Magneto-roton theory of collective excitations in the fractional quantum Hall effect

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We present a theory of the collective excitation spectrum in the fractional quantum Hall effect which is closely analogous to Feynman's theory of superfluid helium. The predicted spectrum has a large gap at $k=0$ and a deep magneto-roton minimum at finite wave vector, in excellent quantitative agreement with recent numerical calculations. We demonstrate that the magneto-roton minimum is a precursor to the gap collapse associated with the Wigner crystal instability occurring near $\nu=\frac{1}{7}$. In addition to providing a simple physical picture of the collective excitation modes, this theory allows one to compute rather easily and accurately experimentally relevant quantities such as the susceptibility and the ac conductivity.

I. INTRODUCTION

The quantum Hall effect[1-4] is a remarkable macroscopic quantum phenomenon occuring in the two-dimensional electron gas (inversion layer) at high magnetic fields and low temperatures. The Hall resistivity is found to be quantized with extreme accuracy[5] in the form

$$\rho_{xy}=h/e^2 i \ . \tag{1.1}$$

In the integral Hall effect the quantum number i take on integral values. In the fractional quantum Hall effect (FQHE) the values of i are rational fractions with odd denominators.

Associated with and of central importance to this quantization of ρ_{xy} is the appearance of exponentially small dissipation,

$$\rho_{xx}\sim e^{-\Delta/2T} \ . \tag{1.2}$$

The activation energy $\Delta\approx 100$ K for the integral case is associated with disorder and the mobility gap between Landau levels and is thus primarily a single-particle effect. The fractional case occurs in low-disorder, high-mobility samples with partially filled Landau levels for which there is no single-particle gap. In this case the excitation gap is a collective effect arising from many-body correlations due to the Coulomb interaction and is therefore smaller in magnitude ($\Delta_{1/3}\approx 6$ K). Considerable progress has recently been made toward understanding the nature of the many-body ground state, which at least for Landau-level–filling factors of the form $\nu=1/m$, where m is an odd integer, appears to be well described by Laughlin's variational wave function.[6] We have recently reported a theory[7] of the collective excitation spectrum which is closely analogous to Feynman's theory of superfluid helium.[8] The spectrum has an excitation gap which is found to be relatively large at zero wave vector but at finite wave vector exhibits a deep "magneto-roton minimum" quite analogous to the roton minimum in helium. The purpose of the present paper is to present a more complete description and derivation of this theory and to use it to make specific experimental predictions.

The outline of the paper is as follows. Section II contains a review of Feynman's arguments for superfluid ^{4}He. In Sec. III we discuss the application of these ideas to fermion systems in a magnetic field and in Sec. IV we specialize to the case of the lowest Landau level. In Secs. V–VII we discuss evaluation of the static structure factor, the collective mode dispersion, and the role of backflow corrections. In Sec. VIII we consider the large-wave-vector limit. In Secs. IX–XI we discuss finite-thickness effects, disorder, and linear response, and in Sec. XII we present a summary of our conclusions.

II. REVIEW OF FEYNMAN'S THEORY OF ^{4}He

In a beautiful series of papers[8] Feynman has laid out an elegantly simple theory of the collective excitation spectrum in superfluid ^{4}He. Even though the underlying ideas were developed for a neutral Bose system in three dimensions, they are sufficiently general that they can be applied *mutatis mutandis* to a charged Fermi system in two dimensions in a high magnetic field. Let us therefore now review Feynman's arguments.

Because ^{4}He is a Bose system, the ground-state wave function is symmetric under particle exchange and has no nodes. Using these facts Feynman argues that there can be no low-lying single-particle excitations, so that the only low-energy excitations are long-wavelength density oscillations (phonons). Now suppose that somehow one knew the exact ground-state wave function, ψ. Then one could write the following variational wave function to describe a

density-wave-excited state which still contains most of the favorable correlations occurring in the ground state:

$$\phi_{\mathbf{k}}(\mathbf{r}_1,\ldots,\mathbf{r}_n)=N^{-1/2}\rho_{\mathbf{k}}\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)\ , \tag{2.1}$$

where $\rho_{\mathbf{k}}$ is the Fourier transform of the density

$$\rho_{\mathbf{k}}=\int d^2\mathbf{R}\exp(-i\mathbf{k}\cdot\mathbf{R})\rho(\mathbf{R})\ , \tag{2.2}$$

$$\rho(\mathbf{R})=\sum_{j=1}^{N}\delta^2(\mathbf{R}-\mathbf{r}_j)\ , \tag{2.3}$$

so that

$$\rho_{\mathbf{k}}=\sum_{j=1}^{N}\exp(-i\mathbf{k}\cdot\mathbf{r}_j)\ . \tag{2.4}$$

Note that since the ansatz wave function contains the ground-state wave function as a factor, favorable correlations are automatically built in. Nevertheless, the excited state is orthogonal to the ground state as required. This may be seen by writing

$$\langle\psi\,|\,\phi_{\mathbf{k}}\rangle=N^{-1/2}\int d^2\mathbf{R}\exp(-i\mathbf{k}\cdot\mathbf{R})\langle\psi\,|\,\rho(\mathbf{R})\,|\,\psi\rangle\ . \tag{2.5}$$

By hypothesis is the ground state $|\,\psi\rangle$ is a liquid with a homogeneous density. Hence (for $k\neq 0$) the overlap integral in Eq. (2.5) vanishes.

To see that $\phi_{\mathbf{k}}$ represents a density wave, consider the following. $\phi_{\mathbf{k}}$ is a function of the particle positions. For configurations in which the particles are more or less uniformly distributed, $\rho_{\mathbf{k}}$, and therefore $\phi_{\mathbf{k}}$, will be close to zero. Hence such configurations will have a low probability of occurrence in the excited state. On the other hand, configurations with some degree of density modulation at wave vector k will have a finite value of $\rho_{\mathbf{k}}$, and $\phi_{\mathbf{k}}$, will be proportionately large, making the probability of such configurations greater. Hence this represents a density wave. Note, however, that the phase of $\rho_{\mathbf{k}}$, and thus $\phi_{\mathbf{k}}$, will match the phase of the density wave. All phases are equally likely if ψ corresponds to a liquid. Hence the average density is still uniform in the excited state. A simple way of interpreting all of this is to view $\rho_{\mathbf{k}}$ as one of a set of collective coordinates describing the particle configuration. We can then make an analogy with the simple harmonic oscillator in which the exact excited-state wave function ψ_1 is obtained from the ground state ψ_0 by multiplication by the coordinate

$$\psi_1(x)=x\psi_0(x)\ . \tag{2.6}$$

ψ_1 is orthogonal to ψ_0, and ψ_1 vanishes for the configuration ($x=0$) at which ψ_0 is peaked ("uniform density"). ψ_1 is nonzero when the coordinate is displaced ("density wave"), but has a phase that varies with displacement so that the average value of the coordinate is zero ("uniform average density") even in the excited state.

Having established an ansatz variational wave function, we need to evaluate the energy. This requires a knowledge of the norm of the wave function

$$s(k)=\langle\phi_{\mathbf{k}}\,|\,\phi_{\mathbf{k}}\rangle=N^{-1}\langle\psi\,|\,\rho_{\mathbf{k}}^{\dagger}\rho_{\mathbf{k}}\,|\,\psi\rangle\ , \tag{2.7}$$

but this is nothing more than the static structure factor for the ground state, a quantity which can be directly measured using neutron scattering. For later purposes it will be convenient to note that $s(k)$ is also related to the Fourier transform of the radial distribution function for the ground state, $g(r)$:

$$s(k)=1+\rho\int d^2\mathbf{R}\exp(-i\mathbf{k}\cdot\mathbf{R})[g(\mathbf{R})-1]+\rho(2\pi)^2\delta^2(\mathbf{k})\ , \tag{2.8}$$

where ρ is the average density.

The variational estimate for the excitation energy is the usual expression

$$\Delta(k)=f(k)/s(k)\ , \tag{2.9}$$

where

$$f(k)=\langle\phi_{\mathbf{k}}\,|\,H-E_0\,|\,\phi_{\mathbf{k}}\rangle\ , \tag{2.10}$$

and H is the Hamiltonian and E_0 is the ground-state energy. Using Eq. (2.1) we may rewrite Eq. (2.10) as

$$f(k)=N^{-1}\langle\Psi\,|\,\rho_{\mathbf{k}}^{\dagger}[H,\rho_{\mathbf{k}}]\,|\,\Psi\rangle\ , \tag{2.11}$$

which will be recognized as the oscillator strength.[9] Because the potential energy and the density are both simply functions of position, they commute with each other. The kinetic energy, on the other hand, contains derivatives which do not commute with the density. This yields the universal result[9]

$$f(k)=\frac{\hbar^2k^2}{2m}\ , \tag{2.12}$$

making the oscillator-strength sum independent of the interaction potential. We emphasize this point because a rather different result will be obtained for the case of the FQHE.

Combining Eqs. (2.12) and (2.9) yields the Feynman-Bijl formula[8,9] for the excitation energy,

$$\Delta(k)=\frac{\hbar^2k^2}{2ms(k)}\ . \tag{2.13}$$

We can interpret this as saying that the collective-mode energy is just the single-particle energy $\hbar^2k^2/2m$ renormalized by the factor $1/s(k)$ which represents the effect of correlated motion of the particles. We emphasize that since we have invoked the variational principle, $\Delta(k)$ is a rigorous upper bound to the lowest excitation energy at wave vector k.

In order to gain a better understanding of the meaning of $\Delta(k)$ and the underlying assumptions that have been used, let us rederive Eq. (2.9) by a different method. Consider the dynamic structure factor defined by ($\hbar=1$ throughout)

$$S(k,\omega)=N^{-1}\sum_n\langle 0\,|\,\rho_{\mathbf{k}}^{\dagger}\,|\,n\rangle\delta(\omega-E_n+E_0)\langle n\,|\,\rho_{\mathbf{k}}\,|\,0\rangle\ , \tag{2.14}$$

where the sum is over the complete set of exact eigenstates. Using Eq. (2.7) the static structure factor is related to $S(k,\omega)$ by

$$s(k)=\int_0^{\infty}d\omega\,S(k,\omega)\ , \tag{2.15}$$

and using Eq. (2.11) we see that the oscillator strength is related to $S(k,\omega)$ by

$$f(k)=\int_0^\infty d\omega\,\omega S(k,\omega)\ . \tag{2.16}$$

Substitution of these results into Eq. (2.9) shows that the Feynman-Bijl expression for $\Delta(k)$ is actually the exact first moment of the dynamic structure factor. That is, $\Delta(k)$ represents the average energy of the excitations which couple to the ground state through the density. This is consistent with the idea that $\Delta(k)$ is a variational bound on the collective-mode energy, since the average energy necessarily exceeds the minimum excitation energy. Note that if the oscillator-strength sum is saturated by a single mode, then the mean excitation energy and the minimum will be the same and the Feynman-Bijl expression will be exact. This idea is consistent with Feynman's argument that there are no low-lying single-particle excitations. The assumption that $\phi_{\mathbf{k}}$ in Eq. (2.1) is a good variational wave function is equivalent to assuming that the density-wave saturates the oscillator-strength sum rule.

How well does the Feynman-Bijl formula work? Evaluating $s(k)$ in Eq. (2.13) from the experimental neutron-scattering cross section and speed-of-sound data yields[8,9] a collective-mode frequency which vanishes linearly at small k (with a slope corresponding exactly to the speed of sound) and exhibits the famous "roton minimum" near $k=2$ Å^{-1}. The roton minimum is due to a peak in $s(k)$ associated with the short-range order in the liquid. The predicted dispersion curve is in good qualitative agreement with experiment, but the predicted frequency at the roton minimum is about a factor of 2 too high.[8,9] Feynman and Cohen[8] have shown that this problem can be remedied by including back-flow corrections which guarantee that the continuity equation $\mathbf{\Delta}\cdot\langle\mathbf{J}\rangle=0$ is satisfied by the variational wave function. These corrections bring the predicted mode energy into excellent quantitative agreement with experiment.

These considerations of Feynman's arguments lead us to the following conclusions. One expects on quite general grounds that (at least for long wavelengths) the low-lying excited states of any system will include density waves. If because of special circumstances (such as those occurring in superfluid ^{4}He) there are no low-lying single-particle states, then the Feynman-Bijl expression for the collective-mode energy will be valid. The expression is also qualitatively correct even at short wavelengths and yields quantitative agreement with experiment if explicit back-flow corrections are included.

III. APPLICATION TO FERMIONS

The single-mode approximation seems less likely to succeed for Fermi systems. For instance, examination of $S(k,\omega)$ for the three-dimensional electron gas (jellium) shows the existence of not only a collective mode (the plasmon) but also a large continuum of single-particle excitations.[9] The low-lying continuum is due to the small kinetic energy of excitations across the Fermi surface. Despite this problem Lundqvist[10] and Overhauser[11] have shown that very useful results can be obtained from the single-mode approximation (SMA). It is straightforward to prove from the compressibility sum rule[9] that the plasmon mode saturates the oscillator-strength sum rule in the limit $k\to 0$. Hence the SMA gives the exact plasma frequency $\Delta(0)=\hbar\omega_p$. For finite k the SMA breaks down, but as noted above $\Delta(k)$ is the exact first moment of the excitation spectrum. Thus for large k, $\Delta(k)$ lies at the centroid of the continuum. Hence the SMA is exact at long wavelengths and gives a reasonable fit to the single-particle continuum even at short wavelengths. The same statement is true for the two-dimensional case, where the plasmon dispersion is $\omega_p\sim k^{1/2}$ at long wavelengths.

We are now in a position to ask what the effect of a large magnetic field is on the Fermi system. It is important to note that the neutral excitations are still characterized[12] by a conserved wave vector $\mathbf{k}$, but that in two dimensions the magnetic field quenches the single-particle continuum of kinetic energy, leaving a series of discrete, highly degenerate Landau levels evenly spaced in energy at intervals of $\hbar\omega_c$, where ω_c is the classical cyclotron frequency. Consider first the case of the filled Landau level ($\nu=1$). Because of Pauli exclusion the lowest excitation is necessarily the cyclotron mode in which particles are excited into the next Landau level. Furthermore, from Kohn's theorem[13] we know that in the limit of zero wave vector, the cyclotron mode occurs at precisely $\hbar\omega_c$ and saturates the oscillation-strength sum rule. Hence, once again the SMA is exact at long wavelengths and yields $\Delta(0)=\hbar\omega_c$. To see explicitly that this is so, note that for $\nu=1$ the radial distribution function is known exactly[6,14] (neglecting mixing of Landau levels in the ground state),

$$g(r)=1-\exp(-r^2/2l^2)\ , \tag{3.1}$$

where $l=(eB/\hbar c)^{-1/2}$ is the magnetic length. Using (2.8) and $\rho=\nu/(2\pi l^2)$, we have, for $k\neq 0$,

$$s(k)=1-\exp(-k^2l^2/2)\ . \tag{3.2}$$

Using (2.9) the predicted mode energy is

$$\Delta(k)=\frac{\hbar^2k^2}{2m\,[1-\exp(-k^2l^2/2)]}\ , \tag{3.3}$$

but $\hbar^2/ml^2=\hbar\omega_c$, so that we finally obtain

$$\Delta(k)=\hbar\omega_c\frac{(kl)^2/2}{1-\exp(-k^2l^2/2)}\ , \tag{3.4}$$

which has the correct limit $\Delta(0)=\hbar\omega_c$. We emphasize that this was derived with a ground-state structure factor calculated by neglecting Landau-level mixing, but that Kohn's theorem[13] requires that the same result be obtained (for $k\to 0$) for the exact ground state.

For the FQHE we are interested not in the case $\nu=1$, but rather we need to consider the fractionally filled Landay level. The Pauli principle now no longer excludes low-energy intra-Landau-level excitations. It is these low-lying excitations rather than the high-energy inter-Landau-level cyclotron modes which are of primary importance to the FQHE. Let us therefore consider what the SMA yields for the case $\nu<1$ by recalling the argument leading to Kohn's theorem.[13] At asymptotically

long wavelengths an external perturbation couples only to the center-of-mass (c.m.) motion. The c.m. degree of freedom has the excitation spectrum of a free particle in the magnetic field and is unaffected by the correlations and interactions among the individual particles. Hence once again the cyclotron mode saturates the oscillator-strength sum rule and the SMA yields $\Delta(0)=\hbar\omega_c$ independent of the filling factor and the interaction potential. The SMA thus tells us nothing about the low-energy modes of interest.

The root of this difficulty can be traced back to Eq. (2.1). For small k the variational excited state has most of its weight in the next-higher Landau level. We can greatly improve the variational bound on the excitation energy by insisting that the excited state lie entirely within the lowest Landau level. This can be enforced by replacing Eq. (2.1) by

$$\phi_{\mathbf{k}}=N^{-1/2}\bar{\rho}_{\mathbf{k}}\psi\ , \tag{3.5}$$

where $\bar{\rho}_{\mathbf{k}}$ is the projection of the density operator onto the subspace of the lowest Landau level. Providing that this projection can be explicitly carried out, we may derive a new approximation (the projected SMA) for the low-lying collective-mode frequency,

$$\Delta(k)=\bar{f}(k)/\bar{s}(k)\ , \tag{3.6}$$

where $\bar{f}$ and $\bar{s}$ are, respectively, the projected analogs of the oscillator strength and the static structure factor. Formulation of the projection scheme and the derivation of (3.6) are carried out in the next section.

IV. PROJECTION ONTO THE LOWEST LANDAU LEVEL

The formal development of quantum mechanics within the subspace of the lowest Landau level has been presented elsewhere.[15] We briefly review here the pertinent results. Taking the magnetic length to be unity ($l=1$) and adopting the symmetric gauge (with $\mathbf{B}=-B\hat{\mathbf{z}}$), the single-particle eigenfunctions of kinetic energy and angular momentum in the lowest Landau level are[6,15]

$$\Phi_m(z)=\frac{1}{(2\pi 2^m m!)^{1/2}}z^m\exp(-|z|^2/4)\ , \tag{4.1}$$

where m is a non-negative integer and $z=x+iy$ is the complex representation of the particle position. We wish to focus on the analytic part of the wave function and ignore as much as possible the ubiquitous Gaussian factor. Hence we define, following Bargmann,[15,16] a Hilbert space of analytic functions with an inner product

$$(f,g)=\int d\mu(z)f^*(z)g(z)\ , \tag{4.2}$$

where the Gaussian factors from (4.1) have been absorbed into the measure:

$$d\mu(z)=(2\pi)^{-1}dx\,dy\exp(-|z|^2/2)\ . \tag{4.3}$$

Operators on this space must take analytic functions into analytic functions. Hence a natural pair to consider is

$$a^\dagger=z\ , \tag{4.4a}$$

$$a=2\frac{d}{dz}\ . \tag{4.4b}$$

These are mutually adjoint with respect to the measure defined in (4.3) and represent ladder operators for the angular momentum.[15]

At this point it is useful to note that the adjoint of z is not the same as the Hermitian conjugate of z, which is z^*. z^* is not analytic in z and hence takes functions out of the Hilbert space (it mixes Landau levels). However, for any two states f and g in the Hilbert space,

$$(f,z^*g)=(zf,g)=(f,z^\dagger g)=\left[f,2\frac{d}{dz}g\right]\ . \tag{4.5}$$

Thus the projection of z^* onto the lowest Landau level is

$$\overline{z^*}=z^\dagger=2\frac{d}{dz}\ . \tag{4.6}$$

Since $\overline{z^*}$ and z do not commute, one has to normal order the derivatives to the left.[15] These results are trivially extended to the many-particle case and the projection of the density operator is easily accomplished as follows. Writing the dot product in complex notation, the density operator from Eq. (2.4) becomes

$$\rho_{\mathbf{k}}=\sum_{j=1}^{N}\exp\left[-\frac{ik}{2}z_j^*-\frac{ik^*}{2}z_j\right]\ . \tag{4.7}$$

The projected version is simply (note the normal ordering)

$$\bar{\rho}_k=\sum_{j=1}^{N}\exp\left[-ik\frac{\partial}{\partial z_j}\right]\exp\left[-\frac{ik^*}{2}z_j\right]\ . \tag{4.8}$$

We are now in a position to project the Hamiltonian. The kinetic energy can be ignored since it is an irrelevant constant $N\hbar\omega_c/2$ in the lowest Landau level. The potential energy is

$$V=\tfrac{1}{2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}v(q)\sum_{i\neq j}\exp[i\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j)]\ , \tag{4.9}$$

where $v(q)$ is the Fourier transform of the interaction potential. The projection of V is

$$\bar{V}=\tfrac{1}{2}\int\frac{d^2\mathbf{q}}{(2\pi)^2}v(q)(\bar{\rho}_{\mathbf{q}}\bar{\rho}_{\mathbf{q}}-\rho e^{-q^2/2})\ . \tag{4.10}$$

Note that, just as in the usual case, $\bar{\rho}_{\mathbf{q}}^\dagger=\bar{\rho}_{-\mathbf{q}}$.

In analogy with Eq. (2.11), the projected oscillator strength is

$$\bar{f}(k)=N^{-1}\langle 0|\bar{\rho}_{\mathbf{k}}^\dagger[\bar{H},\bar{\rho}_{\mathbf{k}}]|0\rangle\ , \tag{4.11}$$

where $|0\rangle$ is the ground state (represented as a member of the Hilbert space of analytic functions). Note that previously it was the kinetic energy which contained derivatives and hence failed to commute with the density. Now the kinetic energy is an irrelevant constant, but both the potential-energy and density operators contain derivatives. Thus Eq. (4.11) becomes

$$\bar{f}(k)=N^{-1}\langle 0|\bar{\rho}_{\mathbf{k}}^\dagger[\bar{V},\bar{\rho}_{\mathbf{k}}]|0\rangle\ . \tag{4.12}$$

The meaning of this is that since the kinetic energy has been quenched by the magnetic field, the scale of the

collective-mode energy is set solely by the scale of the interaction potential, which is, of course, as it should be.

In order to evaluate (4.12) it is convenient to note that the projected density operators obey a closed Lie algebra defined by

$$[\bar{\rho}_{\mathbf{k}},\bar{\rho}_{\mathbf{q}}]=(e^{k^*q/2}-e^{kq^*/2})\bar{\rho}_{\mathbf{k}+\mathbf{q}} . \tag{4.13}$$

It is convenient to use parity symmetry in k to rewrite Eq. (4.12) as a double commutator,

$$\bar{f}(k)=\frac{1}{2N}\langle 0|[\bar{\rho}_{\mathbf{k}}^{\dagger},[\bar{V},\bar{\rho}_{\mathbf{k}}]]|0\rangle . \tag{4.14}$$

Using Eq. (4.10), Eq. (4.14) is readily evaluated with the commutation relation given in (4.13),

$$\begin{aligned}\bar{f}(k)=\tfrac{1}{2}\sum_q v(q)(e^{q^*k/2}-e^{qk^*/2})\\ \times[\bar{s}(q)e^{-k^2/2}(e^{-k^*q/2}-e^{-kq^*/2})\\ +\bar{s}(k+q)(e^{k^*q/2}-e^{kq^*/2})] ,\end{aligned} \tag{4.15}$$

where $\bar{s}(q)$ is the projected static structure factor,

$$\bar{s}(q)=N^{-1}\langle 0|\bar{\rho}_{\mathbf{q}}^{\dagger}\bar{\rho}_{\mathbf{q}}|0\rangle . \tag{4.16}$$

Using

$$\overline{\rho_{\mathbf{q}}^{\dagger}\rho_{\mathbf{q}}}=\bar{\rho}_{\mathbf{q}}^{\dagger}\bar{\rho}_{\mathbf{q}}+(1-e^{-|q|^2/2}) , \tag{4.17}$$

one obtains the relation

$$\bar{s}(q)=s(q)-(1-e^{-|q|^2/2}) , \tag{4.18}$$

where $s(q)$ is the ordinary static structure factor given in Eqs. (2.7) and (2.8). From Eq. (3.2) we see that $\bar{s}(q)$ vanishes identically for the filled Landau level. This is simply a reflection of the fact that it is not possible to create any excitations within the lowest Landau level when it is completely filled.

Clearly, Eq. (4.15) is more complicated than its analog, (2.12). Nevertheless, it is still true that knowledge of the static structure factor is all that is required to evaluate (4.15) and (4.16) and hence obtain the projected SMA mode energy:

$$\Delta(k)=\bar{f}(k)/\bar{s}(k) . \tag{4.19}$$

The essence of the Feynman-Bijl result (2.13) is still maintained—namely that one can express a dynamical quantity, the collective-mode energy, solely in terms of static properties of the ground state.

Let us begin our evaluation of (4.19) by consideration of the small-k limit. We assume throughout that the ground state is an isotropic and homogeneous liquid. Direct expansion of Eq. (4.15) shows that, for small k, $\bar{f}(k)$ vanishes like $|k|^4$. Indeed, one can show that this is true, in general, because Kohn's theorem tells us that the total oscillator-strength sum $f(k)=\hbar^2k^2/2m$ is saturated by the cyclotron mode (to leading order in $|k|^2$). Hence the intra-Landau-level contribution [which is $\bar{f}(k)$] must quite generally vanish faster than $|k|^2$ for both solid and liquid ground states. Given that

$$\bar{f}(k)\sim|k|^4 , \tag{4.20}$$

it follows from (4.19) that a necessary (but not sufficient) condition for the existence of a finite direct ($k=0$) gap is

$$\bar{s}(k)\sim|k|^4 . \tag{4.21}$$

Equation (4.21) is a *sufficient* condition only within the SMA, but as the following argument shows, Eq. (4.21) is always a *necessary* condition for a gap. Equation (4.19) gives the exact first moment of the (intra-Landau-level) excitation spectrum. If $\bar{s}(k)$ vanishes slower than $|k|^4$, then the mean excitation energy vanishes as k approaches zero and there can be no gap. If $\bar{s}(k)\sim|k|^4$, then the mean excitation energy is finite for small k. This does not prove that there is a gap; however, it seems plausible that in this system (unlike ordinary jellium) there can be no low-lying single-particle excitations to invalidate the SMA and defeat the gap since the kinetic energy necessary to produce such a continuum has been quenched by the magnetic field.

Having established the importance of the condition $\bar{s}(k)\sim|k|^4$, let us investigate whether or not this condition obtains. Using Eq. (4.18) and expanding Eq. (2.8) for small k shows that $\bar{s}(k)\sim|k|^4$ if and only if $M_0=M_1=-1$, where

$$M_n\equiv\rho\int d^2r(r^2/2)^n[g(r)-1] . \tag{4.22}$$

In general, for a liquid ground state one can express the two-point correlation function in terms of the occupation of the single-particle angular-momentum eigenstates of Eq. (4.1),

$$g(r)=\rho^{-2}\sum_{\alpha,\beta,\gamma,\delta}\phi_\alpha(0)\phi_\beta(r)\phi_\gamma^*(r)\phi_\delta^*(0)\langle c_\alpha^{\dagger}c_\beta^{\dagger}c_\gamma c_\delta\rangle , \tag{4.23}$$

where $c_\alpha^{\dagger}$ is the creation operator for state α. Using Eq. (4.1) and conservation of angular momentum, we have

$$\begin{aligned}\rho[g(r)-1]=(2\pi\nu)^{-1}\sum_{m=0}^{\infty}\frac{(r^2/2)^m}{m!}\exp(-r^2/2)\\ \times(\langle n_m n_0\rangle-\langle n_m\rangle\langle n_0\rangle\\ -\nu\delta_{m0}) ,\end{aligned} \tag{4.24}$$

where $n_m=c_m^{\dagger}c_m$ is the occupation number for state m. Inserting Eq. (4.24) into Eq. (4.22) yields

$$M_0=\nu^{-1}(\langle Nn_0\rangle-\langle N\rangle\langle n_0\rangle)-1 , \tag{4.25}$$

$$M_1=\nu^{-1}[\langle(L+N)n_0\rangle-\langle L+N\rangle\langle n_0\rangle]-1 , \tag{4.26}$$

where

$$N\equiv\sum_{m=0}^{\infty}n_m \tag{4.27}$$

is the total particle number and

$$L\equiv\sum_{m=0}^{\infty}mn_m \tag{4.28}$$

is the total angular momentum. Since L and N are constants of the motion, their fluctuations vanish, leaving $M_0=M_1=-1$.

This general result implies that, for any liquid ground state in the lowest Landau level, $\bar{s}(k)\sim|k|^4$. Hence any liquid state automatically satisfies the SMA gap condition $\Delta(0)>0$ discussed above. Interpreting $\bar{s}(k)$ as the mean-square density fluctuation at wave vector k, the condition $\bar{s}(k)\sim|k|^4$ is a statement of the lack of density fluctuations or the incompressibility of the quantum system at long wavelengths. This is the source of the finite gap.

Within the SMA the existence of a gap for liquid ground states appears to be the rule rather than the exception. The interesting question of whether or not liquid ground states must have a rational filling factor is an entirely separate issue, about which nothing has been proved by these arguments.

Within the SMA, gapless excitations can occur only as Goldstone modes in systems with broken translational symmetry (which therefore violate our assumption of a liquid ground state). It is worth noting in this connection that the SMA analog of Eq. (4.19) yields the correct transverse magneto-phonon dispersion curve for the Wigner crystal.

We can shed additional light on the meaning of Eq. (4.21) and (4.22) by considering the specific case of the Laughlin ground state.[6] Invoking the analogy with the two-dimensional one-component plasma (2DOCP),[6,17,18] we see that $M_0=-1$ is the charge-neutrality sum rule and $M_1=-1$ is the perfect screening sum rule for the 2DOCP.[17,18] Making use of the 2DOCP compressibility sum rule,[17,18] we obtain M_2 and hence the exact leading term in $\bar{s}(k)$,

$$\bar{s}(k)=\frac{1-\nu}{8\nu}|k|^4+\cdots . \tag{4.29}$$

This result emphasizes the profound importance of the existence of long-range forces in the 2DOCP analog system. These long-range forces are responsible for the charge-neutrality and perfect screening sum rules. From these it follows that there is (within the SMA) a finite excitation gap at $k=0$ and from these also follows the exactness of the fractional charge $\pm\nu$ of the Laughlin quasiparticles.[6]

V. STATIC STRUCTURE FACTOR

In order to go beyond the small-k limit in evaluating Eq. (4.19), we need to have $\bar{s}(k)$ for finite k. Lacking the experimental structure factor that was available for the case of ^{4}He, we are forced to adopt a specific model for the ground state. We have chosen to use the Laughlin ground-state wave function[6] since it appears to be quite accurate[19,20] and because the static structure factor is available through the 2DOCP analogy.[6,21,22]

The static structure factor for the 2DOCP has been computed by both Monte Carlo[17,21] (MC) and hypernetted-chain[18,23] (HNC) methods. The MC results for $g(r)$ used in Ref. 7 are shown in Fig. 1. Recall from Eq. (2.8) that we need to Fourier-transform $g(r)$ to obtain $s(k)$. This is most easily accomplished by transforming an analytic function which has been fitted to the MC data. Fortunately, we can take advantage of the known

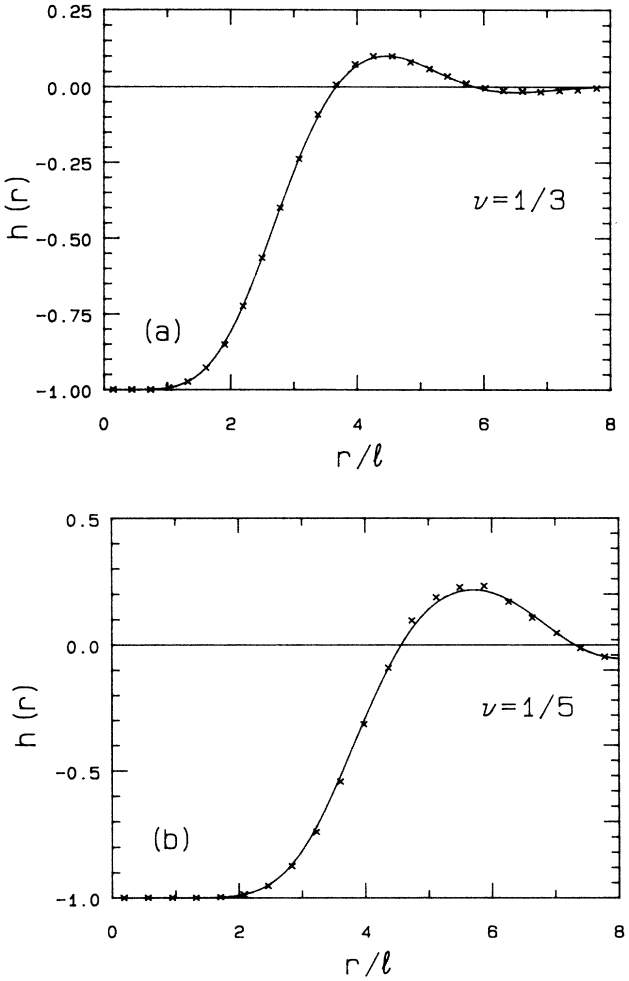

FIG. 1. Crosses are Monte Carlo data for $h(r)=g(r)-1$. Solid line is analytic fit of Eq. (5.1) to the data. (a) $\nu=\frac{1}{3}$; (b) $\nu=\frac{1}{5}$.

analytic form of $g(r)$ for any liquid ground state,[22]

$$g(r)=1-e^{-r^2/2}+\sum_{m=1}^{\infty}{}'\frac{2}{m!}(r^2/4)^m c_m e^{-r^2/4}, \tag{5.1}$$

where the c_m are unknown coefficients and the prime on the sum indicates that it is restricted to odd m only. The latter is a reflection of the Fermi statistics, which requires that pairs of (spin-polarized) particles have odd relative angular momentum. For the Mth Laughlin state ($\nu=1/M$) the coefficients c_m are constrained by the 2DOCP charge-neutrality, perfect screening, and compressibility sum rules[17,18] to obey[22]

$$\sum_{m=1}^{\infty}{}' c_m=(1-M)/4 , \tag{5.2}$$

$$\sum_{m=1}^{\infty}{}' (m+1)c_m=(1-M)/8 , \tag{5.3}$$

$$\sum_{m=1}^{\infty}{}' (m+2)(m+1)c_m=(1-M)^2/8 . \tag{5.4}$$

We fit a finite number (27) of the coefficients to the MC data subject to the constraints (5.2)–(5.4). The best-fit

TABLE I. Coefficients c_m obtained from fitting Eq. (5.1) to the MC data for $g(r)$ subject to the constraints (5.2)–(5.4). The $\nu=\frac{1}{5}$ fit is somewhat less reliable than the $\nu=\frac{1}{3}$ case (see text).

m	$c_m(\nu=\frac{1}{3})$	$c_m(\nu=\frac{1}{5})$
1	−1.000 00	−1.0000
3	+0.510 53	−1.0000
5	−0.020 56	+0.6765
7	+0.310 03	+0.3130
9	−0.490 50	−0.1055
11	+0.201 02	+0.8910
13	−0.009 04	−0.3750
15	−0.001 48	−0.7750
17	+0.000 00	+0.3700
19	+0.001 20	+0.0100
21	+0.000 60	−0.0050
23	−0.001 80	−0.0000
25	+0.000 00	−0.1000
27	+0.000 00	+0.1000

values for $\nu=\frac{1}{3}$ and $\frac{1}{5}$ are displayed in Table I and the resultant analytic $g(r)$ is shown in Fig. 1 along with the MC data. Having obtained an analytic form, the required Fourier transform is readily computed.

An alternative method of obtaining $s(k)$ is to use a modified hypernetted-chain (MHNC) approximation[23] which guarantees that the sum rules[17,18] on $s(k)$ are satisfied. This method gives a value for the energy in the $\nu=\frac{1}{3}$ Laughlin state of $E_{1/3}=-0.4092$, which is quite close to the value of $E_{1/3}=-0.4100\pm0.0001$) from the essentially exact MC method. Figure 2 displays $s(k)$ computed by the MHNC and MC methods.

Having obtained $s(k)$ we compute $\bar{s}(k)$ from Eq. (4.18) and then use this in Eq. (4.15). We also require the interaction potential $v(q)$. Taking the unit of energy to be $(e^2/\epsilon l)$, where ϵ is the dielectric constant of the background medium, the Coulomb potential is $V(r)=1/r$, which has the transform

$$v(q)=\frac{2\pi}{q}\ . \tag{5.5}$$

Using (5.5), the quadratures in (4.15) were computed numerically to obtain the oscillator strength and hence the gap function $\Delta(k)$.

VI. EVALUATION OF THE GAP

Using the results of the preceding section, we have evaluated the collective-mode dispersion for filling factors $\nu=\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, and $\frac{1}{9}$ using the MHNC structure factors. The various gap functions for the case of the pure Coulomb potential are shown in Fig. 3. The MC structure factors for $\nu=\frac{1}{3}$ and $\frac{1}{5}$ yield nearly identical results,[7] except for a small discrepancy in the $\nu=\frac{1}{5}$ curve at small k. We believe that this is due to the difficulty of extracting accurate information on the long-distance behavior of $g(r)$ from the $\nu=\frac{1}{5}$ MC data and we therefore consider the MHNC result more reliable for this case.

Note that, as discussed earlier, the gap is finite at zero

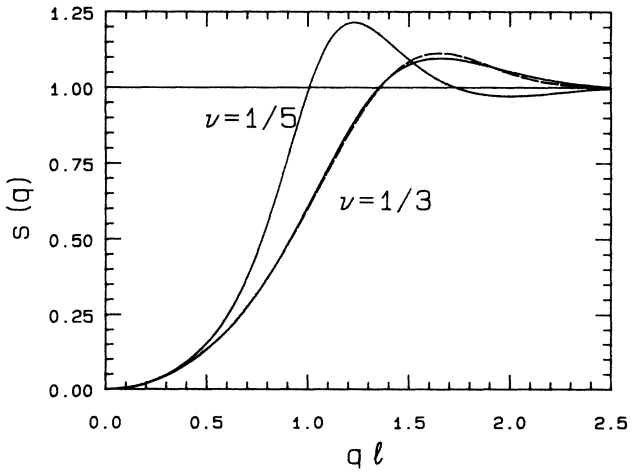

FIG. 2. Static structure factor. Solid line is modified-hypernetted-chain calculation. Dashed line is from fit to Monte Carlo data.

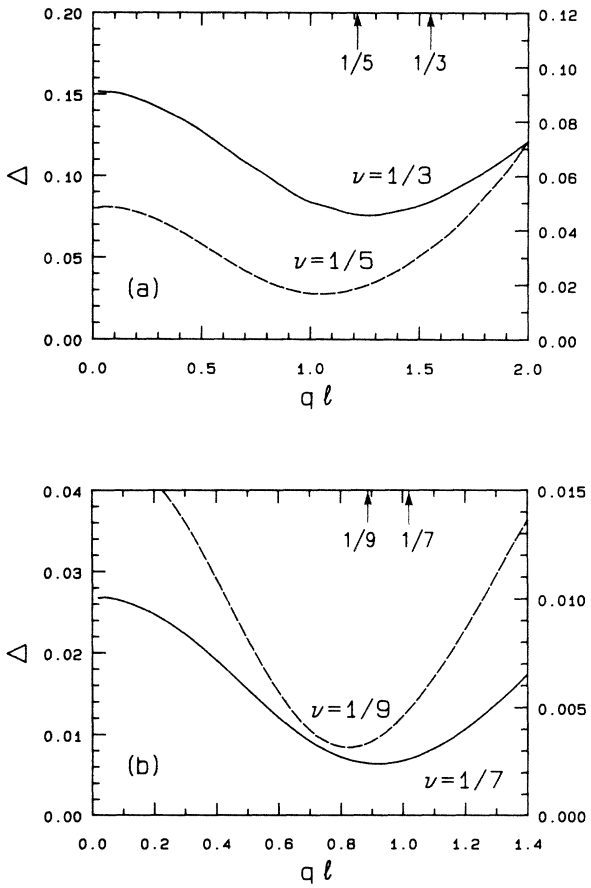

FIG. 3. Collective-mode dispersion. Arrows at the top indicate magnitude of primitive reciprocal-lattice vector of corresponding Wigner crystal. (a) $\nu=\frac{1}{3}$ (scale on left); $\nu=\frac{1}{5}$ (scale on right). (b) $\nu=\frac{1}{7}$ (scale on left); $\nu=\frac{1}{9}$ (scale on right).

wave vector, but exhibits a deep minimum at finite k. This magneto-roton minimum is caused by a peak in $\bar{s}(k)$ and is, in this sense, quite analogous to the roton minimum in helium.[8,9] We interpret the deepening of the minimum in going from $\nu=\frac{1}{3}$ to $\nu=\frac{1}{7}$ to be a precursor of the collapse of the gap which occurs at the critical density ν_c for Wigner crystallization. From Fig. 3 we see that the minimum gap is very small for $\nu<\frac{1}{5}$. This is consistent with a recent estimate[24] of the critical density, $\nu_c=1/(6.5\pm0.5)$. Within mean-field theory, the Wigner crystal transition is weakly first order and hence occurs slightly before the roton mode goes completely soft. Further evidence in favor of this interpretation of the roton minimum is provided by the fact that the magnitude of the primitive reciprocal-lattice vector for the crystal lies close to the position of the magneto-roton minimum, as indicated by the arrows in Fig. 3.

These ideas suggest the physical picture that the liquid is most susceptible to perturbations whose wavelength matches the crystal lattice vector. This will be illustrated in more detail in Sec. XI.

Having provided a physical interpretation of the gap dispersion and the magneto-roton minimum, we now examine how accurate the SMA is. Figure 4 shows the excellent agreement between the SMA prediction for the gap and exact numerical results for small ($N=6,7$) systems recently obtained by Haldane and Rezayi.[20] Those authors have found by direct computation that the single-mode approximation is quite accurate, particularly near the roton minimum, where the lowest excitation absorbs 98% of the oscillator strength.[25] This means that the overlap between our variational state and the exact lowest excited eigenstate *exceeds* 0.98. We believe this agreement confirms the validity of the SMA and the use of the Laughlin-state static structure factor.

Near $k=0$ there is a small ($\sim 20\%$) discrepancy between $\Delta_{\rm SMA}(0)$ and the numerical calculations.[20] It is interesting to speculate that the lack of dispersion near the roton minimum may combine with residual interactions to produce a strong pairing of rotons of opposite momenta leading to a two-roton bound state of small total momentum. This is known to occur in helium.[26] For the present case $\Delta_{1/3}(0)$ happens to be approximately twice the minimum roton energy. Hence the two-roton bound state which has zero oscillator strength could lie slightly below the one-phonon state which absorbs all of the oscillator strength. For $\nu<\frac{1}{3}$ the two-roton state will definitely be the lowest-energy state at $k=0$. It would be interesting to compare the numerical excitation spectrum with a multiphonon continuum computed using the dispersion curves obtained from the SMA.

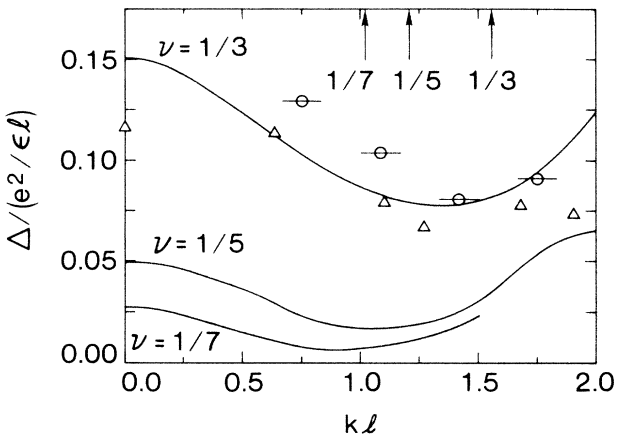

FIG. 4. Comparison of SMA prediction of collective mode energy for $\nu=\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ with numerical results of Haldane and Rezayi (Ref. 20) for $\nu=\frac{1}{3}$. Circles are from a seven-particle spherical system. Horizontal error bars indicate the uncertainty in converting angular momentum on the sphere to linear momentum. Triangles are from a six-particle system with a hexagonal unit cell. Arrows have same meaning as in Fig. 3.

VII. BACKFLOW CORRECTIONS

It is apparent from Fig. 4 that the SMA works extremely well—better, in fact, than it does for helium.[8,9] Why is this so? Recall that, for the case of helium, the Feynman-Bijl formula overestimates the roton energy by about a factor of 2. Feynman[8] traces this problem to the fact that a roton wave packet made up from the trial wave functions violates the continuity equation

$$\nabla\cdot\langle\mathbf{J}\rangle=0\ . \tag{7.1}$$

To see how this happens, consider a wave packet

$$\phi(r_1,\ldots,r_N)=\int d^2k\,\xi(k)\rho_k\psi(r_1,\ldots,r_N)\ , \tag{7.2}$$

where $\xi(k)$ is some function (say a Gaussian) sharply peaked at a wave vector k located in the roton minimum. It is important to note that this wave packet is quasistationary because the roton group velocity $d\Delta/dk$ vanishes at the roton minimum. Evaluation of the current density gives the result schematically illustrated in Fig. 5(a). The current has a fixed direction and is nonzero only in the region localized around the wave packet. This violates the continuity equation (7.1) since the density is (approximately) time independent for the quasistationary packet. The modified variational wave function of Feynman and Cohen[8] includes the backflow shown in Fig. 5(b). This gives good agreement with the experimental roton energy and shows that the roton can be viewed as a smoke ring (closed vortex loop).

A rather different result is obtained for the case of the quantum Hall effect. The current density operator is

$$\mathbf{J}(\mathbf{R})=\frac{1}{2m}\sum_{j=1}^{N}\left[\delta^2(\mathbf{R}-\mathbf{r}_j)\left(\mathbf{p}_j+\frac{e\mathbf{A}(\mathbf{r}_j)}{c}\right)+\left(\mathbf{p}_j+\frac{e\mathbf{A}(\mathbf{r}_j)}{c}\right)\delta^2(\mathbf{R}-\mathbf{r}_j)\right]\ . \tag{7.3}$$

Taking ϕ and ψ to be any two members of the Hilbert space of analytic functions described in Sec. IV, it is straightforward to show that

$$\langle\Phi\,|\,\mathbf{J}(\mathbf{R})\,|\,\Psi\rangle=-\tfrac{1}{2}\nabla\times\langle\Phi\,|\,\mathbf{M}(\mathbf{R})\,|\,\Psi\rangle\ , \tag{7.4}$$

where

$$\mathbf{M}(\mathbf{R})=\rho(\mathbf{R})\hat{\mathbf{z}}\ , \tag{7.5}$$

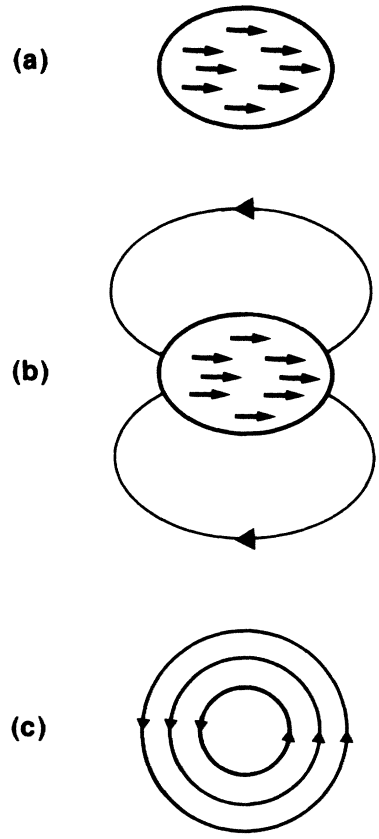

FIG. 5. Schematic illustration of the current distribution in a roton wave packet: (a) helium with no backflow corrections, (b) helium with backflow corrections, and (c) lowest-Landau-level case.

and $\rho(R)$ is the density and $\hat{\mathbf{z}}$ is the unit vector normal to the plane. It follows immediately from (7.4) that

$$\nabla\cdot\langle \mathbf{J}(R)\rangle = 0 \tag{7.6}$$

for any state in the lowest Landau level. Hence the backflow condition is automatically satisfied. The current flow for the magneto-roton wave packet calculated from (7.4) is illustrated schematically in Fig. 5(c). We see that the magneto-roton circulation is rather different from the smoke ring in bulk helium shown in Fig. 5(b).

Equation (7.6) is paradoxical in that it implies $\partial\rho/\partial t = 0$ for every state in the lowest Landau level. This is merely a reflection of the fact that the kinetic energy has been quenched and perturbations can cause particles to move only be means of (virtual) transitions to higher Landau levels. One can resolve this paradox by noting that there are really two different current operators we can consider. The first is the ordinary (instantaneous) current discussed above. The second is the slow (time-averaged) $\mathbf{E}\times\mathbf{B}$ drift of the particles in the magnetic field. Restriction of the Hilbert space to the lowest Landau level eliminates the fast degrees of freedom associated with the cyclotron motion but retains the slow (drift) coordinates.[15] To illustrate this note that for an external potential projected onto the lowest Landau level

$$V = \sum_{\mathbf{q}} V(q)\bar{\rho}_{\mathbf{q}} , \tag{7.7}$$

we have, by virtue of the noncommutivity of the density operators,

$$i\frac{\partial}{\partial t}\bar{\rho}_{\mathbf{k}} = -\sum_{\mathbf{q}} V(q)\bar{\rho}_{\mathbf{k}+\mathbf{q}}(e^{kq^*/2} - e^{k^*q/2}) . \tag{7.8}$$

Expanding this to lowest order in q (smooth potential) gives

$$i\frac{\partial}{\partial t}\bar{\rho}_{\mathbf{k}} = -\sum_{\mathbf{q}} (\mathbf{q}\times\mathbf{k})_z V(q)\bar{\rho}_{\mathbf{k}+\mathbf{q}} . \tag{7.9}$$

Using the continuity equation, this can be rewritten as the usual $\mathbf{E}\times\mathbf{B}$ drift equation

$$\langle \mathbf{J}(\mathbf{r})\rangle = [\nabla V(\mathbf{r})\times\hat{\mathbf{z}}]\rho(\mathbf{r}) . \tag{7.10}$$

We therefore recover the physically correct result even though the Hilbert space has been restricted to the lowest Landau level.

For the case of the magneto-roton wave packet discussed above, we note that the excess particle density is circularly symmetric. Hence the (mean) electric field is radial and the particle drift is purely circular, as illustrated previously in Fig. 5(c). Hence one is once again led to the conclusion that the continuity condition is automatically satisfied by the magneto-roton wave packet. We believe that this accounts for the excellent results obtained using the SMA.

VIII. SMA AT LARGE WAVE VECTORS

We saw in Sec. VI that the SMA is quantitatively accurate out to the magneto-roton minimum. For larger wave vectors the SMA rapidly breaks down as many different states begin to couple to the density. This is simply because the density wave is not a sensible excitation for wavelengths smaller than the interparticle spacing.

Even though Eq. (4.19) does not give a good variational bound on the energy for large k, it still gives the exact first moment of the excitation spectrum. As we shall see below, it is interesting to consider this first excitation moment in the limit of large k:

$$\Gamma = \lim_{k\to\infty} \bar{f}(k)/\bar{s}(k) . \tag{8.1}$$

For large k, Eqs. (2.8), (4.18), and (5.1) yield

$$\bar{s}(k) \sim (1-\nu)e^{-|k|^2/2} . \tag{8.2}$$

Using (8.2) to evaluate $\bar{s}(k+q)$ in Eq. (4.15) and taking advantage of rotational symmetry yields

$$\bar{f}(k) \sim \sum_{q} v(q)e^{-|k|^2/2}[(1-\nu)e^{-|q|^2/2} - \bar{s}(q)] . \tag{8.3}$$

Using (4.18) this may be rewritten as

$$\bar{f}(k) = \sum_{q} v(q)e^{-|k|^2/2}\{[1-s(q)] - \nu[1-s_1(q)]\} , \tag{8.4}$$

where

$$s_1(q) \equiv 1 - e^{-|q|^2/2} \tag{8.5}$$

is the static structure factor for a filled Landau level ($\nu=1$). The sum in (8.4) is simply related to the ground-state energy *per particle*, $E(\nu)$,

$$\bar{f}(k)=-2e^{-|k|^2/2}[E(\nu)-\nu E(1)] , \tag{8.6}$$

or, equivalently,

$$\bar{f}(k)=2e^{-|k|^2/2}E_{\rm coh}(\nu) , \tag{8.7}$$

where $E_{\rm coh}(\nu)$ is the cohesive energy *per particle*. Substitution of (8.7) into (8.1) yields

$$\Gamma=\frac{2E_{\rm coh}(\nu)}{1-\nu} . \tag{8.8}$$

This is the exact first moment of the (intra-Landau-level) density-fluctuation spectrum at $k=\infty$. Note that because the kinetic energy has been quenched, Γ is finite. Also note that this result does not rely on the validity of the SMA or on any particular assumptions about the structure of the ground state (other than the restriction to the lowest Landau level). For the particular case of Laughlin's ground state, $\Gamma(\nu)=0.603,0.508$ at $\nu=\frac{1}{3},\frac{1}{5}$.

We can gain greater insight into the meaning of (8.8) by considering the form of the variational excited state at large k. From (4.8) we have

$$\bar{\rho}_k\psi=\sum_{j=1}^{N} e^{-|k|^2/2}e^{-ik^*z_j/2} \times\psi(z_1,\ldots,z_{j-1},z_j-ik,z_{j+1},\ldots,z_N) . \tag{8.9}$$

We see that $\bar{\rho}_k$ acts, in part, like a translation operator moving particles (transversely) by a distance $|k|$. For large k this corresponds to an incoherent single-particle excitation which leaves a hole and a particle uncorrelated with their immediate surroundings ("undressed"). This explains why the cohesive energy appears in (8.8).

As an aside, we note that Eq. (8.9) can also teach us something about the nature of the collective modes at small k. Recall Halperin's[27] interpretation of the Laughlin wave function which notes that the Laughlin state makes the most efficient use of the zeroes of the wave function by putting them all at the locations of the particles. We see from (8.9) that the density wave is a linear superposition of states with some of their zeroes displaced a distance ik from one of the particles. This accounts for the increased energy of the state.

Equation (8.8) gives the exact first moment Γ of the density-fluctuation excitation spectrum at $k=\infty$. It would also be useful to find not just Γ but $\Delta(\infty)$, the minimum excitation energy at very large wave vectors. This may be done as follows. Equations (8.8)—(8.9) show us that the density wave creates a particle-hole excitation. Recall, however, that for the Laughlin state with $\nu=1/m$, a hole (electron) is precisely a superposition of m fractionally charged quasiholes (quasielectrons).[28,29] This suggests that at large wave vectors an improvement on our density-wave variational state can be obtained by producing a single quasihole-quasielectron pair. An exciton[12,30] formed from such quasiparticles would carry (dimensionless) momentum kl if the particles were separated by a (dimensionless) distance mkl. Hence the energy dispersion for large k is[12,30]

$$\Delta_{\rm ex}(k)=\Delta(\infty)-\frac{1}{m^3kl} , \tag{8.10}$$

where $\Delta(\infty)$ is the sum of the quasihole and quasielectron energies and the second term on the right-hand side of (8.10) represents the Coulomb attraction between the fractional charges. Equation (8.10) is valid for large k, where the particles are widely separated compared to their intrinsic size ($\sim l$). For smaller k we know that the density-wave state is a good description. Suppose we arbitrarily assume that the collective mode crosses over from being a density wave to being an exciton at the wave vector of the roton minimum, $k_{\rm min}$. Then we can equate the SMA and exciton values of the gap,

$$\Delta_{\rm ex}(k_{\rm min})=\Delta_{\rm SMA}(k_{\rm min}) . \tag{8.11}$$

Using (8.10) leads to the prediction

$$\Delta(\infty)=\Delta_{\rm SMA}(k_{\rm min})+\frac{1}{m^3k_{\rm min}l} . \tag{8.12}$$

From the numerical values of $\Delta_{\rm SMA}$ and $k_{\rm min}$ obtained previously, we find $\Delta_{1/3}(\infty)=0.106$ and $\Delta_{1/5}(\infty)=0.025$. These values lie considerably above the results of HNC-approximation calculations of Laughlin,[6,31] $\Delta_{1/3,1/5}(\infty)=0.057,0.014$, and Chakraborty,[32] $\Delta_{1/3,1/5}(\infty)=0.053,0.014$. However, preliminary Monte Carlo results of Morf and Halperin[33] yield a larger value, $\Delta_{1/3}(\infty)=0.099\pm0.009$. In addition, the small-system calculations of Haldane and Rezayi[20] yield a value (extrapolated to $N=\infty$) of $\Delta_{1/3}(\infty)=0.105\pm0.005$, in excellent agreement with the present result. In summary, while the SMA breaks down for large k, it still yields the exact first moment of the excitation spectrum [Eq. (8.8)] and can be used to obtain a good estimate of $\Delta(\infty)$, the sum of the quasihole and quasielectron energies.

IX. FINITE-THICKNESS EFFECTS

The calculations discussed in the preceding sections have all used the Coulomb interaction given by Eq. (5.5). In order to make a comparison with experiment, it is important to recognize that the finite extent of the electron wave functions perpendicular to the plane cuts off the divergence of the Coulomb interaction at short distances. For both GaAs and Si devices, this thickness is on the order of[34] 100 Å, which exceeds the magnetic length at high fields ($l=66$ Å at $B=15$ T).

We have used the Fang-Howard[35] variational form for the charge distribution normal to the plane,

$$g(z)=\frac{b^3}{2}z^2e^{-bz} , \tag{9.1}$$

from which the Stern-Howard[34,36] interaction may be obtained,

$$V(q)=\left[\frac{2\pi}{q}\right]\tfrac{1}{8}(1+q/b)^{-3}[8+9(q/b)+3(q/b)^2] . \tag{9.2}$$

This form for $V(q)$ may be found in Eq. (2.52) of Ref. 34. Figure 6 shows the collective-mode dispersion for four

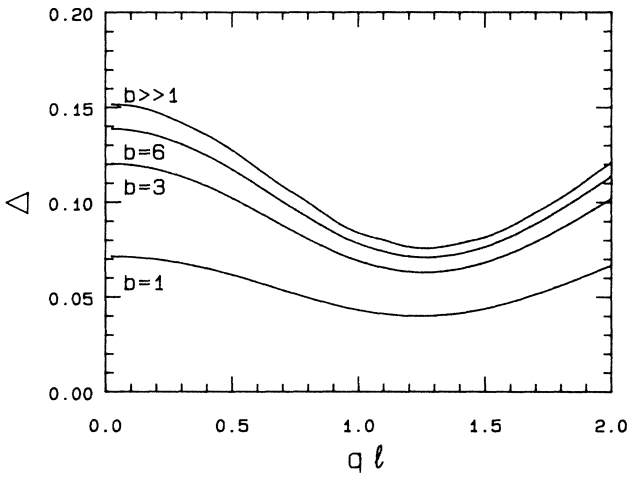

FIG. 6. Effect of finite thickness on collective-mode dispersion. The values of the Fang-Howard parameter bl are ∞, 6, 3, and 1 (going from the top curve to the bottom). All curves are for $\nu=\frac{1}{3}$.

different values of the dimensionless thickness parameter bl. We see that experimentally relevant values of bl cause a significant reduction in the size of the gap. Furthermore, this reduction is B-field dependent. Naively, we expect $\Delta\sim B^{1/2}$ (for fixed ν) since the natural unit of energy is $e^2/\epsilon l$. However, $bl\sim B^{-1/2}$ and therefore Δ does not rise as rapidly as might be expected at large B. This may at least partially (but not fully) account for the saturation in the Δ-versus-B experimental curve recently obtained by Boebinger *et al.*[37]

X. ROLE OF DISORDER

There does not exist at present a good understanding of the temperature and disorder dependence of the conductivity. The experimental activation energy for σ_{xx} is generally believed to be determined by the minimum energy to create charged excitations (quasiparticles). It has been suggested, however, that the scattering of thermally activated rotons may also contribute to the dissipation.[38] The *lowest* excitation energy $\Delta(k)$ occurs at $k=k_{\min}$, corresponding to a neutral density wave with charges of magnitude e separated by a distance $d=kl^2$. For $k>k_{\min}$ and $\nu=1/m$ the excitation is a quasiexciton consisting of charges of magnitude $e^*=e/m$ separated by a distance $d=mkl^2$. Thus the charge-excitation gap corresponds to $\Delta(\infty)$, and from the law of mass action[39] the number of quasiparticles is activated with energy $\Delta(\infty)/2$. The experimentally observed[37,40] activation energies for dissipation are much smaller than the predicted values of $\Delta(\infty)/2$ and $\Delta_{\min}$. Part of the discrepancy is due to finite-thickness effects which (as discussed in Sec. IX) soften the Coulomb repulsion at short distances and hence lower the gap. The remaining discrepancy is presumably due to the disorder in the sample and needs to be further investigated.

It has recently been suggested[41] that, as the disorder is increased, the quasiparticle excitation gap continuously closes, thereby destroying the fractional quantum Hall effect. This picture assumes a liquid state plus independent quasiparticles which interact with the disorder. This may well be correct; however, we wish to point out a possible alternative which may occur for some or even all of the fractional states: The continuous-gap closing transition may be preempted by the formation of a Wigner-glass state[42,43] corresponding to the collapse of the gap near $k_{\min}$ rather than $k=\infty$. If the gap collapses to zero at finite wave vector, then the dissipation is controlled by the generation of rotons as the glassy crystal slides past the impurities, just as occurs for phonon production in charge-density-wave systems.[44] Such a glassy state has been proposed to explain cyclotron-resonance anomalies in Si devices.[43]

The reasons this gap collapse might occur are the following. In the absence of disorder the lowest gap is at $k=k_{\min}$, the wave vector at which the system is already nearly unstable to crystallization. The quasiexciton is made of particles of charge $|e^*|=e/m$ for $\nu=1/m$. Hence it responds much less strongly to the randomly fluctuating Coulomb potential than the roton which consists of charges of magnitude e. Thus the roton gap may collapse first. On the other hand, the charges in the roton are separated by a finite distance $d=k_{\min}l^2$. If the length scale of the random potential is larger than d, then the rotons will respond less strongly to the disorder. These questions will be discussed in greater detail elsewhere.[45]

The lowest filling factor at which evidence for a fractional Hall state has been found is $\nu=\frac{1}{5}$. This state is extremely weak, however, and does not develop a plateau even at very low temperatures.[46] As will be seen in the next section, the $\frac{1}{5}$ state is much more susceptible to external perturbations than the $\frac{1}{3}$ state because of the incipient Wigner crystal instability. Thus the ideas presented here may well explain the weakness of the anomaly at $\nu=\frac{1}{5}$.

XI. STATIC LINEAR RESPONSE

We are now in the position of having rather accurate eigenfunctions $\bar{\rho}_k|0\rangle$ and eigenvalues $\Delta(k)$ for those states which couple to the ground state through the density. The dynamical structure factor is (in the SMA) given by

$$S(q,\omega)=\bar{s}(q)\delta(\omega-\Delta(q))\ . \tag{11.1}$$

From this we can compute the susceptibility $\chi(q,\omega)$, the dielectric function $\epsilon(q,\omega)$ the loss function $\mathrm{Im}[-1/\epsilon(q,\omega)]$, and related quantities such as the coupling between rotons and the substrate phonons. For example, using the memory-function formalism we have computed the ac conductivity in the presence of weak impurity scattering. This work will be reported elsewhere.[47]

As a simple example of what can be done with Eq. (11.1), we consider here the static susceptibility to an external perturbation. Zhang *et al.*[48] and Rezayi and Haldane[49] have recently performed numerical calculations of the charge distribution and relaxation energy of the $\nu=\frac{1}{3}$

state in the presence of a Coulomb impurity of charge Ze. Below we present simple analytical calculations of the same quantities.

The static susceptibility is given by[9]

$$\chi(q)=-2\int_0^\infty \frac{d\omega}{\omega}S(q,\omega)\ . \tag{11.2}$$

Using Eq. (11.1) we have

$$\chi(q)=\frac{-2\bar{s}(q)}{\Delta(q)}\ . \tag{11.3}$$

The quantity $\alpha(q)\equiv\bar{s}(q)/\Delta(q)$ is shown plotted in Fig. 7. Note that the susceptibility is sharply peaked at the wave vector corresponding to the roton minimum and that the magnitude of the susceptibility rises very rapidly as the filling factor approaches $\nu=\frac{1}{7}$. This is consistent with our previous discussion in Sec. VI of the Wigner crystal instability near $\nu=\frac{1}{7}$.

The Fourier transform of the perturbed charge density is (within linear response theory)

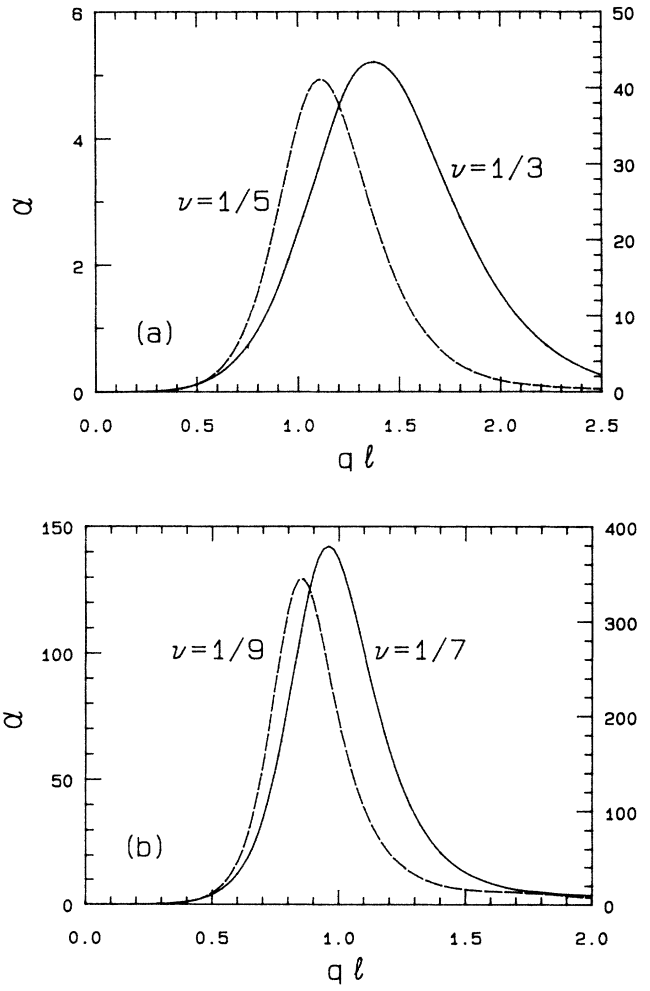

FIG. 7. Susceptibility parameter $\alpha\equiv\bar{s}(q)/\Delta(q)$. (a) $\nu=\frac{1}{3}$ (scale on left), $\nu=\frac{1}{5}$ (scale on right); (b) $\nu=\frac{1}{7}$ (scale on left), $\nu=\frac{1}{9}$ (scale on right).

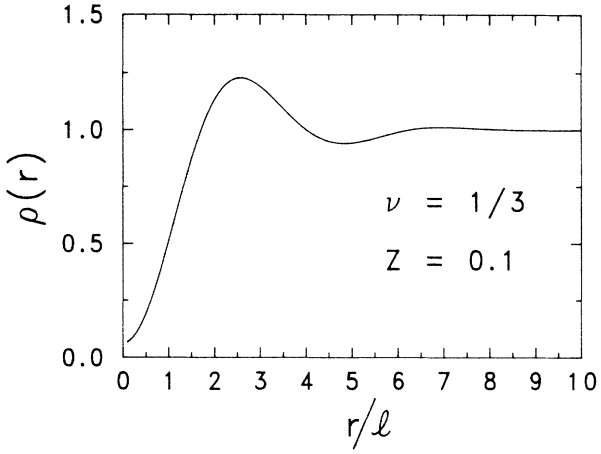

FIG. 8. Normalized charge distribution near a repulsive Coulomb impurity with $Z=0.1$. Note the oscillations at the wavelength corresponding to the roton minimum.

$$\langle\delta\bar{\rho}_q\rangle=\rho v(q)\chi(q)\ , \tag{11.4}$$

where $\rho=e\nu/2\pi l^2$ is the mean density and $v(q)=2\pi Ze^2/\epsilon q$ is the transform of the Coulomb potential. The real space-charge distribution is therefore

$$\frac{\langle\rho(r)\rangle}{\rho}=1+Z\int_0^\infty dq\,\chi(q)J_0(qr)\ . \tag{11.5}$$

This is plotted in Fig. 8 for the case $Z=0.1$. Note that because $\chi(q)$ is sharply peaked at the roton (crystallization) wave vector, the spatial distribution of the charge is oscillatory. This provides a simple physical explanation of the charge oscillations first observed numerically by Zhang *et al.*[48]

The impurity relaxation energy in linear response is

$$\Delta E=\tfrac{1}{2}\int\frac{d^2q}{(2\pi)}v(q)\langle\delta\bar{\rho}_q\rangle\ . \tag{11.6}$$

Evaluation of ΔE using Eqs. (11.3) and (11.4) yields $\Delta E=-1.15Z^2$ for $\nu=\frac{1}{3}$. This compares very favorably with the numerical result $\Delta E=-1.2Z^2$ obtained by Zhang *et al.*[48] for a small system (five particles).

XII. SUMMARY AND CONCLUSIONS

We have presented here a theory of the collective excitations in the fractional quantum Hall regime which is closely analogous to Feynman's theory of superfluid helium. The elementary collective excitations are density waves which exhibit a finite gap at zero wave vector and have a distinct magneto-roton minimum at finite wave vector. This feature is analogous to the roton minimum in helium and is a precursor to the gap collapse associated with Wigner crystallization which occurs near $\nu=\frac{1}{7}$. For larger wave vectors the lowest-lying mode crosses over from being a density wave to being a quasiparticle exciton.[12,30]

Our theory is based on the single-mode approximation, which assumes that 100% of the (intra-Landau-level) oscillator-strength sum is saturated by a single mode. By

direct numerical diagonalization of the Hamiltonian for small systems ($N \leq 8$), Haldane and Rezayi[25] have found that the single-mode approximation is quite accurate, particularly near the roton minimum, where the lowest eigenmode absorbs 98% of the oscillator strength. Another way to state this is that the overlap between our variational excited-state wave function and the exact eigenfunction exceeds 0.98.

We thus have rather accurate eigenvalues and wave functions for those states which couple to the ground state through the density. From these we have calculated the dynamic structure factor and the susceptibility. As an illustration of the utility of this theory we presented a simple calculation of the perturbed charge distribution and relaxation energy due to a Coulomb impurity potential. The results were found to be in good agreement with numerical calculations of Zhang *et al.*[48]

The two main advantages of the present theory are that it gives a clear and useful physical picture of the nature of the collective modes and it allows one to calculate rather easily and accurately experimentally relevant quantities such as the collective-mode dispersion and the susceptibility. There remain, however, many further interesting questions related to the existence of two-roton bound states, the nature of the collective excitations for the compound ground states further down in the hierarchy,[50] the role of higher Landau levels, and so forth.

It would be particularly interesting to see whether states further down in the hierarchy act as multicomponent fluids having more than one collective excitation branch. The projection formalism[15] discussed here can be generalized to discuss fractional states in higher Landau levels[51] and to investigate inter-Landau-level cyclotron-resonance modes[52] as well.

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