# Conductances of filled two-dimensional networks

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Numerical calculations of the conductances of specific two-dimensional networks of conductors whose values are described by a variety of distributions have been carried out. The results are found to differ only slightly from a well-known estimate which predicts that the conductance $G$ of such a network to be given by the value $g_c$, such that a fraction $p_c$ of the conductors have $g<g_c$, where $1-p_c$ is the fraction that can be removed randomly before the network ceases to conduct. In the event that $p_c$ is $\frac{1}{2}$, $g_c$ is the median conductance of the distribution of conductors. These results imply that electrical transport is dominated by bottlenecks which have conductances the order of $g_c$, a result which is important in the discussion of a number of threshold phenomena.

The electrical transport properties of random systems is a subject which has been studied intensively for more than a decade. Such systems are often modeled as a network of current-carrying links described by some wide distribution of conductances. It is often difficult to make quantitative predictions using such a model because of the width of the distribution of the conductances of individual bonds. We present here results of a numerical simulation for random resistor networks which confirms that quantitative statements can be made about the overall conductance of a network from a very general knowledge of the distribution of individual conductances using an estimate of the conductance which can be made based on an argument originally due to Ambegaokar, Halperin, and Langer[1] (AHL). This argument has been employed in a number of other contexts by a variety of investigators.[2–5]

Although the argument is applicable to both two- and three-dimensional networks, we have only tested it numerically for the two-dimensional case where it is extremely simple and predictions based on it turn out to be accurate. AHL (Ref. 1) assert that the overall conductance of a two-dimensional network $G$ is approximately equal to a critical conductance $g_c$ defined as the minimum conductance of the subset of conductances which spans the network when arranged in descending order. This conclusion is obtained in the following way: Remove all of the individual conductors from the network and then insert them in descending order of conductance at their original positions in the network. The overall conductance of the lattice will then be zero until a critical number of bonds are occupied. The conductor, which when placed completes the formation of an infinite cluster, has conductance $g_c$. Transport properties of the system will be dominated by those bonds with conductance of order $g_c$ and the measured conductance of the network $G$, will be approximately $g_c$.

This result is intuitively clear as finite clusters which are connected by conductance $g_c$ are composed of series-parallel combinations of conductance greater than $g_c$ and therefore should not limit the overall transport. Similarly, the bonds remaining to be filled with conductances $g<g_c$ do not affect the overall network conductance because they are shunted by bonds of conductance $g>g_c$.

Although this argument is appealing, it remains to be seen whether it is quantitatively accurate. To test the AHL argument we have performed simulations on random resistor networks of varying lattice size and distribution width. To determine the overall lattice conductance, a renormalization procedure utilizing the $Y$-$\Delta$ transformation[6] was developed to reduce the network to a single effective conductance. The result of this calculation was then compared with the value of the conductance $g_c$ which connects up the infinite cluster. In principle this conductance could have been found for each network by determining when percolation occurred as the network was filled in the manner described above. However, since two-dimensional bond and site networks were used in the simulation, it was possible to equate $g_c$ to the value of the conductance at the appropriate percentile [$(1-p_c)$, where $p_c$ is the percolation threshold] of the distribution. The critical conductance $g_c$ therefore corresponds to the median conductance for bond networks and the 41st percentile for site problems. This simplified the calculation and is highly accurate for the lattice sizes we used.

The simulation was performed on networks with individual conductances generated from four types of distributions: uniform, Gaussian, log normal, and cubic. Typical results are shown in Table I. In general the overall conductance $G$ of the network differs only slightly from $g_c$. This is remarkable considering that in the case of the log-normal distribution of conductances the half-width is approximately $10^8$.

It is interesting to note that the argument of AHL cannot be quantitatively valid for arbitrary conductance distributions. This can be trivially shown in the case of the bimodal distributions often used in the discussion of percolation. However, one need not resort to such extreme distributions to show that $G$ cannot equal $g_c$ for certain distributions. Using the Jensen inequality[7,8] we have $1/\langle g^{-1}\rangle < G < \langle g\rangle$ for any distribution of individual conductances. Here $\langle g\rangle$ is the mean value of the conductance and $\langle g^{-1}\rangle$ is the mean value of the resistance. If a

 

TABLE I. Parent distribution of conductance $g$ and the resultant lattice conductance $G$. $N$, $\langle g \rangle$, and $\sigma_g$ are the number of realizations, mean, and standard deviation of the parent distribution. $G$ and $\sigma_G$ are the overall lattice conductance and standard deviation. $g_c$ is the conductance predicted by AHL.

| Type of distribution | | Parent Distribution $N$ | $\langle g \rangle$ | $\sigma_g$ | Lattice size | Resultant Lattice $g_c$ | $G$ | $\sigma_G$ |
|---|---|---|---|---|---|---|---|---|
| Uniform | bond | 100 | 1.0 | 0.573 | 100×100 | 1.00 | 0.794 | 0.0073 |
| | | 70 | | | 200×200 | 1.00 | 0.794 | 0.0038 |
| | site | 100 | | | 100×100 | 0.82 | 0.703 | 0.0070 |
| Gaussian | bond | 100 | 1.0 | 0.329 | 100×100 | 1.00 | 0.940 | 0.0034 |
| | | 70 | | | 200×200 | 1.00 | 0.941 | 0.0019 |
| | site | 100 | | | 100×200 | 0.92 | 0.917 | 0.0040 |
| Log-normal | bond | 100 | $6\times10^6$ | $10^8$ | 100×100 | 1.00 | 1.004 | 0.1280 |
| | | 100 | | | 200×200 | 1.00 | 1.011 | 0.062 |
| | site | 100 | | | 100×100 | 0.15 | 0.162 | 0.028 |
| | | 100 | | | 200×200 | 0.15 | 0.159 | 0.013 |
| Cubic | bond | 100 | 0.02 | 0.026 | 100×100 | 1.41 | 1.62 | 0.00006 |
| | site | 100 | | | 100×100 | 1.30 | 1.57 | 0.00005 |
| | bond | 70 | 10 | 11.8 | 100×100 | 7.07 | 8.12 | 0.036 |

smooth distribution can be found for which $g_c$ falls outside of these bounds, then $g_c$ cannot equal $G$. One such example is the cubic distribution $P(g)=2g_m^2/g^3$ for $g>g_m$. The mean value of the conductance and the resistance can easily be shown to be $2g_m$ and $2/(3g_m)$, respectively. The critical conductance $g_c$ according to AHL is given by

$$p_c=\int_{g_c}^{\infty}(2g_m^2/g^3)dg$$

or $g_c=g_m/\sqrt{p_c}$. Substituting $g_c$ for $G$ in the Jensen inequality[7] gives $3g_m/2<g_m\sqrt{p_c}<2g_m$. From this it follows that only if $\frac{1}{4}<p_c<\frac{4}{9}$ can the overall conductance $G$ be equal to $g_c$. On the other hand, this is not a serious problem as AHL assert that $g_c$ will be only on the order of $G$, and we indeed find that the cubic distribution for both bond and site geometries (which do not satisfy the condition set by the Jensen inequality) results in a value of $g_c$ close to $G$.

Thus $g_c$ is still extremely useful in the context of physical problems in which the argument might be used. The above results imply that the physical idea that bottlenecks determine the overall conductance of a network is correct. As a check of this we examined the overall conductance $G$ of a lattice filled with conductors positioned using both bond and site procedures. (The occupation of a site results in the placement of half of each of the links to its nearest-neighbor sites.) If bottlenecks were indeed the important factor, then $G$ would decrease when the lattice was filled with sites in place of bonds. Note that only the filling procedure was changed and the distribution of the individual conductances was kept the same. As is seen in Table I, the overall conductance $G$ did decrease when this replacement was made. (In one case, that of the log-normal distribution, the change was a factor of 6.) For each distribution the new value of $g_c$ in the case of sites (41st percentile) was in much better agreement with the computed value of the overall conductance $G$ than was the median, as had been used earlier in the bond problem. This provides strong evidence that the argument of AHL can be quantitatively accurate and that bottlenecks determine the overall conductance of a network.

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