Low-amplitude breather and envelope solitons in quasi-one-dimensional physical models

M. Remoissenet
Laboratoire d'Optique du Réseau Cristallin, Faculté des Sciences, Université de Dijon, 6 Bd. Gabriel, Dijon 21100, France

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In the low-amplitude limit we examine the breather- and envelope-soliton solutions of the generalized nonlinear Klein-Gordon system as nonlinear Schrödinger solitons. The existence of breather and envelope solutions is determined in the continuum and quasidiscrete limit; in this case the oscillations of the carrier in the envelope are treated exactly. The results are applied to the perturbed sine-Gordon and ϕ^4 systems; in both cases the asymmetry of the breathers is controlled by the amplitude of the external force. The study is generalized to the calculations of low-amplitude breather modes in a ferromagnetic chain with a small out-of-plane angle.

I. INTRODUCTION

In quasi-one-dimensional (1D) physical systems, solitons occur in mainly[1] three forms: kinks, pulses, and envelopes. The first two are clearly defined and characterized by the solutions of standard equations: the kink solutions by the sine-Gordon (SG) and related Klein-Gordon (KG) equations, and the pulse solutions by the Korteweg—de Vries or Toda lattice equation. The (topological) kink interpolates between two unconnected degenerate ground states, whereas the pulse does not. Kinks are familiar as dislocations in a metal, domain walls in a ferromagnet or a ferroelectric, or discommensurations; examples of pulses are water waves, electrical pulses in transmission lines, or optical pulses in optical fibers. Envelope solitons, often called breathers, doublets, or bions, appear as solutions to the SG equation or modified Kortweg—de Vries equation.[2] They are sometimes found to be a more difficult concept, and yet in many ways they represent the most important and widespread class of nonlinear excitations.[3] Here an internal carrier wave is modulated by the envelope which constitutes the localized soliton structure. When the wavelength of periodic oscillations of the carrier wave is comparable to the envelope width, the latter looks like it is breathing. Unlike the kinks, the breathers require practically no activation energy; they are important excitations because they can interpolate between extremely nonlinear modes (kinks) and linear modes (phonons or magnons). Furthermore, the internal degrees of freedom of the breathers increase their physical potential: for example, the breather can correspond to an oscillating electric dipole[4] in a 1D condensate.

Although among the generalized KG family the SG system is the only system for which the exact breather-soliton solution can be calculated using the inverse scattering method, it is now well known that in the low-amplitude limit the SG breather is equivalent to a nonlinear Schrödinger envelope soliton.[2] This result can be used to determine the low-amplitude breather solutions of other systems for which it is impossible to directly calculate the breather solutions.[4–7] We have recently extended this approach to determine the necessary conditions of existence of breathers in a generalized KG family. Even for the large-amplitude breathers created by kink-antikink collisions,[7] our numerical experiments remarkably confirmed these results. These results are limited in their applicability to nonperturbed KG systems in the continuum limit.

There are two basic purposes of this paper. First, we present a general methodology for studying breather and envelope solitons in quasi-1D systems, and we illustrate this methodology by applications to familiar systems such as perturbed SG and ϕ^4 systems. Included in this methodology are some novel results in the semidiscrete limit, where the carrier oscillations are very rapid compared to the envelope. Physically, this allows us to take a partial count of lattice effects that will occur in real condensed-matter systems. Second, we present new results on the quasi-1D ferromagnets where small out-of-plane motion is allowed. Although the distinction between a breather and an envelope mode is not sharp, the terminology is useful in the appropriate wavelength limit. In the following, we will use the term breather in the continuum (or small wave-vector) limit and the term envelope in the quasi-discrete limit.

The outline of this paper is as follows. In Sec. II we consider a generalized KG lattice model. The existence of breather solutions is studied in the continuum and low-amplitude limits. The results are applied to the perturbed SG and ϕ^4 systems. Section III is devoted to the determination of the envelope-mode solutions in the quasi-discrete limit. In Sec. IV the calculations of Sec. II are generalized and applied to the determination of breather modes in a ferromagnetic chain model. Section V gives a brief summary and discussion.

II. BREATHER EXCITATIONS IN THE CONTINUUM LIMIT

We consider a KG lattice model where a system of ions of mass m harmonically coupled interact with a nonlinear substrate potential $V(y)$. The Hamiltonian is given by

$$H=\sum_i \Delta a \left[\tfrac{1}{2}\dot{y}_n^2+\frac{1}{2}\frac{C_0^2}{a^2}(y_{n+1}-y_n)^2+\omega_0^2 V(y_n) \right], \qquad (2.1)$$

where y_n is the scalar dimensionless displacement of the nth ion. The constants C_0 and ω_0 are the characteristic velocity and frequency of the system. The constant $\Delta = ma$ sets the energy scale of the system. The corresponding equation of motion of the nth ion is

$$\ddot{y}_n = K(y_{n+1} + y_{n-1} - 2y_n) - \omega_0^2 \frac{dV(y_n)}{dy_n} , \tag{2.2}$$

where $K = C_0^2/a^2$.

We look for small nonlinear collective oscillations in the bottom of the potential wells. For this purpose we assume

$$y_n \to \epsilon\phi_n + \phi_0 ,$$

where $\epsilon \ll 1$ and ϕ_0 is the ground state or potential minimum around which the oscillations will occur. For $\phi_0 = 0$ the potential wells are symmetric (SG system) for $\phi_0 \neq 0$ they are asymmetric (double SG or ϕ^4 systems). Expanding in terms of ϕ in Eq. (2.2) one gets

$$\begin{aligned} \ddot{\phi}_n &= K(\phi_{n+1} + \phi_{n-1} - 2\phi_n) \\ &\quad - \omega_0^2[f\phi_n + \epsilon g\phi_n^2 + \epsilon^2 h\phi_n^3 + O(\epsilon^3)] , \end{aligned} \tag{2.3}$$

where the coefficients $f(\phi_0)$, $g(\phi_0)$, and $h(\phi_0)$ are determined by the shape of the potential and depend on ϕ_0 when the potential wells are asymmetric. Note that $g = 0$ [odd term in $V(\phi)$] if they are symmetric.[6]

We first consider the continuum limit: We restrict ourselves to those excitations which consist of a slowly varying envelope modulating a carrier wave whose dispersion relation is that of a linear wave to order ϵ. In this limit $\phi_n(t) \to \phi(x,t)$ and Eq. (2.3) is approximated by

$$\phi_{tt} - C_0^2\phi_{xx} + (\omega_0')^2(\phi + \epsilon\alpha\phi^2 + \epsilon^2\beta\phi^3) = 0 , \tag{2.4a}$$

where

$$(\omega_0')^2 = \omega_0^2 f, \quad \alpha = g/f, \quad \beta = h/f . \tag{2.4b}$$

We now use a simplified version of the multiple-scale method[4] or derivative expansion method[8] and extend the independent variables $x_0, x_1, \ldots,$ and $t_0, t_1, \ldots,$ where

$$x_n = \epsilon^n x, \quad t_n = \epsilon^n t . \tag{2.5}$$

Accordingly, the displacement field $\phi(x,t)$ in (2.4) is regarded as $\phi(x_0, x_1, \ldots, t_0, t_1, \ldots)$ and the derivative operators $\partial/\partial x$ and $\partial/\partial t$ are expanded as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon\frac{\partial}{\partial x_1} + \cdots , \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \cdots . \tag{2.6}$$

To simplify the terminology hereafter, we write x for x_0, X for x_1, t for t_0, and T for t_1. Consequently, from (2.4) we get

$$\begin{aligned} &\phi_{tt} - c_0^2\phi_{xx} + (\omega_0')^2(\phi + \epsilon\alpha\phi^2 + \beta\phi^3) \\ &\quad + \epsilon^2\phi_{TT} - C_0^2\epsilon^2\phi_{XX} + 2\epsilon\phi_{tT} - 2\epsilon C_0^2\phi_{xX} = 0 . \end{aligned} \tag{2.7}$$

We now look for modulated wave solutions of the form

$$\begin{aligned} \phi &= F_1(X,T)e^{i\theta} + \text{c.c.} \\ &\quad + \epsilon[F_0(X,T) + F_2(X,T)e^{2i\theta} + \text{c.c.}] , \end{aligned} \tag{2.8}$$

which contains a first-harmonic term and also a dc and second-harmonic term.[4,6,7] Here $\theta = kx - \omega t$ and the frequency ω and wave vector k are related by the dispersion relation

$$\omega^2 = (\omega_0')^2 + C_0^2 k^2 \tag{2.9}$$

obtained from Eq. (2.4). Inserting (2.8) into (2.7) and equating dc, first-, and second-harmonic terms we get

$$\epsilon(\omega_0')^2(F_0 + 2\alpha F_1 F_1^*) + O(\epsilon^2) = 0 , \tag{2.10a}$$

$$\begin{aligned} &[-\omega^2 + (\omega_0')^2 + C_0^2k^2]F_1 + 2\epsilon^2\alpha(\omega_0')^2(F_1F_0 + F_1^*F_2) \\ &\quad + 3\epsilon^2\beta(\omega_0')^2(F_1^2F_1^*) - C_0^2\epsilon^2F_{1XX} - 2\epsilon i\omega F_{1T} \\ &\quad - 2\epsilon ikC_0^2F_{1X} + \epsilon^2F_{1TT} + O(\epsilon^3) = 0 , \end{aligned} \tag{2.10b}$$

$$[4C_0^2k^2 - 4\omega^2 + (\omega_0')^2]F_2 + \alpha(\omega_0')^2F_1^2 = 0 . \tag{2.10c}$$

Using (2.9), Eqs. (2.10) give

$$\begin{aligned} &\epsilon^2(\omega_0')^2 \left[-4\alpha^2 + \frac{2\alpha^2}{3} + 3\beta \right] |F_1|^2F_1 - \epsilon_0^2\epsilon^2F_{1XX} \\ &\quad + \epsilon^2F_{1TT} - 2\epsilon i\omega F_{1T} - 2\epsilon ikC_0^2F_{1X} = 0 \end{aligned} \tag{2.11a}$$

with

$$F_0 = -2\alpha|F_1|^2 \tag{2.11b}$$

and

$$F_2 = \frac{\alpha}{3}|F_1|^2 . \tag{2.11c}$$

We now introduce new scales

$$Z = X - V_gT, \quad s = \epsilon T \tag{2.12}$$

with

$$V_g = \frac{d\omega}{dk} = \frac{k}{\omega}C_0^2 . \tag{2.13}$$

Finally from Eq. (2.11a) we get the cubic nonlinear Schrödinger (NLS) equation

$$iF_{1s} + PF_{1ZZ} + Q|F_1|^2F_1 = 0 \tag{2.14}$$

with

$$P = \frac{C_0^2 - V_g^2}{2\omega} \tag{2.15a}$$

and

$$Q = \frac{(\omega_0')^2}{2\omega}\left[\frac{10\alpha^2}{3} - 3\beta\right] . \tag{2.15b}$$

The coefficients P and Q depend on the wave vector k and the potential parameters $(\omega_0')^2$, α, and β. The solutions of (2.14) depend on the sign[9,10] of the dispersive coefficient P and on the nonlinear coefficient Q. If $PQ > 0$, Eq. (2.14) has an envelope-soliton solution which has a vanishing amplitude at $|Z| \to \infty$, and which corresponds to a small-amplitude breather. If $PQ < 0$, a typical solution of (2.14) is a dark (or envelope hole) soliton when the depression of an envelope propagates as a soliton with a finite amplitude at $|Z| \to \infty$. In this case the solution of (2.14) is a dark (or hole) soliton which does not correspond to the small-amplitude limit of breather modes.[6]

For $PQ>0$ the envelope-soliton solution is given by[11,12]

$$F_1(Z,s)=A\,\mathrm{sech}\left[\left[\frac{Q}{2P}\right]^{1/2}A\left[Z-\frac{u_e s}{P}\right]\right]\times\exp\left[i\left[\frac{u_e}{2P}\right]\left[Z-\frac{u_c s}{P}\right]\right], \quad (2.16)$$

where A is the amplitude,

$$A=\left[\left[\frac{u_e^2-2u_eu_c}{2PQ}\right]\right]^{1/2}, \quad (2.17)$$

where $u_e^2-2u_eu_c>0$, and u_e and u_c are the velocities of the envelope and carrier waves. We also note from Eq. (2.15a) that the coefficient P is always positive (small k) and in the following, only the sign of Q will be significant. From Eqs. (2.16), (2.12), (2.11), and (2.8) we then obtain the asymmetric breather solution,

$$\begin{aligned}\phi=\epsilon A\,\mathrm{sech}\left[\epsilon\left[\frac{x-V_et}{L_e}\right]\right]\cos(Kx-\Omega t)\\ -2\alpha\epsilon^2A^2\mathrm{sech}^2\left[\epsilon\left[\frac{x-V_et}{L_e}\right]\right]\\ +\epsilon^2\frac{\alpha}{3}A^2\mathrm{sech}^2\left[\epsilon\left[\frac{x-V_et}{L_e}\right]\right]\cos[2(Kx-\Omega t)]\end{aligned} \quad (2.18)$$

with

$$V_e=V_g+\epsilon(u_e/P), \quad (2.19a)$$

$$K=k+\epsilon(u_e/2P), \quad (2.19b)$$

$$\Omega=(\epsilon u_e/2P)(V_g+\epsilon u_c/P)+\omega, \quad (2.19c)$$

and where the quantity

$$L_e=2P/(u_e^2-2u_eu_c)^{1/2} \quad (2.19d)$$

is the width of the envelope.

From Eq. (2.18) we note that the parameter α in Eq. (2.4), which is related to the asymmetry of the potential wells, controls the weight of the dc and second-harmonic terms.

Let us now illustrate these results by considering two specific examples: The perturbed SG and ϕ^4 systems which are universally used to model a variety of physical phenomenon.

A. Perturbed SG system

In this case (see Fig. 1) we have

$$V(\phi)=1-\cos\phi+\frac{F}{\omega_0^2}\phi \quad (2.20)$$

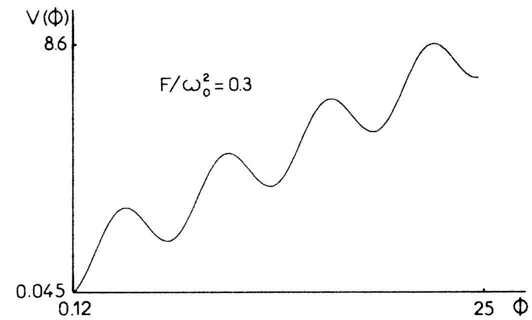

FIG. 1. SG potential perturbed by an external constant force: $F/\omega_0^2=0.3$.

and the corresponding perturbed SG equation of motion

$$\phi_{tt}-C_0^2\phi_{xx}+\omega_0^2\sin\phi+F=0\,. \quad (2.21)$$

For $F=0$ the potential wells are symmetric ($\phi_0=0$) with minima at $\phi=0$ (mod2π); for $F\neq0$ they are asymmetric with minima (Fig. 1) given by

$$\left.\frac{\partial V(\phi)}{\partial\phi}\right|_{\phi_0}=0\,,$$

i.e.,

$$\phi_0=-\sin^{-1}(F/\omega_0^2)\ \ (\mathrm{mod}2\pi)\,. \quad (2.22)$$

We assume $\phi\to\epsilon\phi+\phi_0$ in Eq. (2.20) which is expanded in terms of ϕ, and identified in Eq. (2.4) to give

$$(\omega_0')^2=\omega_0^2\cos\phi_0,\quad \alpha=-\tfrac{1}{2}\tan\phi_0,\quad \beta=-\tfrac{1}{6}\,. \quad (2.23)$$

Inserting these values in (2.15b) gives

$$Q=\frac{\omega_0^2}{4\omega}\cos\phi_0(\tfrac{5}{3}\tan^2\phi_0+1)\,. \quad (2.24)$$

In this case the coefficient Q (which is an increasing function of ϕ_0) is always positive and the solution of (2.14) is an envelope soliton. The corresponding low-amplitude breather is simply obtained from Eq. (2.18), where P and Q are expressed in terms of the coefficients $(\omega_0')^2$, α, and β given by Eq. (2.23). We notice that when $F=0$, we have $\phi_0=0$, $\alpha=0$, and $Q=\omega_0^2/4\omega$, and we recover the small-amplitude breather solution[7] of the nonperturbed SG system.

B. Perturbed ϕ^4 system

The equation of motion corresponding to the potential

$$V(\phi)=\tfrac{1}{4}(\phi^2-1)^2+\frac{F}{\omega_0^2}\phi\,, \quad (2.25)$$

represented in Fig. 2, reads

$$\phi_{tt}-C^2\phi_{xx}+\omega_0^2(\phi^3-\phi)+F=0\,. \quad (2.26)$$

For $F\neq0$ the potential wells are asymmetric with minima given by the solutions of

$$\phi_0^3-\phi_0=\frac{F}{\omega_0^2}\,. \quad (2.27)$$

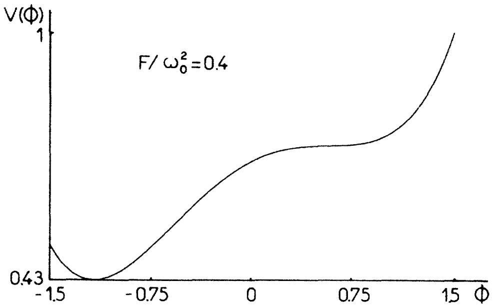

FIG. 2. ϕ^4 potential perturbed by an external constant force: $F/\omega_0^2=0.4$.

For $F=0$ they are asymmetric with $\phi_0=\pm 1$. Using the same technique as in the previous case we get

$$(\omega_0')^2=\omega_0^2(3\phi_0^2-1),\quad \alpha=\frac{3\phi_0^2}{3\phi_0^2-1},\quad \beta=\frac{1}{3\phi_0^2-1}\ . \tag{2.28}$$

Consequently, Eq. (2.15b) gives

$$Q=\frac{3\omega_0^2}{2\omega}\left[\frac{10}{3\phi_0^2-1}\phi_0^4-1\right]. \tag{2.29}$$

For $\phi_0>\frac{1}{3}$ we have $Q>0$ and the two breather solutions corresponding to the two possible minima of the potential wells (see Fig. 2), i.e., two values of ϕ_0, are obtained by replacing in Eq. (2.18) the coefficients $(\omega_0')^2$, α, and β. For $\phi_0=\phi_{01}<\frac{1}{3}$ $(F/\omega_0^2>0.385)$ we have $Q<0$, and in this case we have a breather low-amplitude excitation. In fact we must remark, as represented in Fig. 2 where $F/\omega_0^2=0.4$, that in this case the potential well no longer exists. Note that in this case the kink solutions of the ϕ^4 equation are unstable. The stable small-amplitude oscillations are only possible around the ground state $\phi_{02}=-1.17$.

III. ENVELOPE SOLUTIONS IN THE DISCRETE CARRIER LIMIT

In the preceding section we examined the breather modes, i.e., the slowly oscillating solutions, and used the continuum approximation for both the envelope and carrier wave, but if in Eq. (2.3) the atomic displacements vary greatly, we must treat the phase $\theta=kx-\omega t$ exactly, and only use the continuum approximation for the envelope function F. Thus in the equation of motion (2.3), first we consider θ and F as functions of the discrete variable[13,14] n, and after taking differences we go to the continuum limit for F.

We look for oscillating solutions of the form

$$y_n(t)=F_{1n}(t)e^{i\theta_n}+\text{c.c.}+\epsilon[F_{0n}(t)+F_{2n}e^{2i\theta_n}+\text{c.c.}] \tag{3.1}$$

with $\theta_n=nka-\omega t$. Replacing (3.1) in (2.2) gives

$$\begin{aligned}(\ddot F_{1,n}-2i\omega\dot F_{1,n}-\omega^2F_{1,n})e^{i\theta_n}+\epsilon\ddot F_{0,n}+\epsilon(\ddot F_{2,n}-4i\omega\dot F_{2,n}-4\omega^2F_{2,n})e^{2i\theta_n}\\ =K(F_{1,n+1}e^{ika}+F_{1,n-1}e^{-ika}-2F_{1,n})e^{i\theta_n}+\epsilon K(F_{2,n+1}e^{2ika}+F_{2,n-1}e^{-2ika}-2F_{2,n})e^{2i\theta_n}\\ +\epsilon K(F_{0,n+1}+F_{0,n-1}-2F_{0,n})-\omega_0'^2[F_{1,n}e^{i\theta_n}+\epsilon F_{0,n}+\epsilon F_{2,n}e^{2i\theta_n}+\epsilon\alpha(F_{1,n}^2e^{2i\theta_n}+2F_{1,n}F_{1,n}^*)\\ +2\epsilon^2\alpha(F_{1,n}F_{0,n}+F_{1,n}^*F_{2,n})e^{i\theta_n}+3\epsilon^2\beta F_{1,n}^2F_{1,n}^*e^{i\theta_n}]\ .\end{aligned} \tag{3.2}$$

Since the envelope function varies slowly, we now follow the same procedure as in Sec. II. We use the continuum approximation for F, introduce the slow variable X and T, equate dc, first-, and second-harmonic terms, and get

$$\frac{\epsilon^2}{2\omega}F_{1TT}-i\epsilon F_{1T}-\epsilon^2\left[\frac{Ka^2\cos(ka)}{3\omega}\right]F_{1XX}-i\epsilon\left[\frac{2aK\sin(ka)}{2\omega}\right]F_{1X}-\epsilon^2Q'|F_1|^2F_1=0\ , \tag{3.3a}$$

$$F_0=-2\alpha|F_1|^2\ , \tag{3.3b}$$

$$F_2=\frac{\alpha F_1^2}{3+[16K/(\omega_0')^2]\sin^4\left[\frac{ka}{2}\right]}\ , \tag{3.3c}$$

where

$$Q'=\frac{(\omega_0')^2}{2\omega}\left[4\alpha^2-\frac{2\alpha^2}{3+[16K/(\omega_0')^2]\sin^4\left[\frac{ka}{2}\right]}-3\beta\right]. \tag{3.4}$$

In the above calculations we have used the dispersion relation for the carrier wave,

$$\omega^2=(\omega_0')^2+4K\sin^2(ka/2)\ , \tag{3.5}$$

obtained by linearizing Eq. (2.3).

Then, using the new scales Z and s defined by Eq. (2.12), but now with

$$V_g'=\frac{d\omega}{dk}=\frac{Ka}{\omega}\sin(ka)\ , \tag{3.6}$$

we finally obtain the NLS equation

$$iF_{1s}+P'F_{1ZZ}+Q'|F_1|^2F_1=0\ , \tag{3.7}$$

where

$$P' = \frac{Ka^2}{2\omega}\left[\cos(ka) - \frac{K}{\omega^2}\sin^2(ka)\right] . \quad (3.8)$$

As in the previous case the dispersion coefficient P' and the nonlinear coefficient Q' depend on the wave vector k and the potential parameters $(\omega_0')^2$, α, and β. We note that for small k (continuum limit), $P' \to P$, $Q' \to Q$, and Eq. (3.7) reduces to Eq. (2.14).

Considering the perturbed SG and ϕ^4 cases, we see from Eq. (3.4) that for any k we have $Q' > 0$. From Eqs. (3.8) and (3.5) and with $K = C_0^2/a^2$ we get

$$P' = \frac{C_0^2}{2\omega^3}[(R+2)\cos k - \cos^2 k - 1] , \quad (3.9)$$

where we have assumed $a = 1$ and with $R = (\omega_0')^2/C_0^2$. The sign of P' depends on k and R. For example, we find that for $R = 1$, $P' < 0$ for $k > k_l = 1.2$, and for $R = 10$, $P < 0$ for $k > k_l = 1.5$. These results show us that we only have envelope modes ($P' > 0$) for $0 < k < k_l$, k_l being the limit wave vector of the carrier wave. For $k > k_l$ we have dark soliton modes. Using the solution of (3.7) which is given by Eq. (2.16), and using Eqs. (3.3b), and (3.3c), and Eq. (3.1), we get the asymmetric envelope solution,

$$\begin{aligned} y_n(t) = {} & \epsilon A' \operatorname{sech}\chi \cos(K'na - \Omega' t) - 2\alpha\epsilon^2 (A')^2 \operatorname{sech}^2\chi \\ & + \frac{\epsilon^2 \alpha \operatorname{sech}^2\chi}{3 + [16C/(\omega_0')^2]\sin^4(ka/2)} \cos[2(K'na - \Omega' t)] \end{aligned} \quad (3.10)$$

with

$$\psi = \epsilon(na - V_e' t)/L_e' . \quad (3.11)$$

Here P' and Q' replace P and Q in Eqs. (2.17) and (2.19), defining the quantities A', V_e', K', Ω', and L_e'. For a given set of coefficients ω_0, α, and β which characterize a specific potential, as, for example, the perturbed SG or ϕ^4 system, one easily obtains, from (3.10), the respective envelope-mode solutions.

IV. BREATHERS IN A FERROMAGNETIC CHAIN

Let us consider a 1D ferromagnetic chain described[15] by the following Hamiltonian:

$$H = -J\sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} + A\sum_n (S_n^Z)^2 - g\mu_B \mathbf{B} \cdot \sum_n \mathbf{S}_n , \quad (4.1)$$

where the first term represents the ferromagnetic ($J > 0$) Heisenberg exchange interaction between neighboring spins denoted by the vectors $\mathbf{S}_n$ and $\mathbf{S}_{n+1}$. The second term represents the easy-plane (xy) anisotropy energy ($A > 0$) and the last term represents the Zeeman energy of the spins in a magnetic field ($B_x = B$) perpendicular to the chain axis (Z). g and μ_B are, respectively, the Landé factor and the Bohr magneton. The dynamics of these classical spin vectors is described by the undamped Bloch equation

$$\hbar \dot{\mathbf{S}}_n = \mathbf{S}_n \times [J(\mathbf{S}_{n-1} + \mathbf{S}_{n+1}) + g\mu_B \mathbf{B} - 2AS_n^Z \mathbf{Z}] . \quad (4.2)$$

If the spins with fixed length S have their orientations parametrized by the spherical polar angles ϑ_n (out of plane) and φ_n (in plane),

$$S_n^x = S\cos\vartheta_n \cos\varphi_n, \quad S_n^y = S\cos\vartheta_n \sin\varphi_n, \quad S_n^Z = S\sin\vartheta_n , \quad (4.3)$$

then the equations of motion become[15–17]

$$\begin{aligned} \frac{\hbar}{JS}\dot{\varphi}_n \cos\vartheta_n = {} & \sin\vartheta_n[\cos\vartheta_{n+1}\cos(\varphi_{n+1} - \varphi_n) \\ & + \cos\vartheta_{n-1}\cos(\varphi_{n-1} - \varphi_n)] \\ & - \cos\vartheta_n(\sin\vartheta_{n+1} + \sin\vartheta_{n-1}) \\ & + \frac{2A}{J}\cos\vartheta_n \sin\vartheta_n \\ & + \frac{g\mu_B}{JS}\sin\vartheta_n \cos\varphi_n , \end{aligned} \quad (4.4a)$$

$$\begin{aligned} \frac{\hbar}{JS}\dot{\vartheta} = {} & \cos\vartheta_{n+1}\sin(\varphi_{n+1} - \varphi_n) \\ & + \cos\vartheta_{n-1}\sin(\varphi_{n-1} - \varphi_n) - \frac{g\mu_B}{JS}\sin\varphi_n . \end{aligned} \quad (4.4b)$$

In the continuum limit (where the length-scale ratio $J/B \gg 1$), Eqs. (4.4) become

$$\theta_\tau = Ja^2(\varphi_{ZZ}\cos\vartheta - 2\theta_Z \phi_Z \sin\vartheta) - b\sin\varphi , \quad (4.5a)$$

$$\begin{aligned} \phi_\tau \cos\vartheta = {} & -Ja^2 \vartheta_{ZZ} + 2A\sin\vartheta\cos\vartheta\left[1 - \frac{Ja^2}{2A}\varphi_Z^2\right] \\ & + b\sin\vartheta\cos\varphi \end{aligned} \quad (4.5b)$$

where $\tau = (S/\hbar)t$ and $b = g\mu_B/S$. In the limit $\theta \ll 1$ and $b \ll 2A$, Eqs. (4.5) can be approximated by the SG equation

$$\varphi_{\tau\tau} - 2AJa^2 \varphi_{ZZ} + 2Ab\sin\varphi = 0 \quad (4.6a)$$

with

$$\varphi_\tau = 2A\vartheta , \quad (4.6b)$$

which admits breather solutions which are given by Eq. (2.18) (assuming $2AJa^2 = C_0^2$ and $2Ab = \omega_0^2$) with $\alpha = 0$. To go beyond the SG limit, from Eqs. (4.5) we now consider small oscillations around the in-plane angle φ, $\varphi \to \epsilon\varphi$, and around the out-of-plane angle ϑ_0 which is assumed to be small, $\vartheta \to \epsilon\vartheta_0 + \epsilon^2\vartheta$. Under these conditions Eqs. (4.5) reduce to

$$\epsilon\vartheta_\tau = Ja^2 \varphi_{ZZ} - b\left[\varphi - \epsilon^2\frac{\varphi^3}{6}\right] + O(\epsilon^3) , \quad (4.7a)$$

$$\varphi_\tau = -Ja^2 \epsilon\vartheta_{ZZ} + (2A + b)(\vartheta_0 + \epsilon\vartheta) + O(\epsilon^2) . \quad (4.7b)$$

From Eqs. (4.7) one finally obtains

$$\begin{aligned} & \varphi_{\tau\tau} - 2Ja^2(A+b)\varphi_{ZZ} + (2A+b)b(\varphi - \epsilon^2\varphi^3/6) \\ & \qquad = -Ja^4 \varphi_{ZZZZ} - Ja^2 b\epsilon^2(\varphi^3/6)_{ZZ} . \end{aligned} \quad (4.8)$$

We remark that the first term of (4.8) corresponds to the small-amplitude expansion of Eq. (4.6a) and that the additional term on the right-hand side is of the modified Boussinesq type. When $b \ll 2A$, Eq. (4.8) reduces to Eq. (4.6a).

We now follow the calculations described in Sec. II to reduce (4.8) to a NLS equation. We first introduce the slow variables $T=\epsilon\tau$ and $X=\epsilon Z$, and from (4.8) we obtain

$$\varphi_{\tau\tau}+2\epsilon\varphi_{\tau T}+\epsilon^2\varphi-C_0^2\varphi_{ZZ}-C_0^2\epsilon\varphi_{ZX}-C_0^2\epsilon^2\varphi_{XX} +h[\varphi_{ZZZZ}+4\epsilon\varphi_{ZZZX}+6\epsilon^2\varphi_{ZZXX}+O(\epsilon^3)] +q[\epsilon^2(\varphi^3)_{ZZ}+O(\epsilon^3)]-\omega_0^2\varphi+\epsilon^2 q'\varphi^3=0 \tag{4.9}$$

with

$$h=J^2a^4,\quad q=Ja^2b/6,\quad \omega_0^2=(2A+b)b,\quad q'=\frac{\omega_0^2}{6}\ . \tag{4.10}$$

The nonlinear force terms in Eq. (4.8) are odd (cubic), and they should correspond to even terms in the corresponding potential. Consequently, we look for symmetric modulated wave solutions without constant and second-harmonic terms,

$$\phi=F(X,T)e^{i\delta}+\text{c.c.} \tag{4.11}$$

with $\delta=kZ-\omega\tau$.

Inserting (4.11) in (4.9), keeping terms to order ϵ^2, and introducing the new scales

$$Y=X-V_g''T,\quad s=\epsilon T \tag{4.12}$$

with

$$V_g''=\frac{d\omega}{dk}=\frac{k}{\omega}(hk^2+C_0^2) \tag{4.13a}$$

and

$$\omega^2=\omega_0^2+C_0^2k^2+hk^4\ , \tag{4.13b}$$

we get the NLS equation

$$iF_s+P''F_{YY}+Q''|F|^2F=0\ . \tag{4.14}$$

Here we have

$$P''=[C_0^2+6hk^2-(V_g'')^2]/2\omega\ , \tag{4.15}$$

$$Q''=3(qk^2+q')\ . \tag{4.16}$$

From Eqs. (4.15) and (4.16) we easily see that $P''>0$ and $Q''>0$ for any k ($k\ll1$), and Eq. (4.14) always has the envelope-mode solution of the form (2.16), where P'' and Q'' replace P and Q, and Y replaces Z. Under these conditions one obtains, from (4.11), the symmetric low-amplitude breather solution. Therefore,

$$\phi=A''\text{sech}\left[\left[\frac{Q''}{2P''}\right]^{1/2}\epsilon A''(Z-V_e''\tau)\right]\cos(kZ-\omega\tau)\ , \tag{4.17}$$

where

$$V_e''=V_g''+\epsilon(u_e/P'')\ . \tag{4.18}$$

Here the breather excitation is no longer of the pure SG type as in Eq. (4.6a). This is due to the external field which increases the out-of-plane angle ϑ.

From Eqs. (4.7a) and (4.17) one can calculate (for $k=0$ and $V_e''=0$) the expression for ϑ,

$$\epsilon\vartheta=\frac{A''}{\omega}\left[Ja^2\epsilon(A'')^2\frac{Q''}{2P''}-b\right]\text{sech}(\chi Z)\sin(\omega\tau) +\frac{A''}{\omega}\left[\frac{b\epsilon^2}{8}-Ja^2\epsilon A^2\frac{Q''}{P''}\right]\text{sech}^3(\chi Z)\sin(\omega\tau) +\frac{(A'')^3}{\omega}\frac{b\epsilon^2}{72}\text{sech}^3(\chi Z)\sin(3\omega\tau)\ , \tag{4.19}$$

where

$$\chi=\left[\frac{Q''}{2P''}\right]^{1/2}\epsilon A''\ . \tag{4.20}$$

V. SUMMARY AND DISCUSSION

We have examined the low-amplitude breather and envelope excitations in different models of quasi-one-dimensional physical systems. Considering a generalized Klein-Gordon lattice model we first used the continuum approximation, effectively restricting ourselves to those excitations which consist of a slowly varying envelope modulating a slowly varying carrier wave, whose, to order ϵ, dispersion relation is that of a linear wave. As pointed out in the Introduction, in this small-wave-vector limit the envelope mode looks like it is breathing, and hence we call it a breather. Using the multiple-scale expansion technique, we then reduce the equation of motion to a nonlinear Schrödinger equation. The well-known soliton solutions of the nonlinear Schrödinger equation correspond to a breather in the original nonlinear Klein-Gordon equation. Then our results can be immediately applied to the calculation of breathers in the sine-Gordon and ϕ^4 systems perturbed by an external constant force. In both cases the asymmetry of the breather excitations is controlled by the amplitude of the external force. Both these familiar examples were chosen to illustrate the above expansion and method, which are not so well known in detail by physicists.

We then generalized our study using the semidiscrete limit in which the envelope of a soliton is determined in the continuum limit, while the fast oscillations of the quasiharmonic carrier inside the envelope are treated exactly: Here their wave vector k is not limited to long wavelengths and we call the excitation an envelope mode. In the regime for both perturbed sine-Gordon and for ϕ^4 systems, we have asymmetric envelope solitons which can exist for $0<k<k_1$ (for $k\ll1$ they are the breather modes calculated previously). An interesting related problem is that a nonlinear system which supports envelope solitons is known to exhibit modulational or Benjamin-Feir instability. Qualitatively, it is the tendency of an amplitude-modulated carrier to break into isolated envelope solitons (nonlinear localization effect). Consequently, this instability will exist for $k<k_1$, but above this limit it will disappear because the characteristic excitations are of the dark (hole) soliton type, where the depression of an envelope propagates as a soliton with a finite amplitude (i.e., with infinite energy) at $|Z|\to\infty$.

Although these nodes are interesting in some other physical systems, in this context they cannot be attained by finite-energy excitation from the ground state.

Our approach was generalized to the study of a 1D ferromagnetic chain where the spin orientations were parametrized by the polar in-plane angle φ and out-of-plane angle ϑ. In the continuum and small-amplitude limits we have obtained two equations of motion which can be decoupled. Equation (4.3) for φ, which contains a nonlinear sine-Gordon (small expansion) term, a dispersive term, and a nonlinear modified Boussinesq term, is remarkable in the sense that it can be mapped to the equation of motion of a 1D chain of atoms nonlinearly coupled and interacting with a SG substrate potential. Using the same technique as in Sec. II, we have calculated the symmetric breather (envelope) solitons solutions which are no longer of the pure SG type, as is the case when the out-of-plane angle is very small, i.e., for very low external magnetic field. For any magnetic field the recent numerical simulations of Wysin *et al.*[17] stress that the basic nonlinear excitations in quasi-1D ferromagnets such as $CsNiF_3$ are breathers rather than isolated solitons. This suggests it should be interesting to extend our study to the calculations of breather or envelope structures at high external magnetic field, in such materials, when the spins do not remain close to the easy plane.

Finally, taken together with our earlier results on envelope solitons in nonlinearly coupled lattices,[14] our present results indicate that these nonlinear wave packets called breather or envelope solitons, which can be easily inverted in the low-amplitude limit, can be expected in most quasi-1D nonlinear systems. We hope this work will stimulate further investigations of these widespread, experimentally relevant excitations.

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