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Excluded-volume explanation of Archie's law

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The empirical relation between the electrical conductivity and the porosity of rocks is called Archie's law. Although it has been known for many years, attempts to explain this law are quite recent. These explanations have used effective-medium theories, special percolation models, and fractal pore structures. In this paper it is shown, contrary to previous suggestions, that ordinary percolation theory can account for the zero pore-space threshold and may account for the power-law behavior which was derived from the available experimental data. In this respect simple percolation is more general than the previous explanations.

Because of its importance in soil sciences, the problem of the relation between the electrical resistivity of a rock and its pore fluid content has led to intensive research activity.[1] The large amount of data[2] collected has shown that a simple empirical formula can describe this relation. This formula, widely known as Archie's law,[3] can be written as[1]

$$\rho_t = a\rho_w\phi^{-m} \ , \tag{1}$$

where ρ_t is the electrical resistivity in the bulk of the rock, ρ_w is the resistivity of the conducting water contained in the pore structure, ϕ is the porosity (the pore volume fraction of the rock), and a and m are parameters. Typical values of these parameters are confined[2] to the interval $a=0.62$, $m=1.95$ for the well-cemented sedimentary rocks, and $a=3.5$, $m=1.37$ for weakly cemented (or highly porous) rocks. There are, however, observations of m values outside the above interval.

Although Archie's relation has been known for forty years, attempts to explain it have appeared only in the last few years.[4-7] These attempts were based on capillary tube models[4] and effective-medium theories.[4-7] On the other hand, percolation theory,[8] while used (naturally) to describe rocks,[4] has not been applied to explain Archie's law. Such an application, if possible, would have the advantage of being based on universal behavior and of not requiring *a priori* assumptions regarding the microstructure of the pores in the rocks, as other theories do.

By now, the percolation critical exponents of the conductivity in d dimensions, t_d, are well known[9-11] ($t_2 \approx 1.25$ and $t_3 \approx 1.95$) to be in the above range of m. Hence, one can claim that for the systems for which $m \approx t_d$ Archie's law is consistent with percolation theory. The main difficulty with a percolation model (and probably the reason it has not been used until now on this problem[12]) is that the continuum percolation models discussed so far[8,9] require a critical (threshold) porosity, ϕ_c, which is distinctly higher than the $\phi_c = 0$ porosity implied by Eq. (1). In a very recent paper[12] this difficulty was overcome by suggesting a special new percolation model, which is in a different universality class than ordinary percolation. While yielding a $\phi_c = 0$ percolation, this model has the drawback that its corresponding two- and three-dimensional critical exponents t_0 are not in the range of the data[1] $1.95 \geqslant m \geqslant 1.37$ (if we identify m with t_0) as quoted above, but rather in the range $t_0 \geqslant 2.94$. There is little data[4,12] for m in the range $m \geqslant 2.94$ and thus, if applicable at all to the rock problem, this special model does not describe the bulk of the data. Another more recent work,[13] which was based on finding fractal sandstone pores, has suggested that Eq. (1) (with $a=1$) can be derived by using the conjecture that the conductivity exponent is related to the geometrical exponents.[14] It was argued then[13] that for sandstones with $1.5 < m < 2.5$, the practice of representing conductivity data with a zero percolation threshold is unfounded, and that one cannot draw conclusions about the transport properties of one rock from those of another (if the latter has a different pore geometry). The difficulties with this approach are that it is based only on sandstones [while Eq. (1) appears to be found in many more types of rocks[1,2]], that it is based on a debated conjecture,[12] and most importantly, that it does not address the main issue (as does Ref. 12), i.e., whether or not one can get percolation with a diminishing pore space. Then the question is raised again of whether or not Archie's law can be explained by ordinary percolation in the continuum (i.e., is this theory consistent with the very small critical porosity ϕ_c and with the observed values of m?).

The purpose of this paper is to show, by an extremely simple argument, that this is possible and that the explanation is consistent with our knowledge[1,7] concerning the channel-like or sheet-like pores in sandstones and rocks. Hence (contrary to the claims of Refs. 12 and 13) "ordinary percolation provides a realistic model for the pore space in rocks."

We start by considering the recent excluded-volume theory[15] of percolation thresholds in the continuum, which dealt with a system of interpenetrating (or "soft-core") objects. The present problem is mapped onto the system by the identification of the pores as the soft-core objects and the stone as the insulating matrix.

Let us first determine the critical porosity, i.e., the critical total volume of the pores, ϕ_c (or the critical area of all the punched-out pieces in two dimensions). The system is assumed to be made of interpenetrable, equal-volume pores, each of which has a volume v (all the volumes here are given in terms of fractional volumes, i.e., it is assumed that the system is of a unit volume such as a unit cube[15]). Hence, the probability that a particular randomly chosen point lies outside a given pore is $1-v$. At the percolation threshold there are N_c pores and thus the probability of the particular point being in the matrix, i.e., not in any of the pores, is $(1-v)^{N_c}$. On the other hand, this probability is also $1-\phi_c$ and thus

$$1-\phi_c = (1-v)^{N_c} \ . \tag{2}$$

In Eq. (2) we may use the identity $v = v_{ex} N_c v / v_{ex} N_c$, where v_{ex} is the excluded volume[15] of a pore of volume v. Hence,

$$\phi_c = 1 - [1 - (N_c v_{ex} v / v_{ex})(1/N_c)]^{N_c} \quad . \tag{3}$$

In the limit of an infinite system ($v \to 0$, $N_c \to \infty$) v/v_{ex} and the total excluded volume, $c = N_c v_{ex}$, have been shown[15,16] to be constants of the particular system under consideration. Hence, for the infinite system

$$\phi_c = 1 - e^{-(cv/v_{ex})} \quad . \tag{4}$$

Indeed, in the well-known[15] case of spherical pores, $c = 2.8$, $v_{ex}/v = 8$, and $\phi_c = 0.29$. Knowing the dependence of c on the parameters of the system, it is easy to see how the $\phi_c \to 0$ limit can be approached. Since c is known[15,16] to be a constant of order 1, a small v/v_{ex} ratio means a small value of ϕ_c.

A small v/v_{ex} ratio can be obtained when the geometrical shape of the pores is different from the geometrical shape of their excluded volumes. This is not the case for spherical pores (or other parallel pores where v/v_{ex} is always 1/8), but is the case of elongated channels (length L, radius r) for which it has been shown[15] that while $v = \pi r^2 L$, the excluded volume is given by $v_{ex} = \pi r L$.[2] In the latter case the above ratio can be made as small as the inverse of the aspect ratio, L/r. Hence, for a large aspect ratio, Eq. (4) can be approximated by

$$\phi_c = cr/L \quad , \tag{5}$$

where for this system, which is described by interpenetrable sticks,[15,16] $c = 1.4$. In some recent works the pores were modeled by randomly aligned thin disks.[17] For the disks of radius R and thickness d, $v = \pi R^2 d$ while[17,18] (for $R >> d$) $v_{ex} = \pi^2 R^3$. Hence, for large R/d ratios

$$\phi_c = (c/\pi)(d/R) \quad , \tag{6}$$

where c has been estimated[17,19] to be between 1.4 and 2.7. The important message of Eqs. (5) and (6) is that the critical occupied volume can be made diminishingly small if the "channels" (the grains' separation) or the "cracks" in the rocks are made sufficiently narrow (in comparison with their span). In particular, the result (6) is what one would intuitively expect for consolidated sandstones, i.e., that the pores essentially form a "skin" (or an "envelope") layer around the stones,[7] and it is this sheet-like surface through which the water flow takes place. Since $c/\pi \approx 1$ and d/R can be of the order of 10^{-2}, for all practical purposes ϕ_c may appear as 0. Further, the precision of the ρ_t vs ϕ measurements is such[1,2] that one cannot distinguish 10^{-2} from 0 in ϕ_c.

In the above discussion we have assumed all the channels (or the disks) to have the same size. If this is not the case, one has to introduce the average over the corresponding geometrical parameters. Equation (5) has to be replaced then by[15,16]

$$\phi_c = c \langle r^2 \rangle \langle L \rangle / \langle r \rangle \langle L^2 \rangle \quad , \tag{7}$$

while Eq. (6) has to be replaced by

$$\phi_c = (c/\pi) \langle R^2 \rangle \langle d \rangle / \langle R^3 \rangle \quad . \tag{8}$$

As was shown in Ref. 15, the excluded volume is conserved under variations in the pore size distributions, and thus we have the same c factors as for Eqs. (5) and (6). Considering various distributions one can see, for example, that for disks (or the "thin-skin" pore model) for which one can assume that the "skin thickness" d has a narrow distribution (i.e., $\langle d \rangle \approx d$), the value of ϕ_c will be further reduced. This is due to the distribution in the size of the "sheets," i.e., due to the $\langle R^2 \rangle / \langle R^3 \rangle$ term. In particular, the log-normal pore-size distribution has been previously suggested.[20] Using the corresponding averages associated with this distribution[15] (as well as with other distributions) one finds that $\langle R^2 \rangle / \langle R^3 \rangle$ is smaller than $1/R$ where R corresponds to the case [Eq. (6)] where all the disks have the same radius. Hence, the $\phi_c \to 0$ limit is easier to obtain in the realistic model of "thin skin" pores which have a wide size distribution.

In the preceding section we have shown that the critical volume of all pores can be made as small as desired, provided that a "generalized aspect ratio" (such as L/r or R/d) is made as large as possible. Hence, this implication of Archie's law is fulfilled by the excluded volume theory.

The other consequence of Archie's law is the value of m. We have previously shown[21] that the same system which yielded the above percolation thresholds (or excluded volumes[15,16]) also yielded the universal critical exponents of ordinary percolation. This means that, for the value of $m = 1.9 \pm 0.2$, the data for $\rho_t = \rho_t(\phi)$ and its presentation by Archie's law [Eq. (1)] are consistent with ordinary percolation theory. While this consistency exists, the fact that the continuous range of m values found in the experiments[1,2] (see above) is significantly wider than this "error bar" interval of m appears to indicate a contradiction with ordinary percolation theory or its universality. Before considering this problem, we have to consider a more fundamental issue and that is whether the attempt to correlate m with t_d is justified altogether. This is because the critical exponent t_d has not been proven to govern the conductivity over the wide range for which the experimental data[1,2] exist. Hence, one may even expect a crossover to mean-field exponents. In view of the fact that such a crossover has not been found in the experimental data,[1,2] and in view of the theoretical suggestion[22] that the critical behavior in percolating systems is maintained over a much wider range than in phase transitions, we may assume that m has to do with the conductivity critical exponent. We note, of course, that the evidence from percolation experiments[9-11,23] shows that this is the case for $(\phi - \phi_c)/\phi_c \leq 1$, while Archie's law is applicable over a much wider range. It is not impossible then that the observed m values have to do with effective-medium "exponents," but our argument is that if the critical range is significantly wider than $(\phi - \phi_c)/\phi_c \approx 1$, percolation-theory exponents can account for the data. Let us then consider sources which may cause m to deviate from the single universal value given above, but which still yield results in accordance with ordinary percolation. Below, we suggest two such sources which can account for the entire m range found in the experiments.

The first source we consider is a possible macroscopic anisotropy in the system. This source which has been realized to affect the value of m was concluded from the experimental correlation between the values of m and the deviations from sphericity of the grains which make the rock.[5] It was argued then that, in general, a rock made of nonspherical (i.e., elongated) grains, will have a macroscopic anisotropy. Using effective-medium theory it has been shown that by considering the above deviation and the above macroscopic anisotropy, one can obtain the above continuous range of $m \geq 1.5$ values.[5] Recently, it turned out that macroscopic anisotropy can also account for a continuous (and even wider, in particular at the low end) range of t_d values when the ordinary percolation approach is taken.[21,23] Using computer simulations we have indeed shown that in a system of nonspherical pores, which has a macroscopic anisotropy, apparent critical exponents will appear.[21] This is found to be the case even for data of high accuracy when the data are presented on a log-log plot of ρ_t vs $\phi - \phi_c$. The results of that study for a three-dimensional system, as well as previous results for a two-dimensional system,[23] clearly show that in the direction of anisotropy the apparent critical exponent of the electrical conductivity is larger than the universal exponent t_d, while perpendicular to this direction the apparent exponent is smaller than the universal exponent. It follows then that it is not necessary to invoke a new universality percolation class to explain the $m > 2$ values as suggested in Ref. 12.

Unfortunately, there is as yet no data which correlate the m values with the macroscopic features of the rock texture.[5] It seems worthwhile, however, to carry out such experiments since, if our conjecture is confirmed, one will have a tool to derive information on the textures from the relatively simple measurement of the electrical resistivity. At present, we base our conjecture on computer simulations[21] and physical experiments[23] which suggest a correlation between the value of m and the macroscopic anisotropy.

The second source for variable m values can be the distribution of the "neck" sizes in the pores. This distribution yields a distribution in the values of the resistors which are associated with the pores.[24] The latter distribution is known[25,26] to yield conductivity exponents which are higher than the universal t_d. For the observed $m < 2$ values there is an additional percolation-theory explanation. This explanation is applicable to rocks which can be described as having sheet-like pores, all of which are perpendicular to the same plane. Such a model[27] shows that the system essentially presents a two-dimensional percolation problem and that it can be described by an ensemble of widthless sticks.[27-29] For the latter system, we may think of a crack of length L, width d, and depth h. The volume of the pore is Ldh while the excluded volume[15] in this "thin-sheet" case[15,16] $(L >> d)$ is $(2/\pi)L^2h$ and c is $3.6h$. Hence $\phi_c = (\pi c/2) \times (d/L)$ and in the case of a large aspect ratio $(L >> d)$ it yields a ϕ_c value which can be made as small as desired. The resistivity exponent in an isotropic two-dimensional system is[9] $t_2 = 1.25$ and the two-dimensional macroscopic anisotropy may yield (in this case too,[23] see above) higher and lower m values than this value of t_2. Hence, the broad range of observed m values is consistent with expectations from ordinary percolation theory. If our conjectures are proved correct, the m value may be telling us the dimensionality, the neck size distributions, and the degree of anisotropy of the percolating systems. We must point out, however, that we do not claim that this conclusion rules out the special cases of fractal structures,[13] a special universality class,[12] or effective medium behavior[6] in some rocks. We stress, however, the fact that all the data can be explained in terms of ordinary percolation theory.

In conclusion, we have shown that in rocks where the pores have a large aspect ratio the onset of percolation can be at a very small pore space. In addition to previous explanations, we point out that the pore-space dependence of the rocks' resistivity may be accounted for by the dimensionality and macroscopic anisotropy of the percolating pore system. If this relationship is experimentally proved to be correct, ordinary percolation will become the most general explanation of Archie's law.

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