Sec. IV C must also enter in second order. In addition, we have, in (4.48), (4.60), and (4.72) explicitly evaluated the amplitudes E_0, E_2, and E_3, respectively. The first two provide exact checks for general interactions on the scaling predictions (3.17); the last one has the predicted form and leads to a value of the scaling function coefficient, X_2. Further labor would be needed to evaluate E_1 and E_4 in reasonably compact form although, by subtracting the leading singularities found here, they could be expressed explicitly. Their values, however, would yield only the nonuniversal terms of order v_0^2 in the scaling fields g_μ and g_v. To make further progress on calculating the scaling function, particularly in the region of its nonclassical singularity at x_c, we show, in the next section, how the near-critical dilute Bose fluid can be mapped quantitatively onto a spin system in its critical region: then known ϵ-expansion results for $X(x)$, etc. can be called upon.

V. MAPPING TO SPIN MODELS

The only systematic method currently available to calculate scaling functions for critical crossover in systems between their upper and lower critical dimensions is the method of renormalization-group dimensionality expansions. However, the detailed results in the literature are, for the most part, confined to classical, i.e., nonquantal systems, the Hamiltonians being expressed in the language of spin systems. (Some renormalization-group studies of quantal systems *per se* are reviewed below in Sec. VII.) To take advantage of the available definitive calculations,[9–11] therefore, we establish, in this section, a mapping of the Bose gas Hamiltonian (2.1) onto an appropriate classical-spin Hamiltonian, which is *exact* in the critical region of the two models.

A. The spherical model

As a first step we consider the spherical model,[35,36] whose critical behavior is known to be closely similar to that of the ideal Bose gas.[14] In its generalized form the spherical model Hamiltonian, $\mathscr{H}_S$, entails n-component classical-spin vectors $\vec{s}(\mathbf{r})$ located at the sites of a d-dimensional lattice of spacing, say, $\tilde{a}$ and volume V_Ω. The basic trace operation is then an integral over the spin components, $-\infty < s^\lambda(r) < \infty$, the partition function being defined by

$$Z=\mathrm{Tr}^s\{\exp(-\beta\mathscr{H})\}$$
$$\equiv\prod_{\mathbf{r}}\sum_{\lambda=1}^{n}\int_{-\infty}^{\infty} ds^\lambda(\mathbf{r})\exp\{-\beta\mathscr{H}[\vec{s}(\mathbf{r})]\} . \tag{5.1}$$

The lattice structure does not affect the universal features of the critical behavior. Accordingly in writing the Hamiltonian, etc., we shall treat $\vec{s}(\mathbf{r})$ as a continuous spin field; however, the corresponding spin variables in momentum space, namely,

$$\vec{s}_{\mathbf{k}}=V_\Omega^{-1/2}\int_\Omega d^d r\, e^{-i\mathbf{k}\cdot\mathbf{r}}\vec{s}(\mathbf{r}) , \tag{5.2}$$

must be subject to a *cutoff* of magnitude $k_\Lambda \simeq \pi/\tilde{a}$.

The spherical model Hamiltonian may now be written[37]

$$\mathscr{H}_S=\mathscr{H}_G-\tfrac{1}{2}z\sum_{\mathbf{k}}|\vec{s}_{\mathbf{k}}|^2 , \tag{5.3}$$

where the Gaussian (or "free field") Hamiltonian[14,35] is

$$\mathscr{H}_G=-\tfrac{1}{2}\sum_{\mathbf{k}}J_{\mathbf{k}}|\vec{s}_{\mathbf{k}}|^2-V_\Omega^{1/2}\vec{\mathrm{H}}\cdot\vec{s}_0 . \tag{5.4}$$

Here $J_{\mathbf{k}}$ represents the Fourier transform of the spin-spin or exchange coupling, $J(\mathbf{r}-\mathbf{r}')$. For sufficiently short-range interactions it has the expansion

$$J_{\mathbf{k}}=J_0(1-R_0^2k^2+j_2R_0^4k^4+\cdots), \quad (5.5)$$

in which R_0 measures the *range of interaction*. The vector $\vec{H}\equiv(H,H,\ldots,H)$ represents the external magnetic field (coupled, for convenience, equally to each spin component).

The pure Gaussian model described by $\mathscr{H}_G$ alone is easily solved exactly.[14,35] For $2<d\leq 4$ it has critical exponents $\alpha_G=\frac{1}{2}$, $\gamma_G=1$, $\nu_G=\frac{1}{2}$, and $\eta_G=0$; these are the same as for an ideal Bose gas at fixed chemical potential.

The Gaussian model ceases to be defined below T_c and, furthermore, the mean-square value of a single-spin component, namely,

$$m_2\equiv\langle[s^\lambda(\mathbf{r})]^2\rangle=(1/nV_\Omega)\left\langle\sum_{\mathbf{k}}|\vec{s}_{\mathbf{k}}|^2\right\rangle, \quad (5.6)$$

is a strong function of T and H. The spherical model *per se* is defined by removing this rather unphysical freedom of variation with the aid of the last term in (5.3). In this the *spherical field, z*, is adjusted to satisfy the *spherical constraint*

$$m_2(\beta,z,H)\equiv\langle(s^\lambda)^2\rangle=M_2, \quad (5.7)$$

in which M_2 is fixed. Normally[35,36] one sets $M_2=1$ but we keep it as a free parameter since, by comparison with Sec. II A, it is clear that the spherical constraint is quite analogous to the constant density condition in an ideal Bose gas so that M_2 corresponds to the density ρ. Similarly z is analogous to the chemical potential μ.

The total free-energy density (per spin component) for the spherical model is found to be[14,36]

$$f(\beta,z,H)=\tfrac{1}{2}H^2/(z+J_0)+\tfrac{1}{2}\beta^{-1}\int_{\mathbf{k}}\ln[-\beta(z+J_{\mathbf{k}})/2\pi], \quad (5.8)$$

where the cutoff $k_\Lambda\simeq\pi/\tilde{a}$ must be remembered. The reduced spin susceptibility in zero field is thus

$$\tilde{\chi}\equiv-\frac{1}{\beta}\left[\frac{\partial^2 f}{\partial H^2}\right]_{H=0}=\frac{-1}{\beta(z+J_0)}, \quad (5.9)$$

from which we see that $(z+J_0)$ can never be negative.

The spherical constraint is easily handled and one finds the exponents $\alpha_S=-\frac{1}{2}\epsilon/(1-\frac{1}{2}\epsilon)$, $\beta_S=\frac{1}{2}$, $\gamma_S=2\nu_S=1/(1-\frac{1}{2}\epsilon)$, $\eta=0$ for $2<d\leq 4$. These are simply the constraint-renormalized Gaussian exponents (see Ref. 28 and Sec. III B); they coincide with those for the ideal Bose gas at constant density. Indeed, if one uses (5.5) in (5.8) and likewise expands the ideal-Bose-gas integrand in (2.18) for low momentum, one sees that the parallel between the two models in the critical region is very strong. Before quantifying it, however, we will introduce the analog of the particle-particle interactions into the spin model.

B. The interacting n-vector model

To obtain a physically more reasonable model than the Gaussian or spherical model it is natural to add a fourth-order term in the spins to the Hamiltonian. It is customary in the first instance merely to add a local term proportional to $\int d^dr\,|\vec{s}(\mathbf{r})|^4$, but in the present case we would like to mimic more closely the Bose interaction term $\mathscr{H}_2[\psi]$ in (2.4). Accordingly we add to $\mathscr{H}_S$ the term

$$\mathscr{U}=\int d^dr\int d^dr'\,|\vec{s}(\mathbf{r})|^2u(\mathbf{r}-\mathbf{r}')\,|\vec{s}(\mathbf{r}')|^2$$
$$=V_\Omega^{-1}\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\sum_{\mathbf{q}}u_{\mathbf{q}}(\vec{s}_{\mathbf{k}+\mathbf{q}}\cdot\vec{s}_{-\mathbf{k}})(\vec{s}_{\mathbf{k}'-\mathbf{q}}\cdot\vec{s}_{-\mathbf{k}'}), \quad (5.10)$$

in which the four-spin coupling has Fourier transform

$$u_{\mathbf{q}}=\int d^dr\,e^{i\mathbf{q}\cdot\mathbf{r}}u(\mathbf{r}). \quad (5.11)$$

The purely local $|s|^4$ coupling then corresponds to $u(\mathbf{r})=u_0\delta(\mathbf{r})$ or $u_{\mathbf{q}}\equiv u_0$. As in the Bose case, and as supported by extensive calculations for the spin models, one anticipates that details of the potential beyond u_0 will contribute only corrections to scaling in the critical region.

Again as in the interacting Bose gas, one must resort to perturbation theory to study the effects of $\mathscr{U}$ in a systematic way. The susceptibility and other properties can be obtained as a power series in $u_{\mathbf{q}}$ by developing a diagrammatic expansion for the Green's function or correlation function

$$G_{\kappa\lambda}(\mathbf{r},\mathbf{r}')=\langle s^\kappa(\mathbf{r})s^\lambda(\mathbf{r}')\rangle, \quad (5.12)$$

with $\kappa,\lambda=1,2,\ldots,n$. For an isotropic, translationally invariant system in zero field one may deal with the Fourier transforms

$$\hat{G}_{\kappa\lambda}(\mathbf{k})=\hat{G}(\mathbf{k})\delta_{\kappa\lambda},\quad \hat{G}(\mathbf{k})=\frac{1}{n}\left\langle\sum_{\lambda=1}^{n}s_{\mathbf{k}}^\lambda s_{-\mathbf{k}}^\lambda\right\rangle. \quad (5.13)$$

The reduced spin susceptibility is then

$$\tilde{\chi}=\int d^dr\,G(\mathbf{r})=\hat{G}(\mathbf{0}). \quad (5.14)$$

For zero interactions one has simply

$$\hat{G}^{\,0}(\beta,z;\mathbf{k})=-1/\beta(z+J_{\mathbf{k}}), \quad (5.15)$$

which, on putting $\mathbf{k}=0$, immediately yields the spherical model result (5.9).

The *diagrams* entering the perturbation theory are now precisely the same as for the Bose gas. Thus Figs. 1 and 2 give the first-order and second-order contributions to $\tilde{\chi}$ with, now, a solid line representing the propagator $\hat{G}^{\,0}(\mathbf{k})$ while the dotted lines represent the spin vertex coupling $u_{\mathbf{k}}$. The weight to be attached to a given diagram, however, differs in the two cases. Before discussing this in detail let us examine the first-order corrections to the spin susceptibility. This will actually enable us to achieve the desired mapping between the models in their critical regions.

C. First-order perturbations and matching

As before let us write the spin susceptibility as

$$\tilde{\chi}(\beta,z;u)=\tilde{\chi}^{(0)}(\beta,z)+\tilde{\chi}^{(1)}(\beta,z;u)+\cdots, \quad (5.16)$$

where $\tilde{\chi}^{(n)}$ is of nth order in $u_{\mathbf{k}}$. By (5.9) and (5.15) we have

$$\tilde{\chi}^{(0)}=1/(-\beta\bar{z})\ \text{ with }\ \bar{z}=z+J_0. \quad (5.17)$$

The first-order term is found to be

$$\tilde{\chi}^{(1)}=\frac{-8}{(-\beta\bar{z})^2}\int_{\mathbf{k}}\tfrac{1}{2}\beta(nu_0+2u_{\mathbf{k}})\hat{G}^0(z,\mathbf{k}) . \quad (5.18)$$

This should be compared with the Bose expression (4.4). Notice, in particular, the factor n: this arises from a free summation over the spin index $\lambda=1,\ldots,n$, in the loop in Fig. 1(a) which carries the product $s^{\lambda}_{\mathbf{k}}s^{\lambda}_{-\mathbf{k}}$ for all λ; since the external line carries a fixed spin index, say κ, no such factor n can arise in Fig. 1(b). We see immediately that the contributions of u_0 and $u_{\mathbf{k}}$ will enter here in the same proportions as do v_0 and $v_{\mathbf{k}}$ in the Bose case *provided* one takes $n=2$. This identification is hardly surprising since it has been known for a long time[26] that a superfluid, having a two-component order parameter $\Psi\equiv(\Psi',\Psi'')$, with $\Psi'=\mathrm{Re}(\Psi)$ and $\Psi''=\mathrm{Im}(\Psi)$, should be in the same universality class of critical behavior as the XY or $(n=2)$-component spin model. It is valuable, however, to see how perturbation theory essentially forces the correspondance: see further below.

Now the singularities of the integral in (5.18) can be evaluated in precisely the same fashion as undertaken in Sec. IV B, the only difference being the presence of the lattice cutoff k_Λ here. In parallel to (4.15) one finds

$$\tilde{\chi}^{(1)}\approx\frac{\tilde{D}_0u_0}{(-\beta\bar{z})^2}-\frac{4\Gamma(1-\frac{1}{2}d)}{(-\beta\bar{z})^{1+\epsilon/2}}\frac{\beta u_0(n+2)}{(4\pi\beta J_0R_0^2)^{d/2}} , \quad (5.19)$$

with, in comparison with (4.7), an amplitude

$$\tilde{D}_0=-4\int_{\mathbf{k}}(n+2\tilde{\varphi}_{\mathbf{k}})/(J_0-J_{\mathbf{k}}) , \quad (5.20)$$

in which $\tilde{\varphi}_{\mathbf{k}}=u_{\mathbf{k}}/u_0$. Comparing this and (5.17) with the general scaling expectations (3.11) to (3.14) yields the identifications

$$\tilde{C}=1,\ g_z^0=-\beta\bar{z},\ \gamma_0=1 , \quad (5.21)$$

where, as above, a tilde distinguishes spin-model amplitudes, and more fully,

$$g_z(\beta,z;u_0)=-\beta\bar{z}-\tilde{D}_0u_0,\quad \phi=\tfrac{1}{2}\epsilon ,$$
$$g_u(\beta,z;u_0)=2(n+2)\beta u_0/(4\pi\beta J_0R_0^2)^{d/2} , \quad (5.22)$$
$$\tilde{B}=-2\Gamma(1-\tfrac{1}{2}d) ,$$

where we have chosen $\tilde{B}$ to equal B in (4.19).

Now if two distinct systems belong to the same universality class, as we claim for the $(n=2)$-vector model and the dilute Bose fluid, their critical properties can differ only through nonuniversal parameters. But these enter *only* through the *scaling fields* and related amplitudes, the normalized scaling functions being universal. Thus two models can be mapped onto one another if their scaling fields can be brought into one-to-one correspondance. Furthermore, to leading order near criticality it suffices to map only the *linear* scaling fields onto one another.

Thus if we compare the results (5.22) with (4.16), (4.19), and (4.20) we can read off the mapping from spin model to Bose gas as

$$n=2,\ \beta\Longleftrightarrow\beta,\ \bar{z}\equiv z+J_0\Longleftrightarrow\mu,\ u_{\mathbf{k}}\Longleftrightarrow\tfrac{1}{8}v_{\mathbf{k}} ,$$
$$J_0R_0^2\Longleftrightarrow\hbar^2/2m^*\equiv\Lambda_T^2/4\pi\beta,\ \tilde{D}_0\Longleftrightarrow16D_0 . \quad (5.23)$$

There is some arbitrariness in the numerical coefficients but, in particular, the choice of the factor $\frac{1}{8}$ leads to the correspondance for $J_0R_0^2$ which, by comparing (5.4) and (5.5) with (2.13) and (2.16), yields the further natural correspondances

$$\vec{s}_k\Longleftrightarrow(\tfrac{1}{2}(a_{\mathbf{k}}+a^{\dagger}_{-\mathbf{k}}),\tfrac{1}{2}i(a_{\mathbf{k}}-a^{\dagger}_{-\mathbf{k}})),\ M_2\Longleftrightarrow\rho ,$$
$$\vec{s}(\mathbf{r})\Longleftrightarrow(\tfrac{1}{2}[\psi(\mathbf{r})+\psi^{\dagger}(\mathbf{r})],\tfrac{1}{2}i[\psi(\mathbf{r})-\psi^{\dagger}(\mathbf{r})]) . \quad (5.24)$$

It is worthwhile to explore further the relation between $\tilde{D}_0$ and D_0, since the former depends on the lattice cutoff k_Λ while the latter was evaluated more explicitly in (4.42) and, as $T_c\to0$, depends only on βv_0 and Λ_T. A comparison between (4.7) and (5.20) shows that the differences arise only from the difference between the Bose factor $n_B(\varepsilon_{\mathbf{k}})$ which, as seen, provides an effective momentum cutoff of order Λ_T^{-1}, and the factor $1/(J_{\mathbf{k}}-J_0)$ with cutoff at k_Λ. For matching we therefore anticipate a relation

$$k_\Lambda=\Gamma_d/\Lambda_T , \quad (5.25)$$

which, if Γ_d is a fixed number, means $k_\Lambda\to0$ as $T_c\to0$. To determine Γ_d we evaluate $\tilde{D}_0$ for $n=2$, as in Sec. IV C, and find

$$\tilde{D}_0=-\frac{32k_\Lambda^{d-2}[1+O(j_2R_0^2k_\Lambda^2,\tilde{\varphi}_2\tilde{a}_0^2k_\Lambda^2)]}{(4\pi)^{d/2}\Gamma(\frac{1}{2}d)(d-2)J_0R_0^2} , \quad (5.26)$$

where j_2 enters in (5.2) and we have supposed that

$$\tilde{\varphi}_{\mathbf{k}}\equiv u_{\mathbf{k}}/u_0=1-\tilde{\varphi}_2\tilde{a}_0^2k^2+\cdots$$

so that $\tilde{a}_0$ represents the range of the coupling $u(\mathbf{r})$; by (5.23) we may take $\tilde{a}_0=a_0$. Comparing with (4.42) yields

$$\Gamma_d=2\sqrt{\pi}[\tfrac{1}{2}(d-2)\Gamma(\tfrac{1}{2}d)\zeta(\tfrac{1}{2}d)]^{1/(d-2)} . \quad (5.27)$$

In *addition* we see that by choosing the coefficients j_2 and $\tilde{\varphi}_2$ judiciously in the spin model one can *also* match the higher-order corrections in $\tilde{D}_0$ and D_0 which vanish as powers of $k_\Lambda^2\sim\Lambda_T^{-2}\sim T_c$.

D. Higher-order perturbation theory

The first-order calculations just presented have enabled us to match the $(n=2)$-vector and Bose systems as regards linear scaling fields and amplitudes. By going to second order we could, clearly, match the quadratic parts of the *nonlinear* scaling fields and so on. Physically this is hardly worthwhile; however, it is important to be assured that the *full* perturbation expansions will continue to match to leading orders in the critical region. If so, one can conclude that the expansions for the susceptibility, and other scaling functions, are identical so that the models truly lie in the same universality class.

It was already observed that the *diagrams* entering the perturbation expansions of the n-vector and Bose models are identical although the weights differ in general. More explicitly, at nth order in the Bose case each diagram carries a weight 2^n due to the interchangeability of the arguments $\mathbf{r}$ and $\mathbf{r}'$ of the perturbation in (2.4). However, this factor is conveniently canceled by the conventional prefactor $\frac{1}{2}$ carried by $\mathscr{H}_2$. Each loop in a Bose diagram effectively contributes *twice* since both $\langle \mathscr{T}\psi(\mathbf{r},\tau)\psi^\dagger(\mathbf{r}',\tau')\rangle^{(0)}$ and $\langle \mathscr{T}\psi(\mathbf{r}',\tau')\psi^\dagger(\mathbf{r},\tau)\rangle^{(0)}$ enter (where $\mathscr{T}$ is the time ordering operator[30]). On the other hand, in the spin case each loop contributes a factor n, from the free spin sum on spin components, as seen. However, $\langle \vec{s}(\mathbf{r})\cdot\vec{s}(\mathbf{r}')\rangle^{(0)}$ is identical to $\langle \vec{s}(\mathbf{r}')\cdot\vec{s}(\mathbf{r})\rangle^{(0)}$ so the relative factor is only $\frac{1}{2}n$ which is unity when $n=2$. By a similar token each diagram in the spin system gains a factor 8^n in nth order arising, first, from the interchangeability of $\mathbf{r}$ and $\mathbf{r}'$ in the perturbation $\mathscr{U}$ in (5.10) and, second, from the identity of each of the two spins in the factors $|\vec{s}(\mathbf{r})|^2$ and $|\vec{s}(\mathbf{r}')|^2$; on the contrary, in (2.4) $\psi^\dagger(\mathbf{r})$ and $\psi(\mathbf{r})$ are *not* equivalent. The factor $\frac{1}{8}$ in the correspondence (5.23), together with $n=2$, thus ensures a matching of the two expansions in *all* orders.

Of course, the expansions also differ in that each solid line in a spin diagram carries a simple momentum-dependent factor $\hat{G}^{(0)}(\mathbf{k}) \propto (z+J_{\mathbf{k}})^{-1}$, whereas in the Bose case the Matsubara propagator $\mathscr{G}^{(0)}(\mathbf{k},ik_n)$ enters, which also carries a frequency $k_n = 2\pi n k_B T$. Momentum and frequency are conserved at each vertex and, finally, one must integrate on the internal momenta *and,* in the Bose case, sum on the internal frequencies. As we have observed, the critical singularities in the perturbation terms arise as $\mu \to 0$ solely from momenta close to $\mathbf{k}=0$. Further, since there is a gap of $2\pi k_B T$ between the Matsubara frequencies, only the $n=0$ mode of $\mathscr{G}^{(0)}$ is actually singular at $\mathbf{k}=0$ and $\mu=0$; in fact, one then has $\mathscr{G}^{(0)}(\mathbf{k},0) \propto \hat{G}^{(0)}(\mathbf{k})$ for $k\Lambda_T$ and k/k_Λ small. This indicates that the Matsubara frequencies are, in a renormalization-group sense, *irrelevant* and cannot change critical behavior for $T_c > 0$.[38] They do not, thus, enter directly into the mapping. However, one cannot simply neglect all the nonzero Matsubara frequencies since, as seen in first order, the corresponding modes add up to give Bose-like factors which provide effective cutoffs of order Λ_T^{-1} on all internal momentum integrals. On the other hand, the nonzero frequencies can, thereby, be traded in for a cutoff $k_\Lambda \sim \Lambda_T^{-1}$, as we have seen explicitly in first order. These considerations complete our discussion of the mapping between spin and Bose systems.

VI. ANALYSIS OF THE ϵ EXPANSION

We proceed now to calculate the superfluid density for the dilute Bose gas and its scaling function via the renormalization-group $\epsilon = 4-d$ expansion technique.[9–12,26] We will also, incidentally, present results for the free energy and susceptibility. Finally, we will review the comparison with experiment.

A. Thermodynamics

The free energy of the n-vector spin model has been calculated to first order in ϵ for general n both above and below T_c by Nicoll and Chang.[9] Their results correctly reproduce all known exact limits: $n \to \infty$ (spherical model); $u \to 0$ (Gaussian model); and infinite range (the Kac—van der Waals limit). In addition, they embody correctly all critical exponents to $O(\epsilon)$. In applicable regimes, in particular for $n=1$ (Ising-like), they agree with expressions obtained by other authors.[10–12] Furthermore, in zero-field, which is all we require here, they have been checked by one of us.[11]

Nicoll and Chang[9] work with the reduced Hamiltonian

$$\overline{\mathscr{H}}_\sigma \equiv \mathscr{H}_\sigma / k_B T = \tfrac{1}{2}\int_{\mathbf{q}} (r+q^2)\vec{\sigma}_{\mathbf{q}}\cdot\vec{\sigma}_{-\mathbf{q}} + \tfrac{1}{4}u \int_{\mathbf{q}}\int_{\mathbf{q}'}\int_{\mathbf{q}''} \vec{\sigma}_{-\mathbf{q}'}\cdot\vec{\sigma}_{\mathbf{q}'+\mathbf{q}}\vec{\sigma}_{-\mathbf{q}''}\cdot\vec{\sigma}_{\mathbf{q}''-\mathbf{q}}\,, \tag{6.1}$$

for spins $\vec{\sigma}(\mathbf{x})$ with transforms $\vec{\sigma}_{\mathbf{q}}$, while u is nonnegative and a momentum cutoff $q_\Lambda \equiv 1$ is understood. We bring our spin Hamiltonian, (5.3), (5.4), plus (5.10), into this form by rescaling spin and space variables according to

$$\mathbf{q} = \mathbf{k}/k_\Lambda,\quad \mathbf{x} = k_\Lambda \mathbf{r},\quad \vec{\sigma}_{\mathbf{q}} = (\beta J_0 R_0^2 k_\Lambda^{d+2})^{1/2}\vec{s}_{\mathbf{k}}\,, \tag{6.2}$$

and, for the two relevant thermodynamic fields,

$$r = -(z+J_0)/J_0 R_0^2 k_\Lambda^2,\quad u = 4u_0 k_B T k_\Lambda^{d-4}/J_0^2 R_0^4\,. \tag{6.3}$$

[Note that u as defined in (6.1) is 4 times the parameter u in Ref. 6.] The correspondences for the Bose system follow from (5.23)—(5.25) which yield the simple relations

$$r = -(4\pi/\Gamma_d^2)\beta\mu,\quad u = (8\pi^2/\Gamma_d^\epsilon)(\beta v_0/\Lambda_T^d)\,, \tag{6.4}$$

in which Γ_d is defined in (5.27). Thus r is the chemical potential or temperaturelike variable. For the density we also find, from (5.24), the correspondence

$$\rho = M_2 = (4\pi/n\Gamma_d^2)\int_{\mathbf{q}} \langle |\vec{\sigma}_{\mathbf{q}}|^2\rangle\,. \tag{6.5}$$

Now let the reduced "Helmholtz" free-energy density as a function of the magnetization density $m = \langle |\vec{\sigma}_0|\rangle/(k_\Lambda^d V_\Omega)^{1/2}$ be defined by

$$A(r,m) = (k_\Lambda^d V_\Omega)^{-1}\ln[\mathrm{Tr}^\sigma_{\mathbf{q}\neq 0}\{\exp(-\overline{\mathscr{H}}_\sigma)\}]\,, \tag{6.6}$$

where the factors k_Λ reflect the spatial rescaling in (6.2). Then, to leading order in ϵ, Nicoll and Chang[9] show the singular part of the free energy can be written

$$A_s(r,m) = P^{-1}(\tfrac{1}{2}t_0 m^2 Q^{6/(n+8)} + \tfrac{1}{4}um^4) + \frac{t_0^2}{4u}\left[\frac{n}{n-4}(Q^{(4-n)/(n+8)}-1) + \left[\frac{1}{P}-\frac{1}{Q}\right]Q^{12/(n+8)}\right], \tag{6.7}$$

where the linear scaling field for r is[9,12]

$$t_0 = r + g_n \epsilon \tilde{u} \quad\text{with}\quad g_n \equiv \frac{1}{2}\left[\frac{n+2}{n+8}\right], \tag{6.8}$$

In fact t_0 turns out, to $O(\epsilon)$, also to measure the deviation from the critical line. Thus $t_0 > 0$ specifies the disordered phase above T_c; the ordered phase below T_c is described by $t_0 < 0$ in the limit that the magnetic field, $h = (\partial A/\partial m)_r$, vanishes. The auxiliary functions $P(t_0,m)$ and $Q(t_0,m)$ are given implicitly through the equations[39]

$$P=1+\frac{9\tilde{u}}{n+8}(S^{\epsilon/2}-1)+\frac{\tilde{u}(n-1)}{n+8}\left[\left(\frac{m}{h}\right)^{\epsilon/2}-1\right], \quad (6.9)$$

$$Q=1+\tilde{u}(S^{\epsilon/2}-1), \quad (6.10)$$

$$S=Q/(t_0Q^{6/(n+8)}+3um^2), \quad (6.11)$$

$$m/h=P/(t_0Q^{6/(n+8)}+um^2), \quad (6.12)$$

in which

$$\tilde{u}=u/u^* \text{ with } u^*=8\pi^2\epsilon/(n+8). \quad (6.13)$$

Note that $u^*\equiv u^*(d)$ is the value of the coupling constant u at the nontrivial n-vector fixed point:[26] like the critical exponents, it has an expansion in powers of ϵ although, in higher orders, u^* has some residual dependence on the details of the renormalization group used. We will return later to the question of its value for $\epsilon=1$ or $d=3$. The susceptibility is $\chi=(\partial m/\partial h)=1/(\partial^2 A/\partial m^2)$: above T_c it follows directly from (6.12) when $h\to 0$.

To obtain a more explicit expression for the free energy in the *ordered phase* ($t_0<0$, $h\to 0$) we eliminate Q between (6.10) and (6.11) to obtain

$$S=\frac{1-\tilde{u}+\tilde{u}S^{\epsilon/2}}{t_0(1-\tilde{u}+\tilde{u}S^{\epsilon/2})^{6/(n+8)}+3um^2}. \quad (6.14)$$

From this one can show that S remains bounded for $t_0<0$ when $\epsilon<2$. Thus when $h\to 0$ in (6.9) the first two terms on the right can be neglected which yields

$$P\approx\tilde{u}[(n-1)/(n+8)](m/h)^{\epsilon/2} \text{ as } h\to 0. \quad (6.15)$$

Substitution into (6.12) then gives

$$\tilde{u}[(n-1)/(n+8)](h/m)^{1-(\epsilon/2)}\approx t_0Q^{6/(n+8)}+um^2. \quad (6.16)$$

Taking the limit $h\to 0$ yields the spontaneous magnetization, $m_0(t_0)$, which is proportional to $[n_0(T)]^{1/2} = |\Psi_0(T)|$ in the Bose system: one finds

$$m_0^2=(-t_0/u)Q_-^{6/(n+8)}, \quad (6.17)$$

the subscript minus denoting the ordered phase. This in turn leads to

$$S_-(t_0)=(-1/2t_0)Q_-^{2g_n}, \quad (6.18)$$

with g_n as in (6.8), and hence to an equation for $Q_-(t_0)$, namely,

$$Q_-=1-\tilde{u}+\tilde{u}Q_-^{\epsilon g_n}/(-2t_0)^{\epsilon/2}. \quad (6.19)$$

In terms of the solution of this equation the free energy in the ordered state is simply

$$A_{s-}(t_0)=[t_0^2/(n-4)u](Q_-^{(4-n)/(n+8)}-\tfrac{1}{4}n). \quad (6.20)$$

It is worth remarking that the limit of this expression when $n\to 4$ is perfectly finite and yields the correct $n=4$ result.

B. The susceptibility and scaling

For reference below, and because this is the function whose perturbation expansion we have examined in such detail, we obtain next the scaling form for the susceptibility, χ, above T_c in zero field. In this case we have $h\to 0$, $m\to 0$, and $m/h\to\chi$.

Putting $m=0$ in (6.11) and (6.12) and eliminating S yields the equation

$$Q_+(t_0)=1-\tilde{u}+\tilde{u}Q_+^{g_n\epsilon}/t_0^{\epsilon/2}, \quad (6.21)$$

for $Q_+(t_0)\equiv Q(t_0>0,\ h=0)$. From (6.12) with $m=0$ and (6.9) with S eliminated in favor of Q one gets

$$\chi=t_0^{-1}Q_+^{-6/(n+8)}\left[\frac{n-1}{n+8}(1-\tilde{u}+\tilde{u}\chi^{\epsilon/2})+\frac{9}{n+8}Q_+\right]. \quad (6.22)$$

These two equations suffice to determine χ as a function of t_0 and u. However, they can be cast into a more transparent form if we introduce the *nonlinear scaling fields*

$$\dot{t}_+=t_0/(1-\tilde{u})^{2g_n} \text{ and } \dot{u}=\tilde{u}/(1-\tilde{u}). \quad (6.23)$$

We may then write the susceptibility in standard *scaled form* just as

$$\chi=\dot{t}_+^{-1}X(x_+) \text{ with } x_+=\dot{u}/\dot{t}_+^{\epsilon/2}. \quad (6.24)$$

Note that this exhibits the anticipated exponent $\gamma_0=1$ and, likewise, the crossover exponent $\phi=\frac{1}{2}\epsilon$; both exponents are, in fact, correct to all orders in ϵ as we have seen. Then, if we put

$$Q_+(t_0,u)=\dot{Q}(x_+)(1-\tilde{u})$$

we find from (6.24) the basic equation

$$\dot{Q}(x)=1+x[\dot{Q}(x)]^{g_n\epsilon}, \quad (6.25)$$

so that $\dot{Q}$ is, indeed, a function *only* of the scaled variable x. Finally, the scaling function $X(x)$ is, via (6.24), determined by the equation

$$X=\frac{n-1}{n+8}(1+xX^{\epsilon/2})/\dot{Q}^{6/(n+8)}+\frac{9}{n+8}\dot{Q}^{2g_n}. \quad (6.26)$$

It is not difficult to check that the solution of this equation is simply $X(x)=[\dot{Q}(x)]^{2g_n}$.

One cannot, for general ϵ, solve the equation for $Q(x)$ explicitly, but one can expand for small x and large x which yields

$$X(x)=1+\frac{n+2}{n+8}x+\tfrac{1}{2}\left[\epsilon-\frac{6}{n+2}\right]\left[\frac{n+2}{n+8}\right]^2x^2+\cdots \text{ as } x\to 0,$$
$$=x^{\omega_n}(1+\omega_n x^{-1/(1-g_n\epsilon)}+\cdots) \text{ as } x\to\infty, \quad (6.27)$$

in which the exponent for large x is

$$\omega_n=2g_n/(1-g_n\epsilon)=(\gamma-\gamma_0)/\phi[1+O(\epsilon)], \quad (6.28)$$

where the n-vector susceptibility exponent has the expansion[9,26]

$$\gamma=1+\frac{n+2}{2(n+8)}\epsilon+O(\epsilon^2)\ . \tag{6.29}$$

The form of $X(x)$ for $x\to\infty$ is thus just what is required to yield the correct crossover behavior to $\chi\sim t_0^{-\gamma}$ for $u>0$. Notice, further, that we are in a situation where $x_c=\infty$: compare with (3.22).

C. Imposition of the constraint

By (6.4) and (6.5) the required constant density constraint can be written

$$2k_\Lambda^d(\partial A/\partial r)=n\rho\Gamma_d^2/4\pi\ . \tag{6.30}$$

Note the important factor k_Λ^d arising from the spatial rescaling in (6.6). If we use $k_\Lambda=\Gamma_d/\Lambda_T$ with Γ_d given by (5.27) and recall from (2.23) that the critical point, T_c^0, of the ideal Bose gas is determined by

$$\rho\Lambda^d_{T=T_c^0}=\rho\Lambda_T^d(T/T_c^0)^{d/2}=\zeta(\tfrac{1}{2}d)\ , \tag{6.31}$$

we can rewrite this simply as

$$2(d-2)K_d^{-1}(\partial A/\partial r)=n(T_c^0/T)^{d/2}\ , \tag{6.32}$$

where K_d is defined in (4.12). Furthermore, since t_0 differs from r only by a shift we may replace $(\partial A/\partial r)$ by $(\partial A/\partial t_0)$. Notice that we have eliminated the density in favor of the (ideal) critical temperature.

Now (6.32) involves the total free energy, A, whereas (6.7) or (6.20) give the singular part A_s, which vanishes on the critical line $t_0=0$. As in Sec. III B the regular part of the free energy, say A_0, thus determines the constraint *at* criticality. This should have a contribution proportional to $\epsilon\tilde{u}$; but the work of Rudnick and Nelson[12] shows that its coefficient actually vanishes. Thus although there is an $O(\epsilon\tilde{u})$ shift in the unconstrained T_c we can expect

$$T_c(\rho)=T_c^0(\rho)+O(\epsilon^2,\tilde{u}^2)\ .$$

Putting $T=T_c$ in (6.32) thus indicates that we have

$$2(d-2)K_d^{-1}(\partial A_0/\partial t_0)=n\ , \tag{6.33}$$

correct to $O(\epsilon)$. Subtracting this from (6.32) gives an expression for $(\partial A_s/\partial t_0)$. Now, as can be verified from (6.20), $A_s(t_0)$ vanishes faster than t_0 (the critical exponent α being less than unity); thus the derivative should also vanish at criticality. We can achieve this requirement by incorporating the shifted T_c into the constraint by writing it in the final form

$$2(d-2)K_d^{-1}(\partial A/\partial t_0)\approx n[(T_c/T)^{d/2}-1]\equiv n\bar{t}\ . \tag{6.34}$$

Although in higher orders, this specific form for $\bar{t}$ may be modified, $\bar{t}$ must always vanish linearly with $T-T_c$ and the form specified here will give enhanced accuracy as the ideal Bose limit ($v_0\to 0$, $T_c\to 0$) is approached.

Finally we record here the result following from (6.20) for the free-energy derivative below T_c, namely,

$$\frac{\partial A_{s-}}{\partial t_0}=\frac{2t_0}{(n-4)u}\left[Q_-^{(4-n)/(n+8)}\times\frac{1-\frac{1}{4}\epsilon[1-(1-\tilde{u})Q_-^{-1}]}{1-g_n\epsilon[1-(1-\tilde{u})Q_-^{-1}]}-\frac{n}{4}\right]. \tag{6.35}$$

D. The helicity modulus

The last result we need is an expression for the helicity modulus[40] which, for $n=2$, is proportional to the superfluid density. (One must, of course, recall that in an interacting Bose system the superfluid density, ρ_s, is *not* equal to the condensate density n_0; indeed, unless $\eta=0$, the two quantities have different critical exponents.[40,41]) We may appeal to the work of Rudnick and Jasnow[10] who have calculated the helicity modulus $\Upsilon(t_0,u)$ to order ϵ. With $n>1$ they obtain

$$\beta\Upsilon=e^{-(d-2)l^*}[m^2(l^*)+\tfrac{1}{4}K_4]\ , \tag{6.36}$$

where $K_4=1/8\pi^2$, as before, while (noting that the parameter u in Ref. 10 is $\frac{1}{4}$ times that used here and in Ref. 9) we have

$$m^2(l^*)=-t_0(l^*)/4u(l^*)\ ,\qquad u(l^*)=ue^{\epsilon l^*}/4Q(l^*)\ , \tag{6.37}$$

$$Q(l^*)=1+\tilde{u}(e^{\epsilon l^*}-1)\ ,\qquad t_0(l^*)=t_0e^{2l^*}[Q(l^*)]^{-2g_n}\ . \tag{6.38}$$

Furthermore, the ordered phase, in which Υ is defined, is determined by the condition[10]

$$t_0(l^*)=-\tfrac{1}{2}\ , \tag{6.39}$$

which serves to fix $l^*(t_0,u)$. Using this and eliminating l^* in (6.38) yields an equation for $Q(l^*)\equiv Q_-$ which is identical to (6.19). In terms of this function one finds that $e^{-(d-2)l^*}m^2(l^*)$ is equal to m_0^2 as given in (6.17) while the helicity modulus can be written

$$\beta\Upsilon=(-t_0/u)Q_-^{6/(n+8)}\times\{1+\tfrac{1}{2}\epsilon(n+8)^{-1}[1-(1-\tilde{u})Q_-^{-1}]\}\ . \tag{6.40}$$

This equation together with (6.19), (6.34), and (6.35) provide all the ingredients necessary to calculate ρ_s and the corresponding scaling function under the constraint of constant density. Readers interested only in the final expression should omit the balance of this subsection: the results for the dilute Bose gas (for which we need only $n=2$) are given in (6.58)–(6.67) below.

To proceed, we again look directly for scaling forms. We may still use (6.23) to define the nonlinear scaling field $\dot{u}$ but it proves more appropriate below the transition to use a modified scaling field for t_0. Accordingly, we put

$$\dot{t}_-=-2t_0/(1-\tilde{u})^{2g_n}\quad\text{and}\quad x_-=\dot{u}/\dot{t}_-^{\,\epsilon/2}\ . \tag{6.41}$$

Then if $\dot{Q}(x)$ is defined, *as before,* to be the solution of (6.25) we find from (6.19) that

$$Q_-(t_0,u)=(1-\tilde{u})\dot{Q}(x_-)\ ,$$

i.e., the basic functional equation applies *both* above and below T_c. The helicity modulus may now be cast in scaled form as

$$\beta\Upsilon=(\dot{t}/\dot{u})W(x_-)\ , \tag{6.42}$$

with scaling function

$$W(x)=(1/2u^*)\dot{Q}^{\,6/(n+8)}[1+\tfrac{1}{2}\epsilon(n+8)^{-1}(1-\dot{Q}^{\,-1})]\ . \tag{6.43}$$

For small x and large x we thus find

$$2u^*W(x)=1+\frac{12+\epsilon}{2(n+8)}x+O(x^2) \quad \text{as } x\to 0\ ,$$
$$\approx\left[1+\frac{\epsilon}{2(n+8)}\right]x^{\omega_n'}\left[1+\frac{6[1+O(\epsilon)]}{(n+8)x^{1/(1-g_n\epsilon)}}\right] \quad \text{as } x\to\infty\ , \tag{6.44}$$

with $\omega_n'=6/(n+8)(1-g_n\epsilon)$. It follows that when $t_0\to 0-$ for $u>0$ (i.e., when $x_-\to\infty$) one obtains

$$\beta\Upsilon\sim|t_0|^{\,v} \text{ with } v=(1-\tfrac{1}{2}\epsilon)/(1-g_n\epsilon)\ , \tag{6.45}$$

where v is the critical exponent for the n-vector helicity modulus correct to $O(\epsilon)$. Since[26] $\eta=O(\epsilon^2)$ we have[40,41] $v=2\beta+O(\epsilon^2)$ so that, to first order in ϵ, both Υ and m_0^2 or, equivalently, $\rho_s(T)$ and $n_0(T)$ have equal exponents. Nevertheless, the crossover scaling functions are *different* as follows by comparing (6.40) with (6.17).

When $u\to 0$ the helicity modulus evidently diverges thereby reflecting the instability of the Gaussian model below the transition. This divergence is, of course, removed if the *constraint* is imposed. Again we would like to find a scaling form: to this end let us put

$$\tilde{t}=\bar{t}(1-\tilde{u})^{g_n} \text{ and } y=\dot{u}/\tilde{t}^{\,\epsilon/(2-\epsilon)}\ , \tag{6.46}$$

where $\bar{t}$ is the constrained critical-temperature variable introduced in (6.34). Combining (6.34) and (6.35) yields, in the first place,

$$\frac{nK_d u^*}{2(d-2)}\tilde{t}=\frac{\dot{t}}{(4-n)\dot{u}}\left[\dot{Q}^{\,(4-n)/(n+8)}\frac{1-\frac{1}{4}\epsilon(1-\dot{Q}^{\,-1})}{1-g_n\epsilon(1-\dot{Q}^{\,-1})}-\frac{n}{4}(1-\tilde{u})^{(n-4)/(n+8)}\right]\ , \tag{6.47}$$

where the argument of $\dot{Q}$ here is $x_-=\dot{u}/\dot{t}_-^{\,\epsilon/2}$. This equation relates $\dot{t}$ to $\tilde{t}$ and $\dot{u}$ [and is analogous to (3.34) in Sec. III B].

If one now divides through by $\dot{u}^{\,(2/\epsilon)-1}$ and substitutes for $\tilde{t}$ and $\dot{t}$ in terms of the scaled variables y and x_-, the constraint would be entirely in scaling form were it not for the term proportional to n. In the crossover region, however, $u\to 0$ and so $\tilde{u}$ is small; thus, at the cost of neglecting one of the corrections to scaling, we can set $\tilde{u}=0$ in the troublesome term. This leads to the fully scaled form

$$\frac{nK_d u^*}{2(d-2)}y^{1-(2/\epsilon)}=\frac{x_-^{-2/\epsilon}}{4-n}\left[\dot{Q}^{\,(4-n)/(n+8)}\frac{\dot{Q}-\frac{1}{4}\epsilon(\dot{Q}-1)}{\dot{Q}-g_n\epsilon(\dot{Q}-1)}-\frac{n}{4}\right]. \tag{6.48}$$

The constraint now represents an equation for x_- in terms of y with a solution, say, $x_-=\Xi(y)$. Solving the equation for y small and y large yields

$$\Xi(y)=\Xi_0 y^{1-(\epsilon/2)}\{1+[d\Xi_0/(n+8)]y^{1-(\epsilon/2)}+\cdots\} \quad \text{as } y\to 0\ ,$$
$$=\Xi_\infty y^{[1-(\epsilon/2)]/(1-\alpha_n)}[1+O(y^{-\tau_n})] \quad \text{as } y\to\infty\ , \tag{6.49}$$

where the amplitudes are

$$\Xi_0=[(d-2)/2nK_d u^*]^{\epsilon/2}\ ,$$
$$\Xi_\infty=n(4-n)K_d u^*(1-g_n\epsilon)/2(d-2)(1-\tfrac{1}{4}\epsilon)\ , \tag{6.50}$$

while

$$\tau_n=(1-\tfrac{1}{2}\epsilon)/[1-3\epsilon/(n+8)]\ . \tag{6.51}$$

Furthermore, the parameter

$$\alpha_n=\tfrac{1}{2}(4-n)\epsilon/(n+8)(1-g_n\epsilon) \tag{6.52}$$

is equal to the n-vector specific-heat exponent up to a correction factor $[1+O(\epsilon)]$.[26] Thus we see that when $y\to\infty$ a standard exponent renormalization[28] takes place as anticipated in Sec. III B.

Finally, using (6.42), the constrained helicity modulus can be written in scaled form as

$$\beta\Upsilon=[nK_d/(d-2)]\tilde{t}\,Y(y)\ , \tag{6.53}$$

where the scaling function is given by

$$Y(y)=(d-2)W[\Xi(y)]y^{(2/\epsilon)-1}/nK_d[\Xi(y)]^{2/\epsilon} . \qquad (6.54)$$

From (6.44) and (6.48) we obtain the expansions

$$\begin{aligned} Y(y)&=1+\frac{6\Xi_0}{n+8}[1+O(\epsilon)]y^{1-(\epsilon/2)}+O(y^{2-\epsilon}) \quad \text{as } y\to 0 , \\ &=Y_\infty y^{2g_n\tau_n}[1+O(y^{-\tau_n})] \quad \text{as } y\to\infty , \end{aligned} \qquad (6.55)$$

where, recalling (6.49), we have

$$Y_\infty=[(d-2)/2nK_d u^*][1+\epsilon/2(n+8)]\Xi_\infty^{\tau_n} , \qquad (6.56)$$

and remark, for use below, that by (6.50) we have $Y_\infty \propto (u^*)^{\tau_n-1}$. The prefactors appearing in (6.53) thus serve to normalize $Y(0)$ to unity. Notice, however, that $Y(y)$ does *not* have a regular expansion in powers of y; equivalently, then, the constrained superfluid density does not have a regular perturbation expansion in powers of v_0. This reflects the fact that even in the ideal limit the constraint induces exponent renormalization[28] via the factor $1/(1-\alpha_G)$ where the pure Gaussian exponent is $\alpha_G=\frac{1}{2}\epsilon$. (See also Sec. III B.)

This completes our general derivation of the constrained ϵ-expansion expressions. To study the superfluid density we may set $n=2$ and use the correspondences with the Bose system, including[40]

$$\rho_s(T)=(m^*/\hbar)^2k_\Lambda^{d-2}\Upsilon(T) . \qquad (6.57)$$

Note that the factor of k_Λ^{d-2} again results from the spatial rescalings (6.2) and (6.6): Υ has dimensions $1/L^{d-2}$ since it is an energy density divided by the gradient of the phase squared.[40]

E. Superfluid density scaling function

For ease of reference, here we collect and restate (for the case of $n=2$) all the results needed for expressing the superfluid density in scaled form. From (6.25), (6.31), (6.34), (6.43), (6.48), (6.53), and (6.54) we can write the superfluid density for $\epsilon\equiv 4-d>0$ as

$$\begin{aligned} \rho_s(T)&=m^*\rho(T/T_c)^{d/2}\tilde{t}\,Y(y) \\ &\approx\rho_s(0)[1-(T/T_c)^{d/2}]Y(\tilde{u}/\bar{t}^{\tilde{\phi}}) , \end{aligned} \qquad (6.58)$$

where the renormalized crossover exponent (see Secs. III B and IV B) is

$$\tilde{\phi}=\tfrac{1}{2}\epsilon/(1-\tfrac{1}{2}\epsilon)=(4-d)/(d-2) , \qquad (6.59)$$

while the temperature deviation has been taken as

$$\bar{t}=(T_c/T)^{d/2}-1 . \qquad (6.60)$$

In writing the second line of (6.58) we have used (6.46) for $\tilde{t}$ and replaced factors $1-\tilde{u}$ by unity as appropriate in the crossover region where $v_0\sim u_0\sim u\to 0$. In the first line the density ρ reentered through the ideal-Bose-gas critical-point relation (6.31). In this limit we may certainly identify $m^*\rho$ as equal to $\rho_s(0)$ as in the second line. More generally, however, there will be corrections proportional to v_0 of the sort dropped in going from $\tilde{t}$ to $\bar{t}$. In connection with the experiments in Vycor,[1,2] it should also be recalled from Sec. I, that the density, ρ, of the excess, nonlocalized helium, is *not* accessible to direct observation whereas $\rho_s(0)$ and T_c can be measured.

The interaction variable, $\tilde{u}$, can be expressed as

$$\tilde{u}=u/u^*, \quad u^*=\epsilon/10K_4$$

with

$$K_d=2/(4\pi)^{d/2}\Gamma(\tfrac{1}{2}d) , \qquad (6.61)$$

so that $K_4=1/8\pi^2$, and

$$u=\frac{8\pi^2}{\Gamma_d^\epsilon}\left[\frac{\beta v_0}{\Lambda_T^d}\right]=\frac{16\pi^{(d+2)/2}}{\Gamma_d^\epsilon\Gamma(\frac{1}{2}d-1)}\left[\frac{a}{\Lambda_T}\right]^{d-2} , \qquad (6.62)$$

in which we have used (2.17) while

$$\Gamma_d^{d-2}=(d-2)\zeta(\tfrac{1}{2}d)/4\pi K_d . \qquad (6.63)$$

Recall that $\Lambda_T=h/(2\pi m^*k_BT)^{1/2}$ is the thermal de Broglie wavelength while a is the scattering length.

To express the scaling function $Y(y)$, first let $Z(x)$ be the solution of

$$Z=1+xZ^{\epsilon/5} , \qquad (6.64)$$

and then define $\Xi(y)$ to be the solution of

$$Z(\Xi)^{1/5}\frac{1-\frac{1}{4}\epsilon[1-Z(\Xi)^{-1}]}{1-\frac{1}{5}\epsilon[1-Z(\Xi)^{-1}]}=\frac{1}{2}+\frac{K_d\epsilon}{5K_4(d-2)}y^{1/\tilde{\phi}}\Xi^{2/\epsilon} , \qquad (6.65)$$

with K_d as in (6.61). The behavior of $\Xi(y)$ for y large and small is given by putting $n=2$ in (6.49). Lastly take

$$W(x)=\frac{5K_4}{\epsilon}Z(x)^{3/5}\left[1+\frac{\epsilon}{20}xZ(x)^{(\epsilon/5)-1}\right] . \qquad (6.66)$$

Then the scaling function is finally given by

$$Y(y)=\frac{d-2}{2K_d}W[\Xi(y)]y^{1/\tilde{\phi}}/\Xi(y)^{2/\epsilon} . \qquad (6.67)$$

Now Y is normalized so that $Y(0)=1$. Thus putting $\tilde{u}\propto v_0=0$ in (6.58) correctly reproduces the ideal-Bose-gas result. The behavior of $Y(y)$ for small and large y follows from (6.55) with $n=2$. In fact, one finds that Y diverges like $y^{(\zeta_0-\zeta)/\tilde{\phi}}$ as $y\to\infty$, where the ideal superfluid exponent is $\zeta_0=1$ while the constrained interacting exponent is then given by

$$\zeta=(1-\tfrac{1}{2}\epsilon)/(1-3\epsilon/10) . \qquad (6.68)$$

Correct to order ϵ this is equal to $v/(1-\alpha)$ [see (6.45) and (6.52)] and hence ζ exhibits the expected exponent renormalization.[28]

From (6.62) we see that for fixed, or slowly-varying interactions we have $u\sim T_c^{(d-2)/2}$ in the critical region. This shows how, for $d>2$, the system becomes increasingly ideal as the transition temperature drops (with decrease of density). It also becomes clear that we have justified the original scaling ansatz (1.4) with crossover exponent ϕ_T as given in (1.5). The corresponding scaling function is

$$Y_T(y_T)=Y(y_T^{(d-2)/2}) , \tag{6.69}$$

with amplitude

$$E=[(\tfrac{1}{2}d)^{\tilde{\phi}}\Gamma_d^{\epsilon}\Gamma(\tfrac{1}{2}d-1)/16\pi^2]^{2/(d-2)}(2\hbar^2/m^*k_Ba^2) .$$

In this identification we have dropped higher-order differences as between t and $\bar{t}$, etc. We may hope, however, that the specific forms appearing in (6.58) will result in improved accuracy over a somewhat wider temperature range.

In general it is necessary to solve for $Y(y)$ numerically. This is most easily accomplished by treating x as the independent variable, thus solving for Y parametrically by finding $y=\Xi^{-1}(x)$ from (6.65) and $Y(\Xi^{-1}(x))$ from (6.67). Only the calculation of $Z(x)$ is then nontrivial but the defining equation (6.64) is readily solved by iteration. This procedure has been carried out for $d=3$ (i.e., $\epsilon=1$) and the resulting curve has been fitted to the data for helium in Vycor with appreciable success: see Fig. 1 of Ref. 6. It is instructive to discuss some of the quantitative aspects of this fit.

F. Fit to experimental data

Now it follows from (6.58)–(6.62) that only a *single* fitting parameter is entailed in comparing our scaling result to experimental data. Specifically, if m is the true mass of a ^{4}He atom this parameter can conveniently be taken as

$$\bar{a}=a(m^*/m)^{1/2} . \tag{6.70}$$

It must also be recognized at the outset that the exponent ζ when evaluated from (6.68) with $\epsilon=1$, which yields $\zeta\simeq5/7\simeq0.7143$, must *differ* somewhat from the true experimental value, for which one has[1,2] $\zeta\simeq0.674$ as measured in the bulk fluid, or as observed in Vycor, 0.64 ± 0.05. Thus one certainly cannot hope for a perfect fit to the data even though they do scale very well[2,6] with $\tilde{\phi}=1$, which is *exact* even for $d=3$. By the same token the true scaling function must diverge for large y as $y^{1-\zeta\simeq0.33}$ while, by (6.55), our $O(\epsilon)$ result diverges as $y^{2/7\simeq0.29}$. Nevertheless, a remarkably good fit can be achieved[6] by adjusting $\bar{a}$ to match the amplitude of the divergence for y not too large. This yields[6] $\bar{a}_{\rm fit}\simeq200$ Å.

Now for pure gaseous helium one has $m^*=m$ and $a_{\rm gas}\simeq2.2$ Å. The 85- to 95-fold discrepancy between the fitted value of $\bar{a}$ and $a_{\rm gas}$ is, at first sight, rather disturbing even though it seems very difficult to estimate theoretically what ratios m^*/m and $a/a_{\rm gas}$ should apply in Vycor. However, one can actually estimate the *effective mass ratio*, m^*/m, from the data themselves. Specifically, one may examine plots of $T_c(\bar{\rho})$ versus the overall fitting density $\bar{\rho}$ or versus $\rho_s(0)$. For a true ideal Bose gas one would have, from (6.31),

$$\rho_s(0)=\rho=\zeta(\tfrac{3}{2})(2\pi m^*k_B/h^2)^{3/2}T_c^{3/2} , \tag{6.71}$$

so that above the onset density ρ_0 a plot of $T_c^{3/2}$ versus $\rho_s(0)$ should show linear behavior for small enough T_c. In fact such a plot displays noticeable downwards curvature but becomes reasonably linear for $T_c\lesssim25$ mK and suggests a definite, nonzero limiting slope. If this measured slope is attributed to m^* one concludes

$$m^*/m=1.5\pm0.2 . \tag{6.72}$$

This value seems fully acceptable from a theoretical viewpoint and is certainly not inconsistent with the picture of helium in Vycor we have adopted. However, the fitted value of the scattering length, $a_{\rm fit}$, would then be about 160 Å which is still much larger than seems reasonable.

It must be realized, however, that the value of the parameter $u^*=\epsilon/10K_4$ plays a very large role in determining $a_{\rm fit}$ since it enters as a factor (i) in the argument $y=\tilde{u}/\bar{t}$ of the scaling function and (ii) in the amplitude of the divergence of $Y(y)$ on which the fitting was based. To explore this, consider the scaled quantity

$$\mathscr{Y}(T;T_c)=\rho_s(T;T_c)/\rho_s(0;T_c)[1-(T/T_c)^{3/2}] , \tag{6.73}$$

appropriate for $d=3$. Our theory, explicitly (6.58), (6.53), and (6.55) for $d=3$ with $\tilde{\phi}=1$, etc., predicts

$$\mathscr{Y}_{\rm theor}(T;T_c)\approx\mathscr{A}_{\rm theor}(a,u^*)T_c^{1/7}/\bar{t}^{2/7} , \tag{6.74}$$

with

$$\mathscr{A}_{\rm theor}(a,u^*)=\mathscr{A}_0(a/u^{*2})^{2/7} , \tag{6.75}$$

where $\mathscr{A}_0$ depends only on absolute constants and m^*. Now, even though the exponents here are not quite correct, the experimental data[2,6] show that (6.74) provides a good description (for $\bar{t}$ not too small) with an observed amplitude, say, $\mathscr{A}_{\rm expt}$. Fitting $\mathscr{A}_{\rm theor}$ to this gives

$$a_{\rm fit}=(u^*)^2(\mathscr{A}_{\rm expt}/\mathscr{A}_0)^{7/2} . \tag{6.76}$$

Evidently, then, $a_{\rm fit}$ is, all other things being equal, proportional to $(u^*)^2$.

Now as mentioned originally, u^*, as the fixed point value of u, is strongly dependent on d; furthermore, in contrast to the critical exponents, its ϵ expansion is not so reliable in low orders. To first order in ϵ we found (for $n=2$) $u^*=\epsilon/10K_4$ with $K_4=1/8\pi^2$; by examining the origin of u^* in the renormalization-group theory one sees[26] that a rough and ready estimate of this value for larger ϵ is obtained by replacing K_4 by K_d: see (6.61). Since we have $K_3=1/2\pi^2$ this yields a *fourfold* reduction in u^* relative to the original $\epsilon=1$ estimate. Accepting that and (6.72) for m^*/m, the value of $a_{\rm fit}$ is reduced by a factor of 16 to $a_{\rm fit}\simeq10$ Å: that is certainly more reasonable although, perhaps, still somewhat large.

It should be recalled, however, that $u^*(d)$ is, beyond $O(\epsilon)$, *not* a universal quantity, unlike the exponents. Conceptually, and in leading orders, however, u^* is closely linked to the *renormalized coupling constant, g*, which *is* universal. Furthermore, the field-theoretic-based numerical estimates of critical properties in $d=3$ dimensions due to Baker, Nickel, and co-workers[42,43] give values for g. In a normalization which gives $g\approx(n+8)u^*/8\pi$ for small ϵ, the numerical work[42] yields $g\simeq1.406$ for $d=3$, $n=2$. By contrast the truncated ϵ expansion yields $g\simeq\pi$, which is larger by a factor of about $2.23\simeq\sqrt{5}$. A fivefold reduction of $a_{\rm fit}$ gives 32 Å, which is still rather large; but, by both routes, a significant reduction from $a_{\rm fit}\simeq160$ Å is clearly called for.

The arguments just presented, while suggestive, are rather *ad hoc*. A more systematic estimation method is open if one recalls that we know the leading behavior of the susceptibility scaling function, $X(x)$, and the nonuniversal metrical factors for x both in the ϵ expansion *and*, by perturbation theory, for *general* $d<4$. In the latter case the results (3.8), (4.8), (4.16), (4.19), (4,20), and (4.42) show the scaling variable is

$$x=B\left[\frac{\beta v_0}{\Lambda_T^d}\right]\Big/\left[-\beta\mu-2\tilde{D}\left[\frac{\beta v_0}{\Lambda_T^d}\right]\right]^{\phi}, \tag{6.77}$$

with $B=-2\Gamma(1-\frac{1}{2}d)$ and $\tilde{D}=\zeta(\frac{1}{2}d)$. In the former case, (6.24), (6.27), and (3.4) show that x should be compared with $2x_+/5$. By (6.4), (6.8), etc., this yields

$$B=(4\pi)^{d/2}/5u^*$$

and (6.78)

$$\tilde{D}=(d-2)\epsilon\Gamma(\tfrac{1}{2}d)\zeta(\tfrac{1}{2}d)(4\pi)^{d/2}/40u^* .$$

It is easily checked that both expressions for B and $\tilde{D}$ *agree precisely* to order ϵ! This represents yet a further check on the scaling interpretation of perturbation theory and on the model matching procedures. Furthermore, optimal values for $u^*(d)$ for general d can be found by equating the two expressions for B, which yields

$$u^*(d=3)=2\pi/5\simeq u^*(\epsilon=1)/6.28$$

or for $\tilde{D}$, leading to

$$u^*(d=3)=\pi^2/10=u^*(\epsilon=1)/8 .$$

To within $\pm 13\%$ the two routes indicate the same correction factor of about 7.1. (The differences, of course, reflect the fact that the scaling function to $O(\epsilon)$ cannot be completely correct or consistent even if the "correct" value of u^* is inserted.)

Finally, this argument for $u^*(d=3)$ yields a value for $a_{\rm fit}$ of about (160 Å)/50=3.2 Å. This is almost too close to the gas value for comfort! But, by any measure, it is clear that the quantitative aspects of fitting our theory to the data of Reppy, Crooker, and co-workers,[1,2] are quite consistent with a picture of the excess, mobile helium in Vycor acting as a weakly interacting Bose fluid with an effective mass $m^*\simeq 1.5m$, and effective pair-interactions characterized by a repulsive part similar to that between free helium atoms but with a much reduced attractive tail (since no gas-liquid phase separation is found in Vycor). In closing this discussion, however, it should be remembered that we have, as yet, given no account of the effects of the random, amorphous nature of Vycor on the crossover from ideal behavior: further consideration of that issue is deferred.[8] In the next and last section other approaches to generating the ϵ expansion for a Bose fluid are briefly reviewed.

VII. OTHER APPROACHES

As far as we know, calculations of the helicity modulus in the critical region have been restricted to classical spin systems. The mapping of Sec. V was therefore a necessary step, permitting us to make use of these existing calculations. It is interesting, then, to explore alternate methods for achieving this mapping.

A. Quantum-mechanical recursion relations

Several authors have considered the application of renormalization-group methods directly to the interacting quantum-mechanical Bose gas.[44] Thus Singh[44] has used Wilson's momentum shell integration method[26,45] to derive recursion relations to order ϵ. In order to avoid changing the Bose operator commutation relations under renormalization, Singh chose to absorb the normal spin rescaling factor into a renormalized particle mass; this then is found to flow to infinity under the group action. While the fact that, near the fixed point the particles are very massive, gives one an intuitive feel for the essentially classical nature of the critical point, it unfortunately leaves one with a somewhat ill-defined fixed-point Hamiltonian. Alternatively, following Lee,[46] one may allow the Bose operators to be rescaled and one then discovers that their commutators flow towards zero; another indication of classical behavior. However, the quantum-mechanical nature of the fixed-point Hamiltonian remains somewhat obscure. Nevertheless at the level of diagrammatics the two interpretations are entirely equivalent and lead to the same set of recursion relations.

Let us now examine these recursion relations explicitly. It will be seen that the mapping of Sec. V can in fact be derived from them, although not in the way originally envisioned by Singh.[44] It is convenient to change notation and introduce the definitions

$$s(l)=\beta\hbar^2q_\Lambda^2/2m(l) , \tag{7.1}$$

which embodies the intrinsically quantum-mechanical character, and

$$r(l)=-2m(l)\mu(l)/\hbar^2q_\Lambda^2 , \tag{7.2}$$

$$v(l)=8m^2(l)v_0(l)/\beta\hbar^4q_\Lambda^{4-d} , \tag{7.3}$$

where q_Λ is a cutoff. The recursion relations derived by Singh[44] are then, in differential form,

$$ds/dl=-2s , \tag{7.4}$$

$$dr/dl=2r+svh_1(s,r) , \tag{7.5}$$

$$dv/dl=\epsilon v-(5/2)sv^2h_2(s,r) , \tag{7.6}$$

where the functions $h_1(s,r)$ and $h_2(s,r)$ are given below.

The cutoff entering here should be fixed, once and for all at the beginning of the calculation and should be *temperature independent*: it is reasonable, in fact, to take $q_\Lambda\propto 1/a$ where a is the scattering length or effective atomic diameter. However, Singh[44] with no discussion, assigned this cutoff a value $q_\Lambda\sim 1/\Lambda_T$, a step which seems quite unjustifiable. Indeed, as we will show, Singh's assumption actually proves to lead to incorrect answers in the weakly interacting, low-temperature limit. On the other hand, the *effective* $1/\Lambda_T$ cutoff discovered in Sec. V will arise naturally from an analysis of the recursion relations: it does *not* have to be put in by hand. Indeed it is instructive to see how Singh's approach, if properly implemented, will reproduce our results.

To this end we need the functions entering into the recursion relations, namely,

$$h_1(s,r)=K_d/(e^{s(1+r)}-1)\ , \tag{7.7}$$

$$h_2(s,r)=\tfrac{1}{s}K_d\left[\frac{\frac{1}{2}\coth[\frac{1}{2}s(1+r)]}{1+r}+\frac{s}{\sinh^2[\frac{1}{2}s(1+r)]}\right]. \tag{7.8}$$

It should be noted that there is a discrepancy between the first part of the expression for $h_2(s,r)$ here and Singh's Eq. (23) as printed in Ref. 44(a); however, the difference is of no consequence in Singh's further calculations nor in our analysis here. The important features follow by noting from (7.4) that $s(l)$ vanishes as $l\to\infty$ and that

$$\lim_{s\to 0}[sh_1(s,r)]=K_d/(1+r)\ , \tag{7.9}$$

$$\lim_{s\to 0}[sh_2(s,r)]=K_d/(1+r)^2\ . \tag{7.10}$$

Then, if one puts $v=16u$ the recursion relations for r and u in the limit $s\to 0$ are exactly those normally derived to order ϵ for the s^4 model with $n=2$.[12] This establishes the classical nature of the critical behavior of the interacting Bose gas at finite temperature.

Note, however, that since, with our specification of the cutoff, we have $s(0)\propto\Lambda_T^2/a^2$, the low-density, $T_c\to 0$ limit corresponds to large s rather than to $s\to 0$! It appears then that there should be three significant fixed points in the problem: the two familiar ones $G_0\equiv(r=0,\ v=0,\ s=0)$ standard Gaussian, and $C\equiv(r=r^*,\ v=v^*,\ s=0)$ $(n=2)$-criticality, as well as a zero-temperature Gaussian-like fixed point $G_\infty\equiv(r=0,\ v=0,\ s=\infty)$. The reduced set of equations (7.5) and (7.6) with $s\equiv 0$ describe completely the flow near the first two fixed points but, evidently, the limiting dilute Bose gas corresponds to the third one which was not considered by Singh. Indeed, with his assignment of the cutoff one finds the initial value $s(0)=1/4\pi$ which is independent of all physical parameters so that there is *no* way of describing or investigating the low-temperature limit. What must concern us, however, is the scaling behavior about this new, $s=\infty$ fixed point, G_∞.

Accordingly we will calculate s-dependent scaling fields in the limit of large s and small r, v. To first order in r, v we obtain from (7.5) and (7.6)

$$dr/dl=2r+K_d sv/(e^s-1)\ , \tag{7.11}$$

$$dv/dl=\epsilon v\ . \tag{7.12}$$

We require a combination $t=r+A(s)v$ which renormalizes in a purely multiplicative way as

$$dt/dl=\lambda_1 t\ , \tag{7.13}$$

and can therefore be regarded as a linear scaling field. Substituting with (7.11) and (7.12) yields

$$dt/dl=2\{r+\tfrac{1}{2}v[sK_d/(e^s-1)-2s(dA(s)/ds)+\epsilon A(s)]\}\ , \tag{7.14}$$

which implies $\lambda_1=2$, and

$$2(dA/ds)+(d-2)A/s=K_d/(e^s-1)\ . \tag{7.15}$$

As $s\to 0$ the required solution of this equation must remain bounded so that one finds

$$A(s)=\tfrac{1}{2}s^{-(d-2)/2}K_d\int_0^s d\bar{s}\,\bar{s}^{(d-2)/2}/(e^{\bar{s}}-1)\ . \tag{7.16}$$

For large s the integral may be extended to ∞ so that, recalling the value of K_d, we find

$$A(s)=\zeta(\tfrac{1}{2}d)/(4\pi)^{d/2}s^{(d-2)/2}+O(e^{-s})\ . \tag{7.17}$$

The recursion relations finally yield

$$t\equiv r+A(s)v\approx s^{-1}[-\beta\mu+2\zeta(\tfrac{1}{2}d)(\beta v_0/\Lambda_T^d)]\ , \tag{7.18}$$

for $s\gg 1$. For Singh's original assignment, $q_\Lambda\propto\Lambda_T^{-1}$, there would be no grounds for letting $s\to\infty$ in (7.16) and this result for t could not be derived; however, this is just what is required for consistency with our previous results.

To make full contact with the results of Sec. IV, we may suppose that the thermodynamic quantity of interest, say the susceptibility $X(t,v)$, flows under renormalization as

$$dX/dl=\lambda_0 X(t(l),v(l),s(l))\ . \tag{7.19}$$

Following the usual procedures (see the Appendix) we may integrate this using (7.4), (7.12), and (7.13) from small initial values $t_{(0)}\equiv t(0)$, $v_{(0)}\equiv v(0)$, but large $s_{(0)}\equiv s(0)$, out to some noncritical matching locus determined by, say, an equation $C(t,v,s)=0$. This yields a scaling form

$$X_{(0)}\equiv X(t_{(0)},v_{(0)},s_{(0)})\approx t_{(0)}^{-\gamma}X_\infty(v_{(0)}/t_{(0)}^{\phi},s_{(0)}t_{(0)})\ , \tag{7.20}$$

where $\gamma=-\lambda_0/\lambda_1=-\frac{1}{2}\lambda_0$ and $\phi=\epsilon/\lambda_1=\frac{1}{2}\epsilon$ while the G_∞ scaling function is given by

$$X_\infty(x,z)=X(t^\dagger,\ xt^{\dagger\epsilon/2},\ z/t^\dagger)$$

in which $t^\dagger(x,z)$ satisfies

$$C(t^\dagger,\ xt^{\dagger\epsilon/2},\ z/t^\dagger)=0\ .$$

The matching locus, $C=0$, must, of course, lie within a region about G_∞ where the linear relations (7.12) and (7.13) remain valid. The *relevant* scaling combination in (7.20) is, on using (7.6) and (7.18), found to be

$$x=\frac{v_{(0)}}{t_{(0)}^{\phi}}=\frac{2(4\pi)^{d/2}(\beta v_0/\Lambda_T^d)}{[-\beta\mu+2\zeta(\frac{1}{2}d)(\beta v_0/\Lambda_T^d)]^{\epsilon/2}}\ . \tag{7.21}$$

Apart from a constant factor, this is *exactly* the linear scaling field combination derived in Sec. IV. Note, in particular, that all dependence on the cutoff q_Λ has cancelled out. The *effective cutoff*, Λ_T^{-1}, arises from the exponential character of the integral in (7.16). The second, *irrelevant* scaling combination in (7.20) is

$$z=s_{(0)}t_{(0)}=-\beta\mu+2\zeta(\tfrac{1}{2}d)(\beta v_0/\Lambda_T^d)\ . \tag{7.22}$$

This gives rise to corrections to scaling and is small in the critical region. [Note that $s_{(0)}t_{(0)}$ is *not* of order $s_{(0)}$ as might, at first, have been expected, because of the factor s^{-1} entering (7.18).]

It is not, at this point, clear, however, why the scaling function $X_\infty(x,z)$ for crossover from G_∞ to C should be the same as for crossover from G_0 to C which we have analyzed. To see this one must examine the flows in further detail. If one crossover scaling function is to be mapped onto another it is necessary that the flows from the first fixed point, G_∞ in this case, pass through the neighborhood of the second unstable fixed point, here G_0, before crossing over to the third, stable fixed point, C. Thus if

$$(t_*,v_*,s_*)\equiv[t(l_*),v(l_*),s(l_*)]$$

lies near G_0, and is a known function of the initial point $(t_{(0)},v_{(0)},s_{(0)})$, one need only substitute the functional relationship into the scaling form appropriate near G_0. Hence, we need to elucidate the conditions under which $s(l)$ becomes small, say equal to δs, *before* $v(l)$ becomes too large, say exceeding δv. For $v<\delta v$ the linear recursion relations (7.12) and (7.13) remain valid and fixing l_* via $s_*=s_{(0)}e^{-2l_*}=\delta s$ yields the condition $v_*=v_{(0)}e^{l_*}<\delta v$ and gives

$$t_*=t_{(0)}e^{-2l_*} \text{ and } \chi(l_*)=\chi_{(0)}e^{\lambda_0 l_*} . \tag{7.23}$$

Now by (7.1) and (7.3) we have $v_{(0)}=u_{(0)}/s_{(0)}$ where $u_{(0)}=2mv_0/\hbar^2q_\Lambda^{2-d}$ may be regarded as fixed. The conditions are thus satisfied if

$$s_{(0)}\equiv\hbar^2q_\Lambda^2/2k_BTm_{(0)}>(u_{(0)}/\delta v\, \delta s^{\epsilon/2})^{2/(2-\epsilon)} . \tag{7.24}$$

Consequently, if $T\simeq T_c$ (or $m_{(0)}$) are sufficiently small the flows have the required property.

Now the analog of (7.20) for flows starting near G_0 is just

$$\chi(t_*,v_*,s_*)\approx t_*^{-\gamma}X_0(v_*/t_*^{\epsilon/2},s_*t_*) . \tag{7.25}$$

If we suppose (7.24) is satisfied and use (7.23), etc., we obtain

$$\chi_{(0)}\equiv\chi(t_{(0)},v_{(0)},s_{(0)})$$
$$=\left[\frac{s_{(0)}}{\delta s}\right]^\gamma \chi\left[\frac{t_{(0)}s_{(0)}}{\delta s},\frac{v_{(0)}s_{(0)}^{\epsilon/2}}{(\delta s)^{\epsilon/2}},\delta s\right] , \tag{7.26}$$

the arguments on the right-hand side being just t_*, v_*, and s_*. On using (7.25) this yields

$$\chi_{(0)}\approx t_{(0)}^{-\gamma}X_0(v_{(0)}/t_{(0)}^{\epsilon/2},s_{(0)}t_{(0)}), \tag{7.27}$$

which should be compared with (7.20). We may conclude, taking $s_{(0)}$ sufficiently large (or T_c small) and making the identifications (7.21) and (7.22) for the scaled combinations, that the scaling function $X_\infty(x,z)$ is *identical* to $X_0(x,z)$.

As a final point suppose the condition (7.24) is *not* satisfied, so that the crossover occurs "directly" to the critical fixed point C with the flows not passing close to G_0. Then the full nonlinear equations (7.5) and (7.6) must be used. Under renormalization $s(l)$ will approach zero but $v(l)$ may be large enough to lie outside the crossover region from G_0 to C. Nevertheless, the system will eventually map onto the nonlinear scaling fields $\dot{u}$ and $\dot{t}$ of Sec. VI although these will now be more complicated functions of the starting parameters $t_{(0)}$, $v_{(0)}$, and $s_{(0)}$. Indeed, owing to the complexity of the functions h_1 and h_2 the precise relations are probably intractable; however, they can matter only when v_0 is not small. We have thus demonstrated how the explicit mapping from a Bose fluid to a spin model achieved in Sec. IV can be established on a more formal renormalization-group basis.

B. Path-integral approaches

The use of Feynman path integrals in statistical mechanics has become increasingly popular.[47] Various representations for the interacting Bose fluid partition function exist,[48] but the most useful seems to be one due originally to Bell.[49] The advantage of a path-integral representation is that everything is written in terms of classical commuting variables so that comparisons with classical-spin models should be more straightforward. More explicitly, one may consider the Hamiltonian in the form

$$\beta\mathcal{H}=\int_\Omega d^dr\int_0^\beta d\tau\,\Psi^*(\mathbf{r},\tau)[(\partial/\partial\tau)-(\hbar^2/2m)\nabla^2-\mu+w(\mathbf{r})]\Psi(\mathbf{r},\tau)$$
$$+\tfrac{1}{2}\int_\Omega d^dr\int_\Omega d^dr'\int_0^\beta d\tau\,|\Psi(\mathbf{r},\tau)|^2v(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}',\tau)|^2 , \tag{7.28}$$

where $\Psi(\mathbf{r},\tau)$ is a *classical* complex field. Of course, the similarity to the original quantal Hamiltonian (2.1)–(2.5) is not accidental: the new imaginary time variable, τ, is reminiscent of the corresponding interaction representation variable.[30] The similarity is further elucidated if one makes the plane-wave decomposition

$$\Psi(\mathbf{r},\tau)=V_\Omega^{-1/2}\sum_{\mathbf{k}}\sum_n a_{\mathbf{k},n}e^{-i\mathbf{k}\cdot\mathbf{r}-ik_n\tau} \text{ with } k_n=2\pi n/\beta . \tag{7.29}$$

The Hamiltonian may then be written

$$\mathcal{H}=-\sum_{\mathbf{k}}\sum_n[ik_n-(\varepsilon_{\mathbf{k}}-\mu)]a_{\mathbf{k},n}^*a_{\mathbf{k},n}+V_\Omega^{-1/2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\sum_n w_{\mathbf{q}}a_{\mathbf{k},n}^*a_{\mathbf{k}',n}\delta_{\mathbf{k},\mathbf{k}'+\mathbf{q}}$$
$$+\tfrac{1}{2}V_\Omega^{-1}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}'}\sum_{l,l',n,n'}v_{\mathbf{q}}a_{\mathbf{k},l}^*a_{\mathbf{k}',l'}a_{\mathbf{q},n}^*a_{\mathbf{q}',n'}\delta_{\mathbf{k}+\mathbf{q},\mathbf{k}'+\mathbf{q}'}\delta_{l+n,l'+n'} . \tag{7.30}$$

Note the appearance of $1/\mathscr{G}^0(\mathbf{k},ik_n)$, the inverse Matsubara Green's function, in the first term. Thermodynamic properties follow from the partition function as given by

$$Z=\prod_{\mathbf{k}}\prod_{m}\int_{-\infty}^{\infty}\frac{d\,\mathrm{Re}\{a_{\mathbf{k},m}\}\,d\,\mathrm{Im}\{a_{\mathbf{k},m}\}}{\pi\nu_{\mathbf{k},m}}e^{-\beta\mathscr{H}} \qquad (7.31)$$

with $\nu_{\mathbf{k},n}=(\zeta_{\mathbf{k}}-ik_n)^{-1}$ in which $\zeta_{\mathbf{k}}$ is defined via

$$2\sinh(\tfrac{1}{2}\beta\zeta_{\mathbf{k}})=\exp[\tfrac{1}{2}\beta(\varepsilon_{\mathbf{k}}-\mu)]\ . \qquad (7.32)$$

This normalization factor has been chosen so that Z reduces to the usual ideal noninteracting result, when $w=v\equiv0$;[48] however, it is of no real consequence since it effectively cancels out in *averages* which are ratios of two path integrals. From a perturbational point of view the equivalence of (7.28) to the original forms (2.1)—(2.5) is now clear: one readily sees that the classical diagrammatic expansion resulting from the last two terms in (7.30) is precisely the same as that for the quantal gas. Momentum and frequency conservation are manifest in the δ functions, and the extra Matsubara index on the Green's function is evident in the first term. The discussion in Sec. V D regarding the symmetry factors associated with each diagram remains valid. The choice $n=2$ is mandated by the complex nature of the $a_{\mathbf{k},l}$. Furthermore, if one ignores the $(\partial/\partial\tau)$ term in (7.28), and takes the real and imaginary parts of Ψ as the two components of a spin variable, $\vec{s}=(s^x,s^y),=(\mathrm{Re}\{\Psi\},\mathrm{Im}\{\Psi\})$, then (7.28) is precisely analogous to the classical-spin Hamiltonian [Eqs. (5.4) and (5.10)]. Such a truncation can, in fact, be justified via renormalization-group theory. An application of Wilson's momentum shell integration approach to (7.30) shows that the nonzero Matsubara frequencies flow to infinity under renormalization. Thus, in the same way that one recovers the classical action from the limit $\hbar\to0$ in the Feynman path-integral formulation of quantum mechanics, so one sees that only those fields $\Psi(\mathbf{r},\tau)$ with no τ dependence contribute to the behavior near a fixed point.

This illustrates the advantage of the path-integral approach: the quantal and classical models may be treated simultaneously within the same framework, and the crossover between them examined in detail. Wiegel,[50] and Creswick and Wiegel[51] have discussed these points, but their calculations require further comments. In neither paper has the rescaling of the field Ψ under renormalization been carried out properly: thus the critical point decay exponent η has not been correctly accounted for. (See, e.g., Ref. 52 for a clear discussion of this point in the analogous classical calculation.) Furthermore, the necessary renormalization of the various n-body interaction potentials has not been allowed for.[52] These shortcomings do not affect the validity of their final approximate closed form, which, although somewhat *ad hoc*, does reproduce the correct recursion relations close to the fixed point to order $\epsilon=4-d$. If carried through more systematically, the approach of Wiegel and Creswick should reproduce the recursion relations of Singh,[44] and therefore the mapping of Sec. V.

The study of the effects of quantum mechanics on critical behavior often goes under the name "zero-temperature crossover" since it is only as $T_c\to0$ that quantal corrections become important and can eventually overwhelm the leading, nonquantal behavior. The standard conclusion is that a phase transition in a quantal system at zero temperature which is driven, say, by some auxiliary parameter, g (e.g., the chemical potential in a Bose gas, or the transverse field in an Ising spin system), will have exponents characteristic of the corresponding classical system in one higher spatial dimension. The reasoning behind this[38] is that as $T_c\to0$, the gap $2\pi k_BT$ between successive Matsubara frequencies vanishes, and the discrete sum over frequencies becomes an integral over a continuous parameter. This parameter may, in certain cases, be interpreted as an extra momentum component; hence an increase in effective dimensionality follows.[38] Indeed Suzuki[53] has used the Trotter product formula to derive an explicit mapping of the $T\to0$ d-dimensional Ising model in a transverse field $g\equiv\Gamma$, onto the $(d+1)$-dimensional classical Ising model with Γ replacing the temperature. For finite T the mapping produces a $(d+1)$-dimensional model, but one which is anisotropic and of *finite* extent in the extra dimension, the "thickness' being proportional to $1/T$.

In spin language, a hard-core Bose gas on a lattice corresponds to a spin-$\frac{1}{2}$ quantal XY model in a transverse field.[54] However, from our present analysis it is clear that the naive add-a-dimension conclusion is not correct: Rather in the limit $T_c\to0$ the behavior must, in fact, become *Gaussian* as we have shown, since the relevant transformed interaction strength, varying as $T^{(d-2)/2}v_0$, vanishes as $T\to0$ for $d>2$. (Observe, however, that for $d<2$, the interactions dominate at low temperatures and nontrivial behavior must result.)

Walasek[55] has used field-theoretic methods to study the Hamiltonian (7.30) (with $w=0$). Indeed he has derived a $T_c\to0$ crossover scaling function for the *equation of state* to first order in ϵ; the results (as far as they go) are consistent with ours. The crossover scaling variable is found to be

$$y\propto T/h^{\epsilon/2}\ \text{ for } \epsilon=4-d>0\ , \qquad (7.33)$$

where $h\propto\mu_c-\mu$ (*not*, in fact, $(\mu_c-\mu)/\mu_c$ as is stated in Ref. 55: see Ref. 56). Multiplying numerator and denominator by $\beta^{\epsilon/2}$ yields

$$y\propto T^{(d-2)/2}/(\beta\mu_c-\beta\mu)^{\epsilon/2}\ , \qquad (7.34)$$

which agrees with the results of Sec. IV. Walasek's crossover scaling function for the equation of state above T_c yields the ideal results

$$\Lambda_T^d(\rho-\rho_c)\sim[\beta\Lambda_T^d(p-p_c)]^{(d-2)/2}$$

outside the crossover region, $\Lambda_T^d(p-p_c)\gtrsim k_BT_c$, and the expected, interacting result

$$\Lambda_T^d(\rho-\rho_c)\sim[\beta\Lambda_T^d(p-p_c)]^{d\nu-1}=[\beta\Lambda_T^d(p-p_c)]^{1-\alpha}$$

for $\Lambda_T^d(p-p_c)\ll k_BT_c$, where, correct to $O(\epsilon)$, ν and α are the correlation length and specific-heat exponent, respectively.

Earlier work by other authors has also served to elucidate the Gaussian nature of the low-temperature critical behavior. Hertz[38] has examined a range of model Hamiltonians of the form (7.30), but with ik_n in the first line replaced by various other expressions such as k_n^2, $|k_n|$, and k_n/k as appropriate to different physical situations. For each case he calculates the dynamical critical exponent z and finds that in dimensions greater than $d_> = 4-z$, the zero-temperature fixed point is Gaussian in character. Although he does not discuss the Bose case explicitly, his scaling arguments for the $|k_n|$ case should apply. For this he finds $z=2$, so that for $d>2$ the behavior should be Gaussian, in agreement with our findings.

Vvedensky and Creswick[57] have used the same techniques to discuss the $T_c \to 0$ behavior using (7.28) (with $w=0$) and do indeed find $z=2$. They derive recursion relations to order ϵ which (subject to minor discrepancies which we suspect represent misprints) are identical to those of Singh;[44] they show how the recursion relations interpolate between the Gaussian and s^4 forms for $T=0$ and $T>0$, respectively.

Finally, Gerber[58] has used a mean-field approximation to study the quantal XY model in the context of magnetism. Within his approximation he also finds a Gaussian-to-XY crossover at low temperatures but he argues for the validity of this qualitative conclusion beyond the mean-field approximation.

In concluding this overview of other approaches to the problem of criticality in an interacting Bose fluid at low temperatures, we stress that none of the earlier authors addressed crossover in the equation of state below T_c nor did they consider the superfluid density.

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APPENDIX: NONLINEAR RENORMALIZATION-GROUP FLOWS AND LOGARITHMIC FACTORS

The perturbation-theory analysis at the end of Sec. IV D reveals a special role for $d=3$: logarithmic functions of μ appear at second order in the Green's-function expansion and, presumably, are present in higher order also. Such behavior is, in fact, to be expected on the basis of renormalization-group arguments whenever the exponent ϕ is a rational number, say p/q. Assuming p and q are mutually prime the denominator q is precisely the order of perturbation theory in which the logarithms will, typically, first appear. In our case we have $1>\phi>0$ for $2<d<4$ and so the smallest value of q is 2; furthermore, we have $\phi=\frac{1}{2}$, for $d=3$, so that further investigation is worthwhile. In order to elucidate the appearance of logarithms in this case we follow closely a derivation of Barma and Fisher[34] based on Wegner's original discussion.[25]

Within a renormalization-group formulation the nonlinear scaling fields are defined as those combinations of the physical variables, T, μ, v_0, etc. which renormalize trivially, i.e., merely by a factor $e^{\lambda l}$, under the group action, where l is the parameter governing flow in the space of Hamiltonians and lengths rescale by a factor of e^{-l}. Thus if $\hat{t}, \hat{g}$ are nonlinear scaling fields, one has, at least formally,

$$\frac{d\hat{t}(l)}{dl} = \lambda_1 \hat{t}(l), \quad \frac{d\hat{g}(l)}{dl} = \lambda_2 \hat{g}(l) , \tag{A1}$$

for appropriate renormalization-group eigenvalues λ_1, λ_2. By contrast the corresponding *linear* scaling fields, say t and g, will satisfy these equations only to linear order: in higher order the renormalization group will specify recursion relations

$$\begin{aligned} \frac{dt}{dl} &= \lambda_1 t + a_1 g^2 + a_2 g t + a_3 t^2 + \cdots , \\ \frac{dg}{dl} &= \lambda_2 g + b_1 g^2 + b_2 g t + b_3 t^2 + \cdots , \end{aligned} \tag{A2}$$

where, for simplicity, we neglect further, irrelevant fields. In the present case we have $g=\beta v_0/\Lambda_T^d$ and $t=-\beta\mu+2\zeta(\frac{1}{2}d)g$ [see (4.16) and (4.42)]. For the purposes of bringing out the essential features we consider here only the special case in which $a_2=a_3=b_1=b_2=b_3=0$ while $a_1 \equiv D_0 \neq 0$. Alternatively, we may assume[34] that t and g are an intermediate set of algebraic (nonlogarithmic) scaling fields which have already absorbed the nonlinearities associated with the coefficients assumed to vanish. Note, however, that in our analysis we know $\hat{g}\equiv 0$ for $v_0=0$ so that b_3 vanishes in any case. [See the discussion following (3.10).] Further, it will become evident below that the $a_1 g^2$ term is the significant one.

In addition to the flow equations for the fields we require one for the thermodynamic function of interest, say, $G(t,g)$, in our case the susceptibility. Again for simplicity suppose this is multiplicatively renormalized according to

$$\frac{d}{dl} G(t(l),g(l)) = \lambda_0 G(t(l),g(l)) \tag{A3}$$

with, to recapitulate,

$$\frac{dt(l)}{dl} = \lambda_1 t(l) + D_0 g^2(l), \quad \frac{dg(l)}{dl} = \lambda_2 g(l) . \tag{A4}$$

Now it is easy to check that

$$\hat{t} = t + \frac{D_0}{(\lambda_1 - 2\lambda_2)} g^2, \quad \hat{g} = g , \tag{A5}$$

are nonlinear scaling fields provided $\lambda_1 - 2\lambda_2 \neq 0$. Integrating the resulting equations (A1) and (A3) up to a matching value, $l^\dagger$, yields

$$\hat{t}^{\dagger} \equiv \hat{t}(l^{\dagger}) = \hat{t}_0 e^{\lambda_1 l^{\dagger}}, \quad g^{\dagger} \equiv g(l^{\dagger}) = g_0 e^{\lambda_2 l^{\dagger}},$$
$$G^{\dagger} \equiv G(t^{\dagger}, g^{\dagger}) = G(t_0, g_0) e^{\lambda_0 l^{\dagger}}, \tag{A6}$$

where $\hat{t}_0 = \hat{t}(t_0, g_0)$. The value of $l^{\dagger}$ is to be chosen so that $\hat{t}^{\dagger}$ lies outside the critical region where the function $G(t^{\dagger}, g^{\dagger})$ can be regarded as known.[12] It is convenient here to fix $l^{\dagger}$ by the condition that $G(t^{\dagger}, g^{\dagger})$ have a *fixed*, noncritical value, $G^{\dagger}$, independent of the starting parameters t_0 and g_0. Substitution in (A6) then shows that $\hat{t}_0 e^{\lambda_1 l^{\dagger}}$ must be a function only of

$$x = \hat{g}_0 / \hat{t}_0^{\phi} \quad \text{with} \quad \phi = \lambda_2 / \lambda_1, \tag{A7}$$

so we can write

$$\hat{t}_0 e^{\lambda_2 l^{\dagger}} = W(x). \tag{A8}$$

Finally, (A6) yields

$$G(t_0, g_0) = G^{\dagger} [W(x)]^{\gamma} / \hat{t}_0^{\gamma}, \quad \text{with} \quad \gamma = -\lambda_0 / \lambda_1, \tag{A9}$$

which is just the standard scaling form

$$G(t_0, g_0) = \hat{t}_0^{-\gamma} \hat{W}(\hat{g}_0 / \hat{t}_0^{\phi}) \quad (\phi \neq \tfrac{1}{2}). \tag{A10}$$

As will be seen shortly $W(x)$ and thence $\hat{W}(x)$ have Taylor-series expansions.

Now: "What goes wrong when $\phi = \frac{1}{2}$?" Clearly the coefficient of g^2 in the nonlinear scaling field $\hat{t}$ diverges; but another, less obvious, consequence is that $W(x)$ in (A8) is no longer well defined. To see this set

$$\delta = \lambda_1 - 2\lambda_2 \tag{A11}$$

and, using (A6)–(A8), note that the equation determining $W(x)$ is

$$G(W - \delta^{-1} D_0 x^2 W^{2\phi}, x W^{\phi}) = G^{\dagger}. \tag{A12}$$

By taking derivatives one generates equations for the Taylor coefficients, $W_0 = W(0)$, $W_1 = W'(0), \ldots$ of W: the first few are

$$G(W_0, 0) = G^{\dagger}, \tag{A13}$$

$$W_1 = -G_g(W_0, 0) W_0^{\phi} / G_t(W_0, 0), \tag{A14}$$

$$W_2 = 2\delta^{-1} D_0 W_0^{2\phi} + O(1), \tag{A15}$$

where the subscripts t and g on G denote derivatives with respect to the first and second arguments, respectively. The last term in (A15) represents an an algebraic set of terms, which remain finite as $\delta \to 0$, depending only on derivatives of G evaluated at $t = W_0$, $g = 0$. Since the matching occurs on a noncritical locus these derivatives are all finite. Evidently, then, (A15) implies a divergence of W_2 and hence, via (A10), of the scaling function derivative $\hat{W}_2 = \hat{W}''(0)$ as $\delta \to 0$.

From the simplified recursion relations one has thus recovered precisely the pathologies encountered in the second-order calculations of Sec. IV D. [See Eqs. (4.69) and (4.74).] It remains to demonstrate why a logarithm appears and to determine what becomes of the scaling function $\hat{W}$ when $\phi = \frac{1}{2}$. To this end consider the recursion relations (A3) and (A4) when $\lambda_1 = 2\lambda_2$ so $\phi = \frac{1}{2}$ and $\delta = 0$. Substituting the solution g and solving the general ($\delta \neq 0$) equation for t yields

$$t(l) = [t_0 - \delta^{-1} D_0 g_0^2 (e^{-\delta l} - 1)] e^{\lambda_2 l}, \tag{A16}$$

provided $\delta \neq 0$; but as $\delta \to 0$ this goes over continuously to the $\delta = 0$ solution

$$t(l) = (t_0 + D_0 g_0^2 l) e^{\lambda_2 l}. \tag{A17}$$

On defining

$$w_0(l) = g_0^2 t_0^{-1} / (1 + D_0 g_0^2 t_0^{-1} l), \tag{A18}$$

the matching condition for $\delta = 0$ becomes

$$G^{\dagger} = G[t^{\dagger}, (t^{\dagger} w_0)^{1/2}] \tag{A19}$$

which implies

$$t^{\dagger} = t_0 e^{\lambda_1 l^{\dagger}} (1 + D_0 g_0^2 t_0^{-1} l^{\dagger})$$
$$= g_0^2 e^{2\lambda_2 l^{\dagger}} / w_0(l^{\dagger}) = \tilde{W}(w_0), \tag{A20}$$

where $\tilde{W}$ is a function only of w_0. Rewriting (A18) as

$$l^{\dagger} = 1/D_0 w_0 - t_0 / D_0 g_0^2, \tag{A21}$$

finally leads to

$$\exp[\lambda_1^{-1} \ln(g_0^2) - t_0 / D_0 g_0^2] = e^{-1/D_0 w_0} [w_0 \tilde{W}(w_0)]^{1/\lambda_1}, \tag{A22}$$

which allows one to conclude that w_0 is a function only of the combination $z_0 \mathscr{L}(t_0, g_0)$, where

$$z_0 \equiv g_0^2 t_0^{-1}, \quad \mathscr{L}(t_0, g_0) = [1 - \lambda_1^{-1} D_0 z_0 \ln(g_0^2)]^{-1}. \tag{A23}$$

Substituting (A20) for $l^{\dagger}$ into the expression (A6) for G, and using this fact we obtain

$$G(t_0, g_0) = g_0^{-2\gamma} \overline{W}(z_0 \mathscr{L}), \tag{A24}$$

where the scaling function $\overline{W}$ follows from (A22) and (A23). The final expression for G is then

$$G(t_0, g_0) = (t_0 \mathscr{L}^{-1})^{-\gamma} \hat{W}[g_0 / (t_0 \mathscr{L}^{-1})^{\phi}], \tag{A25}$$

where now $\hat{W}(x) = x^{-2\gamma} \overline{W}(x^2)$. This result is really a special case of the result of Barma and Fisher.[34]

Some remarks are in order. First, the combination $t_0 \mathscr{L}^{-1}$ is more transparently written

$$t_0 \mathscr{L}^{-1} = \hat{t} = t_0 - (D_0 / \lambda_1) g_0^2 \ln(g_0^2). \tag{A26}$$

Hence, rather than the simple quadratic correction to t_0 observed in the nonlinear scaling field when $\phi \neq \frac{1}{2}$ one now has a logarithmic correction. But to linear order the scaling variable is still $g_0 t_0^{-\phi}$ *for all* ϕ. Second, for $t_0 \neq 0$ the function $G(t_0, g_0)$ must have the Taylor-series expansion

$$G(t_0, g_0) = G_0(t_0) + G_1(t_0) g_0 + G_2(t_0) g_0^2 + \cdots. \tag{A27}$$

At first sight (A25), with its logarithmic dependences on g_0, seems to violate this requirement. But, in fact, the scaling function $\hat{W}(x)$ must have an expansion in x and

$\ln x$ with coefficients determined *exactly* so that all $\ln g_0$ terms *cancel.* The remaining terms will, of course, carry logarithms in t_0, but for $t_0 \neq 0$ that does not matter. By carrying out the necessary inversions to calculate $\widetilde{W}(w_0)$ using (A20) and the relations (A22) and (A24), one actually finds

$$\widehat{W}(x)=\widehat{W}_0[1+c_1x+c_{2,1}x^2\ln x+c_2x^2+O(x^3\ln x)]\ , \tag{A28}$$

where the value found for $c_{2,1}$, namely $-2D_0\gamma/\lambda_1$, causes the leading $\ln g_0$ term to cancel. Thus to second order in g_0 one has

$$G(t_0,g_0)=t_0^{-1}\widehat{W}_0[1+c_1g_0/t_0^{1/2}+c_2g_0^2/t_0 - \tfrac{1}{2}c_{2,1}(g_0^2/t_0)\ln t_0+O(g_0^3)]\ . \tag{A29}$$

Finally, therefore, this demonstrates the mathematical origin of the logarithmic factor in the perturbation expansions when $d=3$.

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