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Quantum size effects in superfluid ^{3}He films

Zlatko Tesanovic*
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455

Oriol T. Valls†
Division of Materials Science and Technology, Argonne National Laboratory, Argonne, Illinois 60439
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We study quantum size effects in superfluid ^{3}He films. We consider the thickness d regime $k_F^{-1} \ll d < \xi$. In this regime, quantum size effects are most prominent, while several theoretical simplifications are possible. The Gor'kov equations for the p-type superfluid are solved in the parallel-plate geometry assuming specularly reflecting walls. A simple decomposition for the pairing potential is used which allows explicit evaluation of the various thermodynamical quantities and the NMR response in the Anderson-Brinkman-Morel state. The transition temperature is monotonically depressed as the thickness decreases, and, contrary to what is found for s-wave pairing, exhibits only weak oscillatory behavior. On the other hand, the shift in the transverse resonance frequency shows characteristic discontinuous jumps. Particular emphasis is placed on experimentally detectable deviations from the bulk behavior. The effects of surface roughness on our results are briefly discussed.

I. INTRODUCTION

The study of superfluidity in He films is an important part of the effort to understand phase transitions and long-range order in systems which exhibit, in some sense, two-dimensional behavior. In the case of ^{4}He films, experimental studies have provided[1,2] one of the best checks of the theoretical ideas involved in the description of defect-mediated phase transitions[3] in two dimensions. Although no superfluidity has yet been observed in ^{3}He films, constant advances in experimental techniques have now brought the relevant temperature and thickness ranges within reach. The variety and richness of phenomena which can be expected for ^{3}He is even larger than for ^{4}He, since in the former case the order parameter is a tensor rather than a scalar, and a more complex behavior can be expected. In addition, since the bulk coherence length ξ in ^{3}He may be quite large ($\simeq$300–400 Å), one must distinguish among different thickness regimes depending on the ratios of the thickness d to ξ and to the average interparticle distance as given by the inverse of the bulk Fermi wave vector k_F, that is, depending on whether only the normal component or both normal and superfluid components of the Fermi liquid behave two dimensionally. In this context, we have recently considered,[4] from the theoretical point of view, the nuclear magnetic resonance (NMR) spectrum of several phases of superfluid ^{3}He films in the thickness regimes $d \simeq k_F^{-1}$ and $k_F^{-1} \ll d \ll \xi$, with particular emphasis on the use of NMR as a diagnostic tool to extract information about the pairing state, as is done for the bulk superfluid.

In this paper our main objective is to investigate quantum finite-size effects in superfluid ^{3}He films. The range of thickness for which these effects are most prominent is for d from roughly 0.05ξ to ξ, corresponding typically to about 5–100 layers of liquid ^{3}He. Within this thickness regime it is safe to assume that the pairing interaction is not significantly changed from that in bulk ^{3}He, because, since $d \gg k_F^{-1}$, the normal fluid component is still effectively three dimensional; at the same time, the film is thin enough so that quantum finite-size effects should be observable. As the thickness increases beyond ξ, these effects will be washed out. Moreover, we can restrict our attention to the Anderson-Brinkman-Morel (ABM) state of the p-type superfluid since strong coupling corrections to the free energy always favor this state as opposed to the two-dimensional state (see Ref. 4 and references therein).

The quantum finite-size effects are due to the fact that quasiparticle wave functions cannot be adequately represented by plane waves in the presence of a boundary. The Fermi sphere degenerates into a set of Fermi circles. We have studied here the influence of these effects in quantities such as the transition temperature T_c and the NMR resonance frequencies. Quantum size effects in T_c, in the analogous thickness regime, have been observed experimentally[5] in superconductors. We find that the behavior of T_c as a function of thickness is very different for p-wave pairing from the oscillatory variation found for ordinary s-wave pairing. T_c, the superfluid gap, and the NMR frequency shifts all display characteristic oscillatory and discontinuous behavior as a function of thickness. This is a direct consequence of the quantum size effects. These effects are particularly pronounced for the transverse resonance frequency shift, which makes NMR measurements an attractive experimental possibility.

The thickness regime considered here must be clearly distinguished from others: The "slab" case, where the thickness $d \gg \xi$ but smaller than lengths associated with textures has been studied by Takagi.[6] In that case there are no quantum size effects whatever. When $d \simeq k_F^{-1}$, the normal-state quasiparticle interactions are likely to be significantly modified with respect to the bulk. This is evidenced by an enhancement in the ferromagnetic susceptibility[7,8] which has been experimentally observed (see, for example, Refs. 9—11). These effects were not included in Ref. 4 and are not present, as explained above, in the regime considered here.

Throughout our calculations, we have assumed that the boundaries of the ^{3}He considered are specularly reflecting walls. Of course, experimentally, some diffusive scattering will be present due to the roughness of the substrate. This diffusive scattering will reduce to some extent the sharpness of the features in the resonance frequency shift, for instance, and depress the value of T_c. However, as long as the mean free path of the quasiparticles remains larger than or comparable to ξ, surface scattering will not obliterate[12] the features discussed here. We will return to this question later in the paper. In the case of slab geometry, a study of the influence of boundary roughness has been made by Buchholtz.[13]

In our calculations of the NMR frequency shift we have not included, in the present work, the influence of quasiparticle renormalization factors.[14] It is well known that for the bulk superfluid these factors contribute only to the overall magnitude of the characteristic dipolar energy and have little effect on the NMR spectrum. We found in Ref. 4 that they may play an important role when $d \ll \xi$. However, this is not the case in the thickness regime considered here, which is in this respect more analogous to the bulk: The quasiparticle renormalization factors will only depress the magnitude of the frequency shift and make the effect discussed by Takagi[15] less pronounced.

The paper is organized as follows. In the next section (Sec. II) we set up and solve the Gor'kov equations for a p-type superfluid film within the assumptions discussed above. We show that through the use of a simple but realistic treatment of the pairing interaction the gap equation can be put into manageable form for each of the circles into which the bulk Fermi sphere degenerates. In Ref. 4 we assumed that we could consider one circle only, but this would have been a very unrealistic assumption in the present case and would have made it impossible to elucidate the quantum size effects as a function of thickness. We then obtain explicit results for the transition temperature, the superfluid gap parameter, and the normal-state magnetic susceptibility and discuss the behavior of the specific heat well below T_c. The following section (Sec. III) contains the calculation of the Takagi frequency shift and a discussion of the NMR response. At the end, we briefly recapitulate our conclusions with emphasis on the experimental implications of our results.

II. THEORY AND RESULTS

In this section we present our results for the transition temperature T_c, the gap, the specific heat, and the magnetic susceptibility of the superfluid ^{3}He films as a function of thickness, within the assumptions discussed in the introduction.

Our starting point is the Gor'kov equations,[16] which are generally valid:

$$\left[i\omega_n+\frac{\nabla^2}{2m}+\mu\right]G_{\sigma\sigma'}(\mathbf{r},\mathbf{r}';\omega_n)+\sum_\rho\int d^3r''\,\Delta_{\sigma\rho}(\mathbf{r},\mathbf{r}'')F^*_{\rho\sigma'}(\mathbf{r}'',\mathbf{r}';\omega_n)=\delta(\mathbf{r}-\mathbf{r}')\delta_{\sigma\sigma'}\,, \tag{2.1a}$$

$$\left[-i\omega_n+\frac{\nabla^2}{2m}+\mu\right]F^*_{\sigma\sigma'}(\mathbf{r},\mathbf{r}';\omega_n)+\sum_\rho\int d^3r''\,\Delta^*_{\sigma\rho}(\mathbf{r},\mathbf{r}'')G_{\rho\sigma'}(\mathbf{r}'',\mathbf{r}';\omega_n)=0\,, \tag{2.1b}$$

where all symbols have their usual meanings, as in Ref. 16. We also have the gap equation:

$$\Delta_{\sigma\sigma'}(\mathbf{r},\mathbf{r}')=TV(\mathbf{r}-\mathbf{r}')\sum_n F_{\sigma\sigma'}(\mathbf{r},\mathbf{r}';\omega_n)\,. \tag{2.2}$$

It is convenient to choose the z axis perpendicular to the film. If we then perform a spatial Fourier transform in the plane of the film, we obtain the following convenient form of the Gor'kov equations:

$$\left[i\omega_n-\varepsilon_{k_\perp}+\frac{\nabla_z^2}{2m}+\mu\right]G_{\sigma\sigma'}(z,z',\mathbf{k}_\perp;\omega_n)+\sum_\rho\int dz''\,\Delta_{\sigma\rho}(z,z'',\mathbf{k}_\perp)F^*_{\rho\sigma'}(z'',z',\mathbf{k}_\perp;\omega_n)=\delta(z-z')\delta_{\sigma\sigma'} \tag{2.3}$$

and

$$\left[-i\omega_n-\varepsilon_{k_\perp}+\frac{\nabla_z^2}{2m}+\mu\right]F^*_{\sigma\sigma'}(z,z',\mathbf{k}_\perp;\omega_n)+\sum_\rho\int dz''\,\Delta^*_{\sigma\rho}(z,z'',\mathbf{k}_\perp)G_{\rho\sigma'}(z'',z',\mathbf{k};\omega_n)=0\,. \tag{2.4}$$

To treat the z dependence, it is useful to expand the functions G and F in terms of the eigenfunctions u_ν of a one-dimensional box:

$$G_{\sigma\sigma'}(z,z',\mathbf{k}_\perp;\omega_n)=\sum_\nu u_\nu(z)g^\nu_{\sigma\sigma'}(z',\mathbf{k}_\perp;\omega_n)\,, \tag{2.5a}$$

$$F_{\sigma\sigma'}(z,z',\mathbf{k}_\perp;\omega_n)=\sum_\nu u_\nu(z)f^\nu_{\sigma\sigma'}(z',\mathbf{k}_\perp;\omega_n)\,, \tag{2.5b}$$

where

$$u_\nu(z)=\left[\frac{2}{d}\right]^{1/2}\sin\left[\frac{\nu\pi}{d}z\right]. \tag{2.6}$$

Physically, we are taking into account the fact that the usual three-dimensional Fermi sphere degenerates in the present case into a set of parallel circles, each of which can be labeled by an index ν running from one to a maximum value $\nu_c=[\nu_0]$, where $\nu_0^2\equiv 2md^2/\pi^2\mu$. μ is the chemical potential, and the symbol $[x]$ denotes the largest integer smaller than x.

Next, we must consider the gap equation,

$$\Delta_{\sigma\sigma'}(z,z',\mathbf{k}_\perp)=T\sum_n \sum_{\mathbf{k}'_\perp} V_{\mathbf{k}_\perp\mathbf{k}'_\perp}(z,z')F_{\sigma\sigma'}(z,z',\mathbf{k}_\perp;\omega_n) , \tag{2.7}$$

and we expand the potential as in Eq. (2.5):

$$V_{\mathbf{k}_\perp\mathbf{k}'_\perp}(z,z')=\sum_{\nu,\nu'} u_\nu(z)u_{\nu'}(z')V_{\nu\nu'}(\mathbf{k}_\perp,\mathbf{k}'_\perp) . \tag{2.8}$$

In the three-dimensional case the gap equation is solved by considering only the part of the potential responsible for the $L=1$ pairing at the Fermi surface; that is,

$$V(\mathbf{k},\mathbf{k}')=-3g\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'=-3\frac{\lambda}{N(0)}\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' , \tag{2.9}$$

where $N(0)$ is the density of states at the Fermi surface. In the regime we are considering, since $d\gg k_F^{-1}$ we can expect that the pair-forming potential is still essentially (2.9). We must now, however, write an expression for this potential, not on the three-dimensional Fermi surface but on the set of circles which form now the Fermi region, as denoted by the $\nu\nu'$ labels in Eq. (2.8). The natural and obvious way of rewriting Eq. (2.9) under the present conditions is

$$V_{\nu\nu'}(\mathbf{k}_\perp,\mathbf{k}'_\perp)=-3g\left\{\hat{\mathbf{k}}_\perp\cdot\hat{\mathbf{k}}'_\perp\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]^{1/2}\left[1-\left[\frac{\nu'}{\nu_0}\right]^2\right]^{1/2}+\left[\frac{\nu}{\nu_0}\right]\left[\frac{\nu'}{\nu_0}\right]\right\} , \tag{2.10}$$

which clearly reduces to (2.9) as the thickness of the film goes to infinity. Of course, the assumption represented by (2.10) is likely to break down at very small thickness ($d\simeq k_F^{-1}$) where interactions different from those in the bulk may prevail.

Combining Eqs. (2.10) and (2.7) we see that the gap can be expanded in terms of $u_\nu(z)$ functions:

$$\Delta_{\sigma\sigma'}(z,z',\mathbf{k}_\perp)=\sum_{\nu,\nu'}\Delta^{\nu\nu'}_{\sigma\sigma'}(\hat{\mathbf{k}}_\perp)u_\nu(z)u_{\nu'}(z') , \tag{2.11}$$

and that the gap equation decouples with respect to the index ν so that $\Delta^{\nu\nu'}_{\sigma\sigma'}(\hat{\mathbf{k}}_\perp)$ is of the form

$$\Delta^{\nu\nu'}_{\sigma\sigma'}(\hat{\mathbf{k}}_\perp)=\Delta_{\sigma\sigma'}(\hat{\mathbf{k}}_\perp)\delta_{\nu\nu'}x_\nu , \tag{2.12}$$

where

$$x_\nu\equiv\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]^{1/2} , \tag{2.13}$$

which corresponds to the usual $\sin\theta$ dependence in three dimensions.

It is then an easy matter to solve Eqs. (2.3) and (2.4). After straightforward algebra and using the usual unitarity condition

$$\sum_\rho \Delta_{\sigma\rho}(\hat{\mathbf{k}}_\perp)\Delta^*_{\rho\sigma'}(\hat{\mathbf{k}}_\perp)=|\Delta(\hat{\mathbf{k}}_\perp)|^2\delta_{\sigma\sigma'} , \tag{2.14}$$

one obtains the results

$$g^\nu_{\sigma\sigma'}(z,\mathbf{k}_\perp;\omega_n)=(-i\omega_n-\varepsilon_{k_\perp}-\lambda_\nu+\mu)u_\nu(z)/E^2_{k_\perp,\nu} \tag{2.15}$$

and

$$f^\nu_{\sigma\sigma'}(z,\mathbf{k}_\perp;\omega_n)=x_\nu\Delta_{\sigma\sigma'}(\hat{\mathbf{k}}_\perp)u_\nu(z)/E^2_{k_\perp,\nu} , \tag{2.16}$$

where

$$\lambda_\nu=\frac{1}{2m}\left[\frac{\nu\pi}{d}\right]^2 \tag{2.17}$$

and

$$E^2_{k_\perp,\nu}=\omega_n^2+(\varepsilon_{k_\perp}+\lambda_\nu-\mu)^2+|\Delta(\hat{\mathbf{k}}_\perp)|^2x_\nu^2$$
$$\equiv\omega_n^2+\tilde{E}^2_{k_\perp,\nu} . \tag{2.18}$$

In all cases the index ν is limited by the maximum value ν_c defined earlier. Proceeding in the usual way, it follows from Eq. (2.16) that the gap equation can be rewritten as

$$\Delta^i_{\sigma\sigma'}=-\tfrac{3}{2}g\sum_\nu x_\nu^2\sum_{|\mathbf{k}_\perp|}\frac{\tanh(\frac{1}{2}\beta\tilde{E}_{k_\perp,\nu})}{\tilde{E}_{k_\perp,\nu}}\Delta^i_{\sigma\sigma'} , \tag{2.19}$$

where the index i equals x or y, and the quantities $\Delta^i_{\sigma\sigma'}$ are defined by

$$\Delta_{\sigma\sigma'}(\hat{\mathbf{k}}_\perp)=\sum_{i=x,y}\Delta^i_{\sigma\sigma'}\hat{\mathbf{k}}_{\perp i} . \tag{2.20}$$

The sum over $|\mathbf{k}_\perp|$ on the right-hand side of (2.19) can be converted in the usual fashion into an integral over $\varepsilon_{k_\perp}$ with a cutoff frequency ω_0 of order $0.1E_F$.[17] The gap equation at zero temperature then reduces to

$$1=\frac{2}{3}\frac{dk_F}{\pi\lambda}\sum_{\nu=1}^{\nu_c}\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]\times\sinh^{-1}\left\{\omega_0/\Delta\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]^{1/2}\right\} , \tag{2.21}$$

where λ is the usual three-dimensional coupling constant and $\Delta=|\Delta(\hat{\mathbf{k}}_\perp)|$, which is independent of $\hat{\mathbf{k}}_\perp$ for the ABM state. Since $\omega_0\gg\Delta$, we have as our final result for the zero temperature gap

$$\Delta = 2\omega_0 \exp\left[-\frac{2dk_F}{3\pi\lambda} - B\left[\frac{\nu_c}{\nu_0}\right] \Big/ \nu_c - \frac{\nu_c(\nu_c+1)(2\nu_c+1)}{6\nu_0^2}\right], \quad (2.22)$$

where

$$B\left[\frac{\nu_c}{\nu_0}\right] = \frac{1}{2}\sum_{\nu=1}^{\nu_c}\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]\ln\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]. \quad (2.23)$$

Note that ν_c is the largest integer less than ν_0. Following standard methods[18] we can write the chemical potential μ in terms of the density n and ν_c:

$$\mu = \frac{\pi d}{m\nu_c}\left[n + \frac{\pi}{6d^3}\nu_c(\nu_c+1)(\nu_c+\tfrac{1}{2})\right]. \quad (2.24)$$

After specifying the number of layers, that is ν_c, the thickness $d(\nu_c)$ can be found by eliminating the chemical potential using (2.24) and the definition of ν_c. It is not hard to see that in the limit $d\to\infty$, when $\nu_c \to dk_F/\pi$, Eq. (2.22) reduces to the usual three-dimensional result.

The transition temperature T_c follows easily from a calculation along the same lines as that leading to (2.22). The result is

$$T_c = 1.14\omega_0 \exp\left[-\frac{2dk_F}{3\pi\lambda} \Big/ \nu_c - \frac{\nu_c(\nu_c+1)(2\nu_c+1)}{6\nu_0^2}\right]. \quad (2.25)$$

We can use this formula and the considerations following (2.24) to find T_c as a function of thickness. The results are displayed in Fig. 1, where we plot T_c/T_c^{bulk} (T_c^{bulk} is the bulk transition temperature), as a function of $k_F d$. Note that this result is quite different from that found in the singlet case,[19] when the ratio Δ/T_c turns out to be independent of thickness, and oscillations in Δ are reflected in oscillations in T_c. It may be possible to use this fact to determine whether superconducting films are in a singlet or triplet pairing state.

The reason for this difference, physically, is the following: Nonanalyticities in T_c as a function of thickness may occur whenever the number of circles constituting the Fermi "surface" increases or decreases by one. The circles may be visualized as "popping out" at the poles of the three-dimensional Fermi sphere as the thickness decreases. Since for p-wave pairing the gap vanishes at the poles, the influence of this sudden change in ν_c on T_c is felt only as a discontinuity in the derivative, while for s-wave pairing, where the gap is isotropic, a change in ν_c produces a discontinuity in T_c itself.

Next, we can calculate the magnetic susceptibility as a function of thickness. For our purposes we need only the susceptibility of the normal fluid, and that is easily found by simply calculating the density of states on the Fermi region. This result is plotted in Fig. 2, again as a function of $k_F d$. It is discontinuous whenever ν_0 is an integer.

Finally, one can also calculate the specific heat in the superfluid state. The important point is that we expect the standard T^3 low-temperature behavior to be replaced by an exponential decay since the gap generally does not vanish on any of the Fermi circles. As $T\to 0$ the dominant part in the specific heat will go as

$$\sim \exp\left[-\frac{\Delta\left[1-\left[\frac{\nu_c}{\nu_0}\right]^2\right]^{1/2}}{T}\right].$$

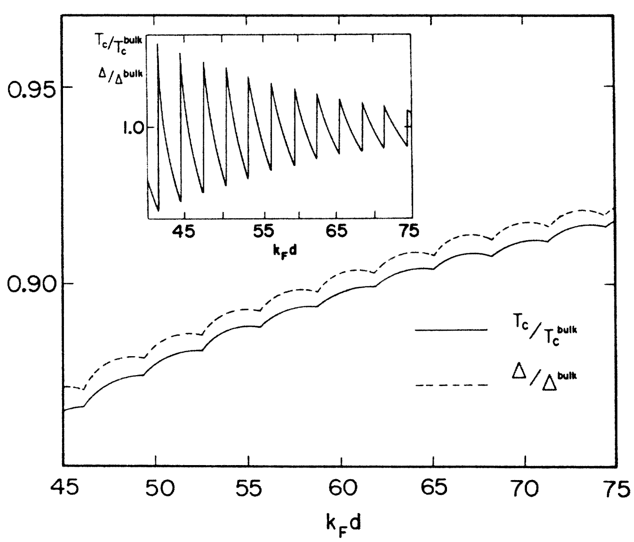

FIG. 1. The transition temperature T_c and the superfluid gap Δ of the superfluid ^{3}He film as functions of $k_F d$ (k_F=0.785 Å^{-1}). Note that $T_c/T_c^{\text{bulk}} \neq \Delta/\Delta^{\text{bulk}}$. The inset schematically shows the same quantities in the s-type superfluid.

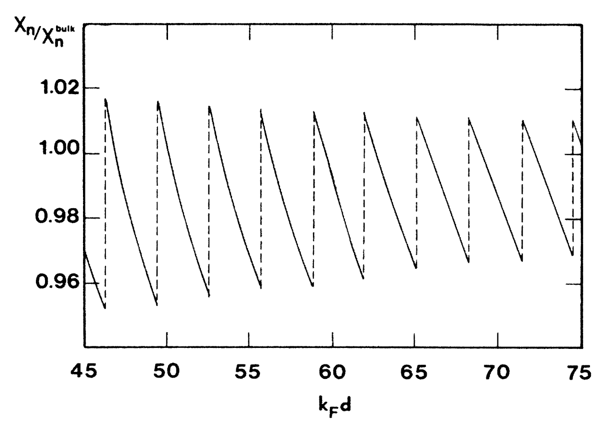

FIG. 2. The normal-fluid magnetic susceptibility as a function of $k_F d$. The discontinuous jumps result from abrupt changes in the density of states which occur when ν_0 equals to an integer.

III. NUCLEAR MAGNETIC RESONANCE SPECTRUM

In this section we will use the results of Sec. II to calculate quantum size effects in the NMR response of superfluid ^{3}He films. We are particularly interested in the NMR technique because it is an accurate and commonly used diagnostic tool in the study of ^{3}He, and, as we will see, quantum size effects are very pronounced in the resonance frequencies. As in Ref. 4, we will use the methods developed by Leggett[20] for the bulk case. The first step is to find the dipolar Hamiltonian $H_{\rm dip}$. The general form of $H_{\rm dip}$ for superfluid film is given in Eqs. (2.20) and (2.21) of Ref. 4. The coefficients $E_{\rm dip}$, g, and h which appear in those expressions can be calculated explicitly as functions of the thickness starting from

$$H_{\rm dip}=-\tfrac{1}{2}\gamma^2\sum_{\mathbf{k}_\perp,\mathbf{k}'_\perp}\sum_{\nu,\nu'}f^*_{\nu\alpha}(\mathbf{k}_\perp)f_{\gamma\beta}(\mathbf{k}'_\perp)K_{\alpha\beta}(\mathbf{k}_\perp,\mathbf{k}'_\perp;\nu,\nu')\ , \tag{3.1}$$

where

$$K_{\alpha\beta}(\mathbf{k}_\perp,\mathbf{k}'_\perp;\nu,\nu')=\int_0^d dz\int_0^d dz'\int d^2\rho\, e^{-i(\mathbf{k}_\perp-\mathbf{k}'_\perp)\cdot\rho}u_\nu(z)u_\nu(z')u_{\nu'}(z)u_{\nu'}(z')\frac{1}{r^3}(\delta_{\alpha\beta}-3\hat{\mathbf{r}}_\alpha\hat{\mathbf{r}}_\beta)\ , \tag{3.2}$$

and $f_{\nu\alpha}(\mathbf{k}_\perp)$ is defined as

$$\mathbf{f}_\nu(\mathbf{k}_\perp)u_\nu(z)=-\frac{i}{2}\sum_n(\sigma_2\sigma)_{\alpha\beta}f^{\nu}_{\alpha\beta}(z,\mathbf{k}_\perp;\omega_n)\ . \tag{3.3}$$

Since $K_{\alpha\beta}$ is a traceless tensor, there are only two independent components.[4] We first perform the integration over ρ in (3.2).[21] The remaining z and z' integrals are lengthy but elementary and we obtain

$$K_{xx}=0 \tag{3.4a}$$

$$K_{zz}(\mathbf{k}_\perp,\mathbf{k}'_\perp;\nu,\nu')=-\frac{8\pi}{d}y_{\nu\nu'}\left[(-)^{\nu+\nu'}e^{-y_{\nu\nu'}}\Pi_{\nu\nu'}-\Pi_{\nu\nu'}+\frac{y_{\nu\nu'}}{4[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}+\frac{y_{\nu\nu'}}{4[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}\right] \tag{3.4b}$$

where

$$y_{\nu\nu'}=d\,|\mathbf{k}_\perp-\mathbf{k}'_\perp|=\sqrt{2m}\,d[2\mu-(\lambda_\nu+\lambda_{\nu'})-2(\mu-\lambda_\nu)^{1/2}(\mu-\lambda_{\nu'})^{1/2}\cos(\phi-\phi')]^{1/2}$$

and

$$\begin{aligned}\Pi_{\nu\nu'}=&-\frac{(\nu+\nu')^2\pi^2}{4[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}+\frac{y^2_{\nu\nu'}}{4[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}-\frac{(\nu-\nu')^2\pi^2}{4[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}+\frac{y^2_{\nu\nu'}}{4[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}\\
&+\frac{\nu'(\nu+\nu')\pi^2}{4(y^2_{\nu\nu'}+\nu'^2\pi^2)[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}+\frac{\nu'(\nu'-\nu)\pi^2}{4(y^2_{\nu\nu'}+\nu'^2\pi^2)[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}-\frac{y^2_{\nu\nu'}}{4(y^2_{\nu\nu'}+\nu'^2\pi^2)[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}\\
&-\frac{y^2_{\nu\nu'}}{4(y^2_{\nu\nu'}+\nu'^2\pi^2)[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}+\frac{\nu(\nu+\nu')\pi^2}{4(y^2_{\nu\nu'}+\nu^2\pi^2)[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}+\frac{\nu(\nu-\nu')\pi^2}{4(y^2_{\nu\nu'}+\nu^2\pi^2)[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}\\
&-\frac{y^2_{\nu\nu'}}{4(y^2_{\nu\nu'}+\nu^2\pi^2)[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}-\frac{y^2_{\nu\nu'}}{4(y^2_{\nu\nu'}+\nu^2\pi^2)[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}\\
&+\frac{1}{4[y^2_{\nu\nu'}+(\nu+\nu')^2\pi^2]}+\frac{1}{4[y^2_{\nu\nu'}+(\nu-\nu')^2\pi^2]}\ .\end{aligned} \tag{3.5}$$

Now, once the above form of $K_{\alpha\beta}(\mathbf{k}_\perp,\mathbf{k}'_\perp;\nu,\nu')$ is known one can manipulate $H_{\rm dip}$ into the standard form, as discussed above, using the procedure outlined in Appendix A of Ref. 4. The coefficients g and h are found from the angular averages $\tilde{K}^0_{zz}$ and $\tilde{K}^1_{zz}$ defined as

$$\tilde{K}^0_{zz}\equiv\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}\tilde{K}_{zz},\quad \tilde{K}^1_{zz}\equiv\int\frac{d\phi}{2\pi}\int\frac{d\phi'}{2\pi}\cos(\phi-\phi')\tilde{K}_{zz}\ , \tag{3.6a}$$

with

$$\tilde{K}_{zz}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\equiv\sum_{\nu\nu'}\left[1-\left[\frac{\nu}{\nu_0}\right]^2\right]^{1/2}\left[1-\left[\frac{\nu'}{\nu_0}\right]^2\right]^{1/2}K_{zz}(\mathbf{k}_\perp,\mathbf{k}'_\perp;\nu,\nu')\ . \tag{3.6b}$$

To find g and h explicitly we perform the angular average and the two sums in (3.6) numerically. $H_{\rm dip}$ is then written as

$$H_{\rm dip}=\tfrac{1}{2}\tilde{E}_{\rm dip}\int\frac{d\phi}{2\pi}[\tilde{K}^{1}_{zz}\,|\,\mathbf{d}(\hat{\mathbf{k}})\,|^2+(-\tfrac{3}{2}\tilde{K}^{1}_{zz}-\tfrac{1}{2}\tilde{K}^{0}_{zz})\,|\,\mathbf{d}_\perp(\hat{\mathbf{k}})\,|^2+\tilde{K}^{0}_{zz}\,|\,\mathbf{d}_\perp(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\,|^2]\,, \tag{3.7}$$

where

$$\tilde{E}_{\rm dip}=\frac{\pi}{15}\gamma^2\tilde{\psi}^2=\frac{4}{25}\left[\frac{T_c}{T_c^{\rm bulk}}\right]^2 l^2(d,\nu_c)g_D/\kappa(d,\nu_c)\;. \tag{3.8}$$

The two quantities $l(d,\nu_c)$ and $\kappa(d,\nu_c)$ appearing in (3.8) are defined by

$$l(d,\nu_c)=\frac{\ln 1.14\dfrac{\omega_0}{T_c}}{\ln 1.14\dfrac{\omega_0}{T_c^{\rm bulk}}}\,, \tag{3.9}$$

$$\kappa(d,\nu_c)=\frac{\dfrac{\nu_c}{\nu_0}-\dfrac{\nu_c(\nu_c+1)(2\nu_c+1)}{3\nu_0^3}+\dfrac{\nu_c(\nu_c+1)(2\nu_c+1)(3\nu_c^2+3\nu_c-1)}{30\nu_0^5}}{\dfrac{\nu_c}{\nu_0}-\dfrac{\nu_c(\nu_c+1)(2\nu_c+1)}{6\nu_0^3}} \tag{3.10}$$

and g_D is given in Ref. 20. $\tilde{K}^{0}_{zz}$ and $\tilde{K}^{1}_{zz}$ are plotted as functions of thickness in Fig. 3.

As explained in the Introduction we are interested in the A phase only, and therefore we proceed to calculate Leggett's frequency

$$\tilde{\Omega}_A^2=\tfrac{3}{2}\gamma^2\tilde{E}_{\rm dip}\,|\,\tilde{K}^{1}_{zz}\,|\,/\chi_n\;. \tag{3.11}$$

Note that χ_n is the susceptibility of the normal fluid as introduced in Sec. II. In Fig. 4 $\tilde{\Omega}_A^2/\Omega_A^2$ is shown plotted versus film thickness. Ω_A denotes the bulk value. The characteristic "saw-tooth" oscillations in $\tilde{\Omega}_A$ are the direct consequence of the discontinuous behavior in the density of states, reflected in χ_n, and give conspicuous evidence of the quantum size effects. The jumps in $\tilde{\Omega}_A^2$ are roughly 5% of the overall value and could be detected in a careful NMR experiment. In fact, these oscillations probably represent the most straightforward experimental signature of the thin film regime in superfluid ^{3}He.

Note that in our calculation of $\tilde{\Omega}_A$ we have ignored the quasiparticle renormalization factors (for their definition see Leggett, Ref. 20). In our previous work[4] we have pointed out that the effect of quasiparticle corrections can be rather important for $d\simeq k_F^{-1}$. However, in the thickness regime considered in this paper we do not expect the quasiparticle corrections to introduce any qualitatively new effects. Their action is probably limited, as d decreases, to a slight change in the magnitude of $\tilde{\Omega}_A$. For very thin films this is expected to change but our present approach is not suitable for detailed calculations at very small thickness as discussed in the Introduction.

The situation will change somewhat in the presence of surface roughness. The effects of surface scattering on p-wave pairing turn out to be rather intricate, and some new possibilities appear. A discussion of some of these effects has been given by one of us[22] in a different context (possible p-wave pairing in heavy-fermion superconductors). A complete presentation will be found in Ref. 12. For the

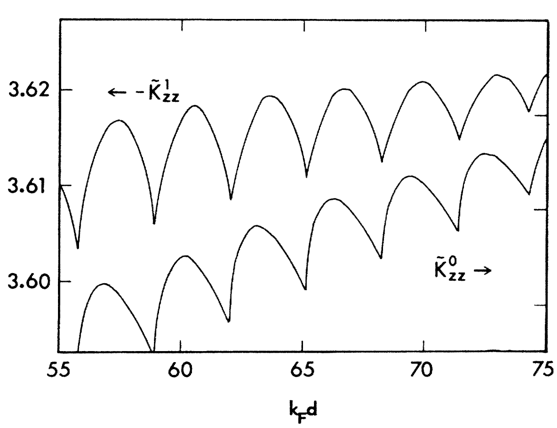

FIG. 3. Coefficients $\tilde{K}^{0}_{zz}$ and $\tilde{K}^{1}_{zz}$ of Eq. (3.6). As d increases they slowly tend to their bulk values, 24.0 and -4.0, respectively.

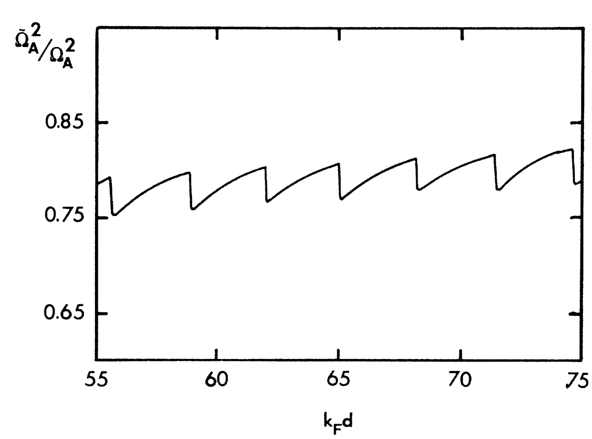

FIG. 4. The shift $\tilde{\Omega}_A^2$ in the transverse resonance frequency. The discontinuities occur through the jumps in χ_n (see Fig. 2). Note that $\tilde{\Omega}_A^2$ (Ω_A^2) are used here without the factors $1-T/T_c$ $[(1-T/T_c^{\rm bulk})]$. They are to be multiplied by these factors in order to obtain the standard measurable frequency shifts.

purposes of the present work we can summarize it as follows: The problem of boundary scattering can be mapped onto that of two-dimensional motion in a random potential. This potential is state dependent: Its strength increases with ν. Its effects on the broadening of the energy levels are described by a state-dependent scattering rate $S(\nu)$ which can be obtained by a fairly straightforward but lengthy calculation.[12] The result is stated in Ref. 22. For reasonably smooth surfaces (which is the case considered in the present paper) surface scattering effects lead only to an overall slight depression of $T_c(d)$ and to some rounding off (but not obliteration) of the discontinuities obtained here. These features should, therefore, be observable. As the roughness increases past a certain critical value which depends on the thickness, a different regime is reached where the superfluid becomes gapless.[12,22] However, before this occurs, the overall reduction in Δ or T_c, or changes in the qualitative features of our present results, are not serious within the thickness regime considered here.

IV. CONCLUSIONS

We have studied quantum size effects in superfluid ^{3}He films in the thickness regime $k_F^{-1} \ll d < \xi$, where normal-fluid properties are well understood and several simplifying assumptions concerning the pairing potential and the quasiparticle renormalization factors are possible. Explicit results were obtained for the variation with thickness of the critical temperature, superfluid gap, and shift in the transverse resonance frequency in the ABM state. The influence of the quantum size effects on the low-temperature specific heat of the superfluid was briefly discussed.

Our results indicate that quantum size effects are quite prominent in the superfluid ^{3}He films and very different from those found for s-wave pairing, which have been experimentally observed in superconducting films.[5] In a careful experiment one could detect the characteristic variation with thickness of various quantities and consequently acquire useful information about the nature of the superfluid state, the variation of the pairing interaction with thickness (which is likely to become important for very thin films) and the influence of boundary scattering.

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*Present address: Department of Physics, Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138.

†Permanent address: School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455.

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