PHYSICAL REVIEW B VOLUME 33, NUMBER 4 15 FEBRUARY 1986

Fractional statistics and the quantum Hall effect of two-dimensional fermion and boson systems

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(Received 25 October 1985)

Using fractional statistics, we discuss the quantum Hall effect (QHE) of two-dimensional fermion and boson systems. If the basic charge carriers in a system are fermions with charge e, a necessary condition to have the QHE at a filling factor $\nu = p/q$ with $\sigma_H = (p/q)e^2/h$ is that mutual primes p and q must satisfy $e^{ipq\pi} = (-1)^{p^2}$, hence the QHE is allowed only when q is odd. If the basic charge carriers are bosons with charge e, such as bound pairs of electrons, a necessary condition to have the QHE at $\nu = p/q$ with $\sigma_H = (p/q)e^2/h$ is that p and q must satisfy $e^{ipq\pi} = 1$, so the possible candidates are $\nu = (2n_1)/(2n_2+1)$ and $(2n_1+1)/(2n_2)$ where n_1 and n_2 are integers.

The odd-denominator rule is one of the important and interesting discoveries in the quantum Hall effect (QHE) of two-dimensional (2D) electron systems: a quantized Hall step is only found at filling factors $\nu = p/q$ with an odd q. The integral QHE (IQHE) has $q = 1$, and the fractional QHE (FQHE) has $q = 3, 5, 7, \ldots$.[1,2]

This rule has aroused considerable general interest.[3,4] Why, so far, is there no quantum Hall step found at $\nu = p/q$ with an even q? Under what conditions can a 2D electron system exhibit the QHE at even denominator fractions? Using fractional statistics,[5,6] we study this issue in the present paper. Our discussion will be limited to systems of one layer. When we refer to a fermion (or boson) system, we mean that the system has only one type of basic charge carriers which are fermions (or bosons). Under this definition, for example, a system consisting of electrons will be classified as a boson system if all electrons form into bound pairs as the basic charge carriers in the QHE. There is no difficulty in extending our discussion to systems having multiple layers and several types of basic charge carriers, but we will not consider this here.

Laughlin claims that the odd-denominator rule comes from the fact that there is no wave function available at $\nu = 1/(2m)$ for a fermion system where m is an integer.[7,8] This is interesting, but we need to study this further. Laughlin's wave function is an approximation for the ground states at $\nu = 1/(2m+1)$. Although, for example, the ground state at $\nu = \frac{1}{2}$ and $\frac{1}{4}$ is not close to Laughlin's wave function, we can still find some other approximation. Generally, in a many-body problem, one can always find some wave function to come near to a true ground state. Why does one approximation, such as Laughlin's, produce the QHE but the other one for $\nu = \frac{1}{2}$ and $\frac{1}{4}$ does not? In addition, for some systems, such as 2D electron gas in a periodic potential,[9] Laughlin's wave function is not applicable, so his argument is not relevant for cases like these. All the above mean that the nature of this rule must be much more profound.

This paper concludes that the nature of the selection rule for the QHE lies in fractional statistics. The connection between these statistics and the FQHE was initially suggested by Halperin.[10] We will show that if a fermion or a boson system has a quantized Hall step at $\nu = p/q$ (p and q are mutual primes) with $\sigma_H = (p/q)e^2/h$ where e is the charge of the fermion or boson (possibly different from the electron charge), identical quasiparticles of a fractional charge $(p/q)e$ can be produced. Then we present a general argument to demonstrate that quasiparticles obey fractional statistics with a phase factor $e^{i\pi p/q}$. From the result in Ref. 11, a necessary condition for fermion systems to exhibit the QHE at $\nu = p/q$ is that p and q satisfy $e^{ipq\pi} = (-1)^{p^2}$, thus establishing proof that the QHE for fermion systems is allowed only when q is odd. By the same argument, this necessary condition for 2D boson systems is that p and q must satisfy $e^{ipq\pi} = 1$, so the possible candidates are $\nu = (2n_1)/(2n_2+1)$ and $(2n_1+1)/(2n_2)$ where n_1 and n_2 are integers. There may be the IQHE, such as $\nu = 2, 4, \ldots$, and the FQHE with an even or an odd denominator, such as $\nu = \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \ldots$, and $\nu = \frac{2}{3}, \frac{4}{3}, \frac{2}{5}, \ldots$. This conclusion is consistent with recent numerical calculations.[12,13] As a result, if in a 2D electron system electrons form into bound pairs, the QHE may occur at even denominator fractions. Because the Landé g factor for electrons in GaAs is relatively small,[14] the possibility to have such pairs has been discussed.[15]

For simplicity, we consider a 2D electron system in the x-y plane in which single electrons are the basic charge carriers. To extend this discussion to other fermion systems is straightforward. We apply a strong magnetic field $\mathbf{B}$ in the z direction. There are N_e electrons total and $N_s = B$ (area)$/\phi_0$ ($\phi_0 = h/e$, a unit flux) Landau orbits in the lowest Landau level. The filling factor is $\nu = N_e/N_s = p/q$ where p and q are mutual primes. If there is a Hall step at $\nu = p/q$ (see Fig. 1), as ν slightly deviates from p/q, σ_H is still given by

$$\sigma_H(p/q \pm \delta\nu) = (p/q)e^2/h \quad . \tag{1}$$

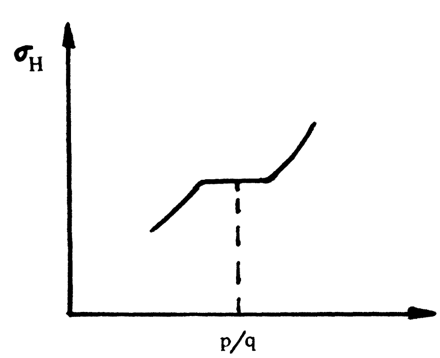

FIG. 1. Quantized Hall step at $\nu = p/q$.

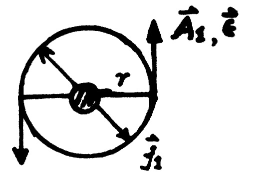

FIG. 2. A solenoid pierces the surface and produces a quasiparticle.

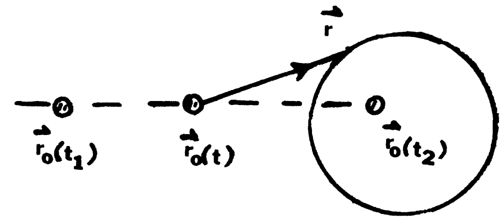

FIG. 3. Quasiparticle moves with the solenoid.

For example, experiments have shown that at $\nu = \frac{2}{3}$, for $\delta\nu$ as big as 0.07, σ_H is still given by the quantized value $\frac{2}{3}e^2/h$.[16]

We use a tiny solenoid to pierce the surface. If the solenoid has magnetic flux $-\phi_0$ (antiparallel to **B**), a localized quasielectron of charge $-(p/q)e$ is formed around the solenoid. A localized quasihole of charge $(p/q)e$ can be produced with a solenoid of flux ϕ_0.[17,18] The proof is simple. Let us assume that initially the solenoid has flux $\phi(0)=0$, and at time T, $\phi(T)=-\phi_0$. We consider a small circle of radius r around the solenoid (see Fig. 2). The solenoid produces a vector potential

$$\mathbf{A}_1 = \phi(t)\hat{\boldsymbol{\theta}}/(2\pi r) \;, \tag{2}$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector along the circumference. An electric field is produced,

$$\boldsymbol{\epsilon} = -\partial \mathbf{A}_1/\partial t = -\dot{\phi}\hat{\boldsymbol{\theta}}/(2\pi r) \;. \tag{3}$$

Therefore, there is a Hall current along the radial direction,

$$\mathbf{j}_1 = \sigma_H \dot{\phi}\hat{\mathbf{r}}/(2\pi r) \;, \tag{4}$$

where $\hat{\mathbf{r}}$ is a unit vector along the radial direction. Then the total charge which flows into the inside of the circle is $-e(p/q)$ because

$$Q = -\int_0^T dt 2\pi r j_1 = \sigma_H \phi(T) = -e(p/q) \;. \tag{5}$$

If the solenoid has flux ϕ_0, the charge is $\sigma_H\phi_0 = e(p/q)$. The quasiparticle is obviously localized. We should point out that physical solenoids are not necessary in our argument. Instead of using solenoids, we can change the density of **B** in a small region to produce quasiparticles. Since solenoids present a good picture for our discussion, we keep using solenoids throughout this paper.

We want to emphasize that the quantized Hall step is a crucial condition to produce identical quasiparticles. Because adding some solenoids to the system slightly changes N_s and ν, only when there is a quantized Hall step, σ_H is unchanged by Eq. (1), and identical quasielectrons can be produced if we use several solenoids of flux $-\phi_0$ to pierce the system. A solenoid of flux $-q\phi_0$ produces q quasielectrons. On the other hand, this solenoid produces p electrons. Therefore, we identify a cluster of q quasielectrons as p electrons.

Now we move a solenoid of flux $-\phi_0$. As seen in Fig. 3, this moving solenoid produces a vector potential

$$\mathbf{A}_2 = \phi_0 \hat{\mathbf{z}} \times [\mathbf{r} - \mathbf{r}_0(t)]/\{2\pi[\mathbf{r} - \mathbf{r}_0(t)]^2\} \;, \tag{6}$$

where $\mathbf{r}_0(t)$ is the position of the solenoid at time t. Because $\mathbf{A}_2$ varies with time, it produces an electric field. The induced Hall current is given by

$$\mathbf{j}_2 = -\sigma_H \hat{\mathbf{z}} \times \partial \mathbf{A}_2/\partial t \;. \tag{7}$$

Let us consider a small closed loop (Fig. 3). At time t_1, the solenoid is outside the loop, but at time t_2 the solenoid reaches a point inside the loop. The total charge following the solenoid into the loop is just $-(p/q)e$, because

$$Q = \int_{t_1}^{t_2} dt \oint \mathbf{j}_2(\hat{\mathbf{z}} \times d\mathbf{l}) = -\sigma_H \frac{\phi_0}{2\pi} \oint \{\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_0(t_2)/[\mathbf{r} - \mathbf{r}_0(t_2)]^2 - \hat{\mathbf{z}} \times [\mathbf{r} - \mathbf{r}_0(t_1)]/[\mathbf{r} - \mathbf{r}_0(t_1)]^2\} d\mathbf{l} = -\sigma_H \phi_0 \;. \tag{8}$$

This proves that when we adiabatically move a solenoid, the quasiparticle is moving with it.

Quasiparticles obey fractional statistics. Let us move a quasielectron of charge $Q = -(p/q)e$ along a counterclockwise loop to make a circle. The size of the quasiparticle is negligible in comparison with the radius of the loop. At first, we assume that there is no solenoid inside the loop. After one circle, the quasiparticle picks up a phase factor,

$$\exp(iQ\Phi/\hbar) \;, \tag{9}$$

where Φ is the total magnetic flux inside the loop. We then place another solenoid of flux $-\phi_0$ inside the loop and move the original quasielectron along the same loop to make a circle again (see Fig. 4). Because the total magnetic flux is now reduced by one unit, the new phase factor is given by

$$\exp[iQ(\Phi - \phi_0)/\hbar] \;. \tag{10}$$

The difference between Eqs. (10) and (9) is a factor

$$\exp(i\eta) = \exp(i2\pi p/q) \;. \tag{11}$$

Since the second solenoid produces an identical quasielectron in the inside of the loop, the phase factor in Eq. (11) is a phase produced after one quasielectron circles around another one by 2π. As one quasielectron is rotating about

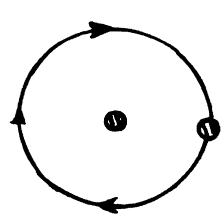

FIG. 4. One quasiparticle circles around another one.

the second one, the angle η in the phase factor $e^{i\eta}$ is linearly increasing. By the Bohm-Aharonov effect, $\eta = 2\pi p/q$ is uniquely determined when one circle is completed. Therefore, η must be $p\pi/q$ as half a circle is completed. As a result, if we make an interchange of two identical quasielectrons along a counterclockwise loop, the phase yielded is also

$$\exp(ip\pi/q) \ . \tag{12}$$

This means that quasielectrons obey fractional statistics.[7,8] By the same argument, quasiholes also obey fractional statistics with the same phase factor. When $\nu = 1/(2m+1)$, the above result is the same as that obtained by Arovas, Schrieffer, and Wilczek who studied the statistics of Laughlin's quasiparticles.[19]

We can show that the odd-denominator rule is the consequence of statistics for fermion systems. Let us first assume that at $\nu = p/q$ there is a Hall step $\sigma_H = (p/q)e^2/h$, so we can produce identical quasiparticles. We consider a counterclockwise interchange of two identical clusters, each having q quasielectrons joined together (see Fig. 5). The size of these clusters is negligible in comparison with their separation. The ordering of quasiparticles inside the cluster is fixed during this interchange. Because every quasielectron in one cluster interchanges q quasielectrons in another cluster, there are $q \times q = q^2$ pair permutations altogether. The phase factor produced in this interchange is

$$\exp[(ip\pi/q)q^2] = \exp(ipq\pi) \ . \tag{13}$$

On the other hand, from Fermi statistics, an interchange of two identical clusters containing p electrons each gives a phase factor $(-1)^{p^2}$ because there are $p \times p = p^2$ interchanges of pairs of electrons. A cluster of q quasielectrons is equivalent to a cluster of p electrons, so we must have

$$e^{ipq\pi} = (-1)^{p^2} \ . \tag{14}$$

Equation (14) is a necessary condition for a fermion system to have the QHE at $\nu = p/q$ with $\sigma_H = (p/q)e^2/h$. Now if q were even, the left side of Eq. (14) would become 1, then p would have to be even to make the right side equal to 1. This immediately contradicts with the fact that p and q are mutual primes. Therefore, the assumption of a Hall step at $\nu = p/q$ with an even q is not compatible with Eq. (14). We must conclude that for a fermion system no quantized Hall step can be formed when q is even. If q is odd, Eq. (14) does not produce any contradiction. Statistics allow the QHE at odd q. This is the odd-denominator rule observed in the experiments of electron systems.

It is easy to see that the above argument can be applied to 2D interacting bosons in a strong magnetic field. A system consisting of singlet bound pairs of electrons is an example of boson systems. If these bound pairs do not break into single electrons in the QHE, they are basic charge carriers, so we must treat them as elematary particles in the QHE. When there is a quantized Hall step at $\nu = p/q$ with

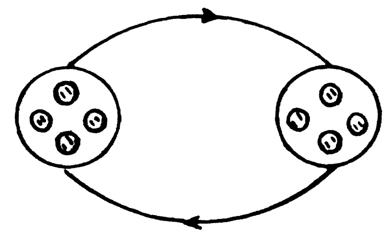

FIG. 5. Counterclockwise interchange of two identical clusters having q quasiparticles each.

$\sigma_H = (p/q)e^2/h$, we can also use solenoids to produce identical quasiparticles and antiquasiparticles which obey fractional statistics with the same phase factor in Eq. (12). Here e is the charge of the boson particle which may be different from the electron charge. Because now a cluster of q quasiparticles is equivalent to a cluster of p bosons and an interchange of two identical clusters of bosons does not produce any phase factor, the necessary condition becomes that p and q satisfy

$$\exp(ipq\pi) = 1 \ . \tag{15}$$

From it, ν with both odd p and q, is excluded from the QHE. There are two possibilites to hold Eq. (15): (1) even p and odd q; (2) odd p and even q. Mutual primes p and q cannot both be even. The possible candidates are $\nu = (2n_1)/(2n_2+1)$ and $(2n_1+1)/(2n_2)$ where n_1 and n_2 are integers. Yoshioka's numerical calculation of 2D boson systems at $\nu < 1$ shows big downward cusps in the ground-state energy at $\nu = \frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$.[12] This is consistent with our prediction from fractional statistics. Therefore, a 2D electron system may have the QHE at even denominator fractions if, for example, electrons form into bound pairs.

As a result of the repulsive Coulomb interaction, one Landau orbit only accommodates one boson when $\nu < 1$. If $\nu > 1$, more than one boson occupies one orbit in the lowest Landau level. For example, $\nu = 2$ does not mean that boson particles fully occupy two Landau levels. Haldane has extended his hierarchy scheme to boson systems.[3] He takes $\nu = 1/(2m)$ from Laughlin's wave function as the basic ones, then builds up the hierarchy. He finds that $\nu = p/q$ with both odd p and q is excluded from the QHE, a result consistent with ours. But there is also an important difference; his hierarchy for bosons only allows $\nu < 1$. In our scheme, $\nu > 1$ is not excluded; for example, $\nu = 2$ and $\frac{3}{2}$ are possible candidates. To resolve this difference, we have carried out a numerical calculation of small boson systems.[13] The result supports our conclusion that there are downward cusps in the ground state energy at $\nu = 2$ and $\nu = \frac{3}{2}$, so $\nu > 1$ cannot be excluded from the QHE of boson systems.

I wish to thank Professor F. D. M. Haldane for many useful discussions and Professor Kazumi Maki for his suggestion to consider boson systems. This work was supported in part by NSF Grant No. DMR-8405347.

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