# Soliton-breather approach to classical sine-Gordon thermodynamics

Kazuo Sasaki
*Department of Physics, University of Southern California, Los Angeles, California 90089-0484*
(Received 21 October 1985)

Thermodynamics of the Boltzmann gas consisting of solitons, antisolitons, and breathers of the classical sine-Gordon system is studied by taking their interactions into account. A compact expression for the thermodynamic potential is obtained, which agrees with that obtained from the classical limit of the Bethe-*ansatz* formulation of the quantum sine-Gordon thermodynamics. We further find that the average number of breathers excited at low temperatures ($m \ll T \ll M$, where $m$ and $M$ are the phonon and the soliton masses, respectively) is half the total number of degrees of freedom of the system.

## I. INTRODUCTION

Recently, the thermodynamics of the quantum sine-Gordon system has been studied by using the Bethe-*ansatz* method.[1,2] The theory is formulated as an appropriate limit[3] of the Bethe-*ansatz* thermodynamics of the *XYZ* spin chain.[4] The free energy of the sine-Gordon system is expressed as a sum of contributions from solitons and breathers. In this paper we shall derive the same result for the free energy based on a soliton-gas picture in the classical limit. We shall consider a grand canonical ensemble of a Boltzmann gas consisting of solitons and breathers.

There are other approaches to the sine-Gordon thermodynamics based on the soliton-gas picture: the ideal-gas phenomenology[5,6] (classical theory) and the path-integral method[7] (quantum theory in the weak-coupling regime). In these theories solitons and phonons are regarded as elementary modes, and a dilute-gas approximation to solitons is introduced. Although their results are consistent with the exact ones,[1,2,8,9] they are valid only at low temperatures[10] (much less than the soliton mass).

Our approach[11] here is basically the same as that of the ideal-gas phenomenology of Currie *et al.*,[6] except that we regard solitons and breathers, instead of solitons and phonons, as elementary modes, and we take into account all interactions between elementary modes. Here we should comment on the independent degrees of freedom of the system. As shown by Dashen *et al.* by semiclassical analysis as well as perturbation analysis,[12] the energy spectrum of the sine-Gordon system is exhausted by solitons and breathers, which is consistent with the Bethe-*ansatz* results.[1,2] Further, they concluded that the lowest-energy state of breathers is nothing but the renormalized phonon and that the higher-energy states are bound states of phonons.[12] This means that the breather and the phonon are not independent degrees of freedom. Therefore it is possible to formulate the sine-Gordon thermodynamics by regarding solitons and breathers as basic modes (the Bethe-*ansatz* and the present approaches) or equivalently by regarding solitons and phonons as basic modes (the ideal-gas phenomenology and the path-integral method). A soliton-gas approach starting with solitons, breathers, *and* phonons[13] is clearly inconsistent.

Very recently, Takayama and Ishikawa[14] established formal relation between the soliton-gas and the Bethe-*ansatz* approaches. Their soliton-gas formulation is different from ours, but the final result agrees with the present result.

The paper is organized as follows. In the next section we summarize some properties of solitons and breathers in the classical sine-Gordon system. In Sec. III the grand canonical ensemble of a soliton-breather gas is considered. Thanks to pairwise additivity of phase shifts (factorizability of the $S$ matrix) for collisions between solitons and breathers, the summation involved in the grand partition function can be performed formally (diagrammatically) and a compact expression for the thermodynamic potential is obtained. This expression is shown to be the same as the Bethe-*ansatz* result[1,2,15] and the one obtained by the extended ideal-gas phenomenology of Takayama and Ishikawa.[14] In the present approach we can obtain the thermal average of numbers of solitons, antisolitons, and breathers, while in the Bethe-*ansatz* method only the difference of soliton and antisoliton numbers (the winding number) has been obtained.[15]

In Sec. IV we demonstrate how the Boltzmann gas of breathers yields the free energy of the classical phonon. This is a classical analogue of Fowler's analysis of the Bethe-*ansatz* thermodynamics in the weak-coupling limit,[16] where he obtained the free energy of a free-boson gas with the phonon mass. We further find that the average number of breathers is half the total number of degrees of freedom of the system at low temperatures where contributions from solitons and antisolitons can be neglected. This result is consistent with the fact that each breather has two degrees of freedom. Concluding remarks are given in Sec. V. Mathematical details of some of analysis are given in Appendixes A–D.

## II. SOLITONS, BREATHERS, AND THEIR INTERACTIONS

The sine-Gordon model we consider is described by the Hamiltonian

 

$$H=\int dx\left[\tfrac{1}{2}\Pi^2+\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2+\frac{m^2}{g^2}[1-\cos(g\phi)]\right], \tag{1}$$

where $\Pi=\partial\phi/\partial t$ is the canonical momentum density of the field $\phi$, $g$ is the coupling constant, and $m$ is the phonon mass. We use a system of units in which $\hbar=k_B=c=1$. In this paper we restrict ourselves to the weak-coupling limit ($g\to 0$), where quantum correction is small. In the rest of this section we shall summarize some results of the classical field theory, which will be used in the subsequent sections.

The soliton solution to the equation of motion derived from (1) is given by

$$g\phi_s(x,t\mid\alpha,q,Q)=4\tan^{-1}\{\exp[Qm\cosh\alpha(x-vt-q)]\}, \tag{2}$$

where $\alpha$ $(-\infty<\alpha<\infty)$ is the rapidity which is related with the velocity $v$ by $v=\tanh\alpha$, $q$ $(-\infty<q<\infty)$ is the position of the soliton at $t=0$, and $Q=\pm 1$ is the topological charge [$Q=+1$ $(-1)$ refers to the soliton (antisoliton)]. The energy $E_s$ $(E_{\bar{s}})$ and the momentum $P_s$ $(P_{\bar{s}})$ of a soliton (an antisoliton) are given by

$$E_s(\alpha)=E_{\bar{s}}(\alpha)=M\cosh\alpha=(M^2+P_s^2)^{1/2}, \tag{3}$$

$$P_s(\alpha)=P_{\bar{s}}(\alpha)=M\sinh\alpha, \tag{4}$$

where $M=8m/g^2$ is the soliton (antisoliton) mass. The momentum $P_s$ $(P_{\bar{s}})$ is the canonical variable to the soliton position $q$.[17,18]

The breather solution is given by

$$g\phi_B(x,t\mid\alpha,q,\theta,\psi)=4\tan^{-1}\left[\tan\theta\frac{\cos[m\cos\theta\cosh\alpha(t-vx)+\psi]}{\cosh[m\sin\theta\cosh\alpha(x-vt-q)]}\right]. \tag{5}$$

The breather has an internal degree of freedom, which is described by the variables $\theta$ $(0<\theta<\pi/2)$ and $\psi$ $(0<\psi<2\pi)$, as well as the translational degree of freedom, which is described by the rapidity $\alpha$ $(-\infty<\alpha<\infty)$ and the position $q$ $(-\infty<q<\infty)$. The energy $E_B$ and the momentum $P_B$ of the breather is given by

$$E_B(\alpha,\theta)=2M\sin\theta\cosh\alpha=(M_B^2+P_B^2)^{1/2}, \tag{6}$$

$$P_B(\alpha,\theta)=2M\sin\theta\sinh\alpha, \tag{7}$$

where $M_B(\theta)=2M\sin\theta$ is the breather mass. The breather mass varies continuously from zero to twice the soliton mass depending on the internal variable $\theta$ in the classical field theory, while $\theta$ takes only discrete values in the quantum field theory[12,18] [$\theta=jg^2/16$ $(j=1,2,\ldots,<8\pi/g^2)$ in the weak-coupling limit]. The translational and the internal momenta $P_B$ and $16\theta/g^2$ are canonical variables to the position $q$ and the internal phase $\psi$, respectively.[17,18]

For small values of $\theta$ $(\theta\to 0)$ the breather solution (5) describes a plane wave ("phonon")

$$g\phi_B(x,t\mid\alpha,q,\theta\to 0,\psi)\to 4\theta\cos(\omega_k t-kx+\psi), \tag{8}$$

where $k=m\sinh\alpha$ is the wave number, and the frequency $\omega_k$ is given by $\omega_k=(m^2+k^2)^{1/2}$. In the opposite limit $(\theta\to\pi/2)$ the breather solution (5) represents an unbound pair of a soliton and an antisoliton.

A scattering process involving arbitrary numbers of solitons and breathers is described as successions of two-body collisions.[19] A breather labeled by $(\alpha,\theta)$ suffers a shift in its position [$q$ in Eq. (5)] by an amount[19]

$$\Delta_{BB}(\alpha,\theta;\alpha',\theta')=\frac{1}{m\sin\theta\cosh\alpha}\ln\left[\frac{[\cosh(\alpha-\alpha')+\sin\theta\sin\theta']^2-\cos^2\theta\cos^2\theta'}{[\cosh(\alpha-\alpha')-\sin\theta\sin\theta']^2-\cos^2\theta\cos^2\theta'}\right] \tag{9}$$

when it collides with a breather $(\alpha',\theta')$. The direction of the shift is the positive (negative) direction of the $x$ axis if $\alpha>\alpha'$ $(\alpha<\alpha')$. Position shifts of "particles" in other two-body collisions can be obtained from $\Delta_{BB}$ as[19]

$$\Delta_{BS}(\alpha,\theta;\alpha')=\tfrac{1}{2}\Delta_{BB}(\alpha,\theta;\alpha',\pi/2), \tag{10}$$

$$\Delta_{SB}(\alpha;\alpha',\theta')=\Delta_{BB}(\alpha,\pi/2;\alpha',\theta'), \tag{11}$$

$$\Delta_{SS}(\alpha;\alpha')=\tfrac{1}{2}\Delta_{BB}(\alpha,\pi/2;\alpha',\pi/2), \tag{12}$$

where $\Delta_{BS}(\alpha,\theta;\alpha')$ denotes a position shift of a breather $(\alpha,\theta)$ due to a soliton $(\alpha')$, and so on. The position shifts are independent of charges of solitons. The relations (10)–(12) follow from the fact that a breather goes to an unbound soliton-antisoliton pair in the limit $\theta\to\pi/2$ as already noted after Eq. (8). The position shift plays an essential role in counting "microscopic states" to obtain the partition function in the next section. The breather suffers also a shift in its internal phase [$\psi$ in Eq. (5)]. However, it has no effects on the thermodynamics.

## III. THERMODYNAMICS OF A SOLITON-BREATHER GAS

Let us consider the grand canonical ensemble of a Boltzmann gas consisting of solitons, antisolitons, and breathers (we shall call it a soliton-breather gas). The thermodynamic potential per unit length of the system, $\Omega$, is a function of the temperature $T$, the chemical potentials $\mu_s$, $\mu_{\bar{s}}$, and $\mu_B$ of the soliton, the antisoliton, and the breather:

$$\Omega(T,\mu_s,\mu_{\bar{s}},\mu_B)=-TL^{-1}\ln\Xi(L,T,\mu_s,\mu_{\bar{s}},\mu_B), \tag{13}$$

where $L$ is the system size and $\Xi$ is the grand partition function.

We note that the sine-Gordon system itself does not approach the thermal equilibrium since it is an integrable system. The presence of a "heat bath" is assumed when we discuss the thermodynamics of this system. Then the thermal equilibrium is achieved by soliton-antisoliton pair creations and annihilations. These processes may be regarded as "chemical reactions"[20]

$$S+\bar{S}\leftrightarrow B \tag{14}$$

and the condition for equilibrium is given by[21] $\mu_s+\mu_{\bar{s}}=\mu_B$. Since we cannot control the number of breathers we have to put $\mu_B=0$. On the other hand, the total topological charge $W=N_s-N_{\bar{s}}$ (the winding number), where $N_s$ and $N_{\bar{s}}$ are the numbers of solitons and antisolitons, can be controlled, so that $\mu_s$ can have a finite value. We thus obtain the equilibrium condition for the soliton-breather gas,

$$\mu_s+\mu_{\bar{s}}=0,\ \mu_B=0\ . \tag{15}$$

Therefore in the thermal equilibrium the thermodynamic potential density $\Omega_{\rm eq}$ is a function of $T$ and $\mu_s$ only,

$$\Omega_{\rm eq}(T,\mu_s)=\Omega(T,\mu_s,-\mu_s,0)\ . \tag{16}$$

However the function $\Omega(T,\mu_s,\mu_{\bar{s}},\mu_B)$ is still useful to calculate the densities $n_s$, $n_{\bar{s}}$, and $n_B$ of solitons, antisolitons, and breathers. They are given by

$$n_a=-\left[\frac{\partial\Omega}{\partial\mu_a}\right]_{\mu_s=-\mu_{\bar{s}},\,\mu_B=0},\quad a=s,\bar{s},B\ . \tag{17}$$

The winding-number density $w=W/L$ is obtained from $\Omega_{\rm eq}$ or Eq. (17) as

$$w=-(\partial\Omega_{\rm eq}/\partial\mu_s)=n_s-n_{\bar{s}}\ . \tag{18}$$

Now we shall calculate the grand partition function $\Xi$ in Eq. (13). It can be expressed as

$$\Xi=\sum_{N_s}\sum_{N_{\bar{s}}}\sum_{N_B}\frac{\exp[\beta(\mu_sN_s+\mu_{\bar{s}}N_{\bar{s}}+\mu_BN_B)]}{(N_s!)(N_{\bar{s}}!)(N_B!)}\times\int d\Gamma(\{P\}\,|\,N,N_B)\times\exp[-\beta E(\{P\}\,|\,N,N_B)]\ , \tag{19}$$

where $\beta=1/T$, $N=N_s+N_{\bar{s}}$, $d\Gamma(\{P\}\,|\,N,N_B)$ is the "phase volume element," and $E(\{P\}\,|\,N,N_B)$ is the energy of the system with $N_s$ solitons, $N_{\bar{s}}$ antisolitons, and $N_B$ breathers. The symbol $\{P\}$ represents a set of variables $\{\alpha_{s1},\alpha_{s2},\ldots,\alpha_{sN},\ (\alpha_{B1},\theta_1),(\alpha_{B2},\theta_2),\ldots,(\alpha_{BN_B},\theta_{N_B})\}$, where $\alpha_s$ is the rapidity of a soliton (an antisoliton), and $\alpha_B$ and $\theta$ are the rapidity and the internal variable of a breather. The energy $E$ is given by[17,18]

$$E(\{P\}\,|\,N,N_B)=\sum_{i=1}^{N}E_s(\alpha_{si})+\sum_{j=1}^{N_B}E_B(\alpha_{Bj},\theta_j)\ , \tag{20}$$

where $E_s$ and $E_B$ are given by Eqs. (3) and (6).

The phase volume element $d\Gamma$ may be written as

$$d\Gamma(\{P\}\,|\,N,N_B)=L^{N+N_B}\prod_{i=1}^{N}\prod_{j=1}^{N_B}d\Gamma_{si}\,d\Gamma_{Bj}\times R(\{P\}\,|\,N,N_B)\ , \tag{21}$$

where $d\Gamma_s$ and $d\Gamma_B$ are the phase volume elements (divided by the system size $L$) of a free soliton and a free breather, which are given by

$$d\Gamma_s=(2\pi)^{-1}dP_s=(2\pi)^{-1}E_s(\alpha)d\alpha\ , \tag{22}$$

$$d\Gamma_B=(2\pi)^{-1}(16/g^2)dP_B(\theta)d\theta=(M/\pi m)E_B(\alpha,\theta)d\alpha\,d\theta\ . \tag{23}$$

The function $R$ in Eq. (21) represents the restriction on the phase space available for the particles due to their interactions; if there is no interaction $R=1$. The function $R$ can be expressed in terms of the position shifts given in Eqs. (9)—(12) as

$$R(\{P\}\,|\,N,N_B)=\det R_{ij}(\{P\}\,|\,N,N_B)\ , \tag{24}$$

where the $(N+N_B)\times(N+N_B)$ matrix $R_{ij}$ is defined by

$$LR_{ii}=\begin{cases}L-\sum_{j=1}^{N}{}'\Delta_{ss}(i;j)-\sum_{j=1}^{N_B}\Delta_{SB}(i;j), & 1\le i\le N \quad (25a)\\ L-\sum_{j=1}^{N}\Delta_{Bs}(i-N;j)-\sum_{j=1}^{N_B}{}'\Delta_{BB}(i-N;j)\ , & N+1\le i\le N+N_B \quad (25b)\end{cases}$$

and

$$LR_{ij}=\begin{cases}\Delta_{ss}(i;j), & 1\le i,j\le N \quad (26a)\\ \Delta_{SB}(i;j-N), & 1\le i\le N,\ N+1\le j\le N+N_B \quad (26b)\\ \Delta_{BS}(i-N;j), & N+1\le i\le N+N_B,\ 1\le j\le N \quad (26c)\\ \Delta_{BB}(i-N;j-N), & N+1\le i,j\le N+N_B \quad (26d)\end{cases}$$

for $i\neq j$. In Eq. (25) the prime on $\sum$ denotes that the term $j=i$ is not involved in the summation. Derivation of Eq. (24) is given in Appendix A.

Noticing that $d\Gamma$ and $E$ in Eq. (19) depend on $N_s$ and $N_{\bar{s}}$ only through $N=N_s+N_{\bar{s}}$, we rewrite Eq. (19) as

$$\Xi=\sum_{N=0}^{\infty}\sum_{N_B=0}^{\infty}(N!N_B!)^{-1}\xi^N\xi_B^{N_B}\times\int d\Gamma(\{P\}\,|\,N,N_B)\times\exp[-\beta E(\{P\}\,|\,N,N_B)]\ , \tag{27}$$

where

$$\xi=\exp(\beta\mu_s)+\exp(\beta\mu_{\bar{s}}),\quad \xi_B=\exp(\beta\mu_B)\ . \tag{28}$$

The summation in Eq. (27) is performed most conveniently by a diagrammatic method (see Appendix B). The result is summarized as

$$
\begin{aligned}
-\beta\Omega &= L^{-1}\ln\Xi \\
&= \xi \int d\Gamma_s \exp[-\beta\epsilon_s(\alpha)] \\
&\quad + \xi_B \int d\Gamma_B \exp[-\beta\epsilon_B(\alpha,\theta)] ,
\end{aligned} \tag{29}
$$

where $\epsilon_s$ and $\epsilon_B$ are determined by the coupled integral equations:

$$
\begin{aligned}
\beta\epsilon_s(\alpha) &= \beta E_s(\alpha) + \xi \int d\Gamma_s' \,\Delta_{ss}(\alpha';\alpha)\exp[-\beta\epsilon_s(\alpha')] \\
&\quad + \xi_B \int d\Gamma_B' \,\Delta_{BS}(\alpha',\theta';\alpha)\exp[-\beta\epsilon_B(\alpha',\theta')] ,
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
\beta\epsilon_B(\alpha,\theta) &= \beta E_B(\alpha,\theta) + \xi \int d\Gamma_s' \,\Delta_{SB}(\alpha';\alpha,\theta) \\
&\qquad \times \exp[-\beta\epsilon_s(\alpha')] \\
&\quad + \xi_B \int d\Gamma_B' \,\Delta_{BB}(\alpha',\theta';\alpha,\theta) \\
&\qquad \times \exp[-\beta\epsilon_B(\alpha',\theta')] .
\end{aligned} \tag{31}
$$

In Eq. (29) the first term represents contribution from solitons and antisolitons, and the second term from breathers. The effective energy spectrums $\epsilon_s$ and $\epsilon_B$ depend not only on $\alpha$ (and $\theta$) but also on $T$, $\mu_s$, $\mu_{\bar{s}}$, and $\mu_B$, although we do not write explicitly the latter variables as arguments of $\epsilon_s$ and $\epsilon_B$. A set of equations (29)—(31), which are the central result of the present work, describes thermodynamics of the soliton-breather gas.

Equations (29)—(31) with $\xi = 2\cosh(\beta\mu_s)$ and $\xi_B = 1$, which will give $\Omega_{eq}$ [see Eq. (16)], agree with the classical limit of the corresponding equations of Hida *et al.*[15] obtained by the Bethe-*ansatz* method. To see this explicitly, we first note the relation

$$
\epsilon_s(\alpha) = \tfrac{1}{2}\epsilon_B(\alpha,\pi/2) , \tag{32}
$$

which comes from Eqs. (3), (6), (9)—(12), (30), and (31). The classical limit of the Bethe-*ansatz* result is achieved by replacing the discrete spectrum of the breather mass by the continuum one (the weak-coupling limit) and replacing the Fermi statistics by the Boltzmann statistics (this means we take the limit $\eta_j \to \infty$ for $j = 1,2,\ldots,\nu_1 - 1$ and $\eta_{\nu_1} \to 0$ in the results of Hida *et al.*[15]). Then we find that Eqs. (2.9a), (2.10), and (4.1) of Hida *et al.*[15] coincide with Eqs. (31), (29), and (32). The classical limit of the Bethe-*ansatz* results obtained by Fowler and Zotos[1] and by Imada *et al.*[2] agree with Eqs. (29)—(31) with $\xi = 2$ and $\xi_B = 1$.

It is not difficult to show that the extended ideal-gas phenomenology of Takayama and Ishikawa[14] yields Eqs. (29)—(31) if solitons and breathers are regarded as elementary modes, and chemical potentials are introduced in their formulation. The proof is left for the reader.

## IV. BREATHER GAS AT LOW TEMPERATURES

We shall solve Eq. (31) at low temperatures ($m \ll T \ll M - |\mu_s|$) where contributions from solitons and antisolitons can be neglected. The free energy of the breather gas then reduces to the free energy of a classical phonon gas.

In this temperature region, only small amplitude ($\theta \ll 1$) breathers are important. The breather energy, Eq. (6), is approximated by $E_B \simeq 2M\theta\cosh\alpha$ and the position shift in breather-breather collision, Eq. (9), by[16]

$$
\Delta_{BB}(\alpha,\theta;\alpha',\theta') \simeq (4\pi/m\theta)\,\mathrm{sech}(\alpha)\delta(\alpha-\alpha')\min(\theta,\theta') \tag{33}
$$

for $\theta,\theta' \ll 1$. Then neglecting the second term (contribution from solitons and antisolitons) in Eq. (31), we have

$$
\begin{aligned}
\beta\epsilon_B(\alpha,\theta) &= 2\beta M\theta\cosh\alpha \\
&\quad + 2(2M/m)^2 \xi_B \int_0^{\pi/2} d\theta' \min\{\theta,\theta'\} \\
&\qquad \times \exp[-\beta\epsilon_B(\alpha,\theta')] .
\end{aligned} \tag{34}
$$

Equation (34) can be solved following Fowler's analysis[16] of the Bethe-*ansatz* thermodynamics in the weak-coupling limit and at low temperatures ($g \to 0$ and $T \ll M$). The solution is found to be (see Appendix C)

$$
\xi_B \exp[-\beta\epsilon_B(\alpha,\theta)] = \left[ \frac{\frac{1}{2}\beta m \cosh\alpha}{\sinh[(\beta M\theta + \frac{1}{2}\beta m \xi_B^{-1/2})\cosh\alpha]} \right]^2 \tag{35}
$$

for $\beta m \ll 1$. Substituting Eq. (35) into Eq. (29) and neglecting contribution from solitons and antisolitons, we obtain

$$
\begin{aligned}
\beta\Omega &\simeq -\xi_B \int_0^{\pi/2} \frac{2M}{m} d\theta \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} 2M\theta\cosh\alpha \\
&\qquad \times \exp[-\beta\epsilon_B(\alpha,\theta)] \\
&= \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} m\cosh\alpha[\ln(\beta m\cosh\alpha) - \tfrac{1}{2}\beta\mu_B] .
\end{aligned} \tag{36}
$$

Now we set $\mu_B = 0$ in Eq. (36) to get the thermodynamic potential density (the free-energy density in the present case) of the breather gas in equilibrium,

$$
\Omega_{eq} = T \int \frac{dk}{2\pi} \ln(\beta\omega_k), \quad \omega_k = (m^2 + k^2)^{1/2} , \tag{37}
$$

where $k = m\sinh\alpha$. This expression agrees with the free energy of a classical harmonic-phonon gas with the dispersion $\omega_k$. This phonon is nothing but a small amplitude limit of a breather given in Eq. (8).

The breather density $n_B$ is calculated from Eq. (17) with $\Omega$ given by Eq. (36), resulting in

$$
n_B = \tfrac{1}{2} \int_{-\pi/a}^{\pi/a} dk \frac{1}{2\pi} , \tag{38}
$$

where we cut off the integral at $\pm\pi/a$ ($a$ is the "lattice constant"). Since $(L/2\pi)\int dk$ can be interpreted as the total number of degrees of freedom of the system, Eq. (38) shows that the number of breathers $N_B = Ln_B$ is half the total number of degrees of freedom. This result is consistent with the fact that each breather has two degrees of freedom.

The distribution function $n_B(\alpha,\theta)$ of the breather in the phase space defined by

$$n_B=\int d\Gamma_B\, n_B(\alpha,\theta) \tag{39}$$

is found to be

$$n_B(\alpha,\theta)=\tfrac{1}{2}\beta m \cosh(\alpha)\coth[\beta(M\theta+\tfrac{1}{2}m)\cosh(\alpha)] \times \exp[-\beta\epsilon_B(\alpha,\theta\,|\,\mu_B=0)]\ , \tag{40}$$

where $\epsilon_B(\alpha,\theta\,|\,\mu_B=0)$ denotes $\epsilon_B(\alpha,\theta)$ given by Eq. (35) with $\mu_B=0$ ($\xi_B=1$). We note that $n_B(\alpha,\theta)$ is not simply given by $\exp(-\beta\epsilon_B)$.

## V. CONCLUDING REMARKS

We have studied thermodynamics of the Boltzmann gas consisting of solitons, antisolitons, and breathers of the classical sine-Gordon model, by taking their interactions into account. A set of equations (29)–(31) determine the thermodynamic potential. This agrees with the result of the Bethe-*ansatz* method[1,2,15] in the classical limit and with the extended ideal-gas phenomenology.[14] We have demonstrated how the breather gas yields the free energy of a classical phonon and found that the average number of breathers is half the total number of degrees of freedom if contributions from solitons and antisolitons can be neglected.

Recently, Maki[22] solved the coupled integral equations (30) and (31) analytically in the case of $\mu_s=\mu_{\bar{s}}=\mu_B=0$ within a "harmonic-phonon" approximation to give multisoliton contribution to the free energy. The same analysis can be performed with finite chemical potentials, details of which will be reported in a separate paper. The analytical evaluation of "anharmonic" corrections in this formulation (or the Bethe-*ansatz* formulation) remains to be done.

It is interesting to note that the formulas (29)–(31) can be applied to the one-dimensional gas of hard rods (see Appendix D), which yields the exact equation of state[23]

$$P=nT/(1-na)\ , \tag{41}$$

where $P$ is the pressure, $n$ is the density of particles, and $a$ is the length of the rod. The analogy between the soliton gas and the hard-rod gas was pointed out by Sasaki.[24]

## ACKNOWLEDGMENTS

The author would like to thank Professor K. Maki for enlightening discussions and comments and for a critical reading of the manuscript. He also thanks Dr. Xenophon Zotos for helpful discussions on the Bethe-*ansatz* thermodynamics and Professor H. Takayama for discussions on the work of Ref. 14 prior to publication. This work is supported in part by the National Science Foundation under Grant No. DMR-82-14525 and by Japan Society for the Promotion of Science.

## APPENDIX A: VOLUME ELEMENT OF THE PHASE SPACE

Integration over phase space in the classical statistical mechanics is defined as the quasiclassical limit of summation over quantum states. Therefore $d\Gamma(\{P\}\,|\,N,N_B)$ in Eq. (19) may be determined as follows.

For simplicity we consider a case of $N_B=0$. Then the quantum state is described by a set of momenta of solitons, $\{P_1,P_2,\ldots,P_N\}$. The allowed values of the momenta may be determined by the periodic boundary condition for the wave function of $N$-soliton state. There is no reflection wave in the soliton-soliton scattering for a special value of the coupling constant $g^2=8\pi/n$ ($n$ is an integer).[25] In this case the periodic boundary condition can be expressed in terms of the two-body phase shift[1,16,18,26,27] $\delta_{ss}(\alpha_{ij})$ as

$$P_iL+\sum_{j=1}^{N}{}'\,\delta_{ss}(\alpha_{ij})=2\pi n_i,\quad i=1,2,\ldots,N \tag{A1}$$

where $\alpha_{ij}=\alpha_i-\alpha_j$, $P_i=M\sinh\alpha_i$, and $n_i$ is an integer. The prime on $\sum$ denotes that the term $j=i$ is not involved. Here we do not need an explicit form of $\delta_{ss}(\alpha_{ij})$, but we note that $\delta_{ss}(\alpha_{ij})=-\delta_{ss}(\alpha_{ji})$ and that in the quasiclassical limit ($n\to\infty$) we have the relation[18]

$$-\partial\delta_{ss}(\alpha_{ij})/\partial P_i=\Delta_{ss}(\alpha_i;\alpha_j)\ , \tag{A2}$$

where $\Delta_{ss}$ is the position shift of soliton given by Eq. (12) in the text. In this limit the summation over quantum states determined by (A1) goes to the integration over momenta as

$$\sum_{n_1}\sum_{n_2}\cdots\sum_{n_N}\to\int dP_1\int dP_2\cdots\int dP_N\frac{\partial(n_1,n_2,\ldots,n_N)}{\partial(P_1,P_2,\ldots,P_N)}=L^N\int d\Gamma_{s1}\int d\Gamma_{s2}\cdots\int d\Gamma_{SN}R(\{P\}\,|\,N,0)\ , \tag{A3}$$

where $d\Gamma_s$ and $R$ are defined by Eqs. (22) and (24) in the text. We have used Eq. (A2) to derive the last equality.

The above discussion is easily extended to the case of $N_B\neq0$. The breather has two quantum numbers; the momentum $P_B$ and the internal variable $\theta$. The way of counting momentum states is the same as above. The quasiclassical quantum theory[12,18] shows that $\theta$ takes discrete values $\theta=jg^2/16$ ($j=1,2,\ldots<8\pi/g^2$; $g\to0$). Therefore the summation over the internal variable of breather is performed as

$$\sum_j\to\frac{16}{g^2}\int d\theta\ . \tag{A4}$$

We thus obtain the phase volume element [Eq. (21)] with Eqs. (22)–(26) in the text, which is appropriate for the classical statistical mechanics.

## APPENDIX B: DERIVATION OF EQS. (29)–(31)

We want to perform the summation in Eq. (27). For simplicity let us consider a system consisting of solitons and antisolitons only ($N_B=0$). Then we have

$$\Xi=\sum_{N=0}^{\infty}\Xi_N\ , \tag{B1}$$

where $\Xi_0=1$ and

$$\Xi_N=\frac{L^N}{N!}\int\prod_{i=1}^{N}d\gamma_i R(\{P\}\,|\,N,0),\quad N\geq 1 \tag{B2}$$

$$d\gamma_i=d\Gamma_{si}\,\xi\exp[-\beta E_s(\alpha_i)]\ . \tag{B3}$$

The function $R(\{P\}\,|\,N,0)$ is expressed in terms of the position shift of soliton $\Delta(i,j)\equiv\Delta_{ss}(\alpha_i;\alpha_j)$, Eqs. (24)–(26). If we represent $\Delta(i,j)$ by a line with an arrow running from a point $j$ to $i$, $\Xi_N$ may be represented diagrammatically as shown in Fig. 1, where the dot denotes the integration $\int d\gamma$. There appear "connected" and "disconnected" diagrams. The summation of Eq. (B1) can be carried out diagrammatically,[28] resulting in

$$\begin{aligned}\ln\Xi&=(\text{sum all over connected diagrams})\\ &=L\sum_{N=0}^{\infty}\int d\gamma\, Y_N(\alpha)\ ,\end{aligned} \tag{B4}$$

where we have introduced the functions $Y_N(\alpha)$ defined by $Y_0=1$,

$$\begin{aligned}Y_1(\alpha)&=-\int d\gamma_1\Delta(1,\alpha)\ ,\\ Y_2(\alpha)&=\tfrac{1}{2}\left[\int d\gamma_1\Delta(1,\alpha)\right]^2\\ &\quad+\int d\gamma_1\int d\gamma_2\Delta(1,2)\Delta(2,\alpha)\ ,\end{aligned} \tag{B5}$$

etc. The functions $Y_N(\alpha)$ may be represented diagrammatically as shown in Fig. 2, where the tails of lines without dots represent $\alpha$ dependence of $Y_N(\alpha)$.

The summation in Eq. (B4) can be carried out again in the same way as before. The result is expressed in terms of $Y_N$ as

$$\sum_{N=0}^{\infty}Y_N(\alpha)=\exp\left[-\int d\gamma'\Delta(\alpha',\alpha)\sum_{N=0}^{\infty}Y_N(\alpha')\right]\ . \tag{B6}$$

It is convenient to introduce a quantity $\tilde{\epsilon}(\alpha)$ defined by

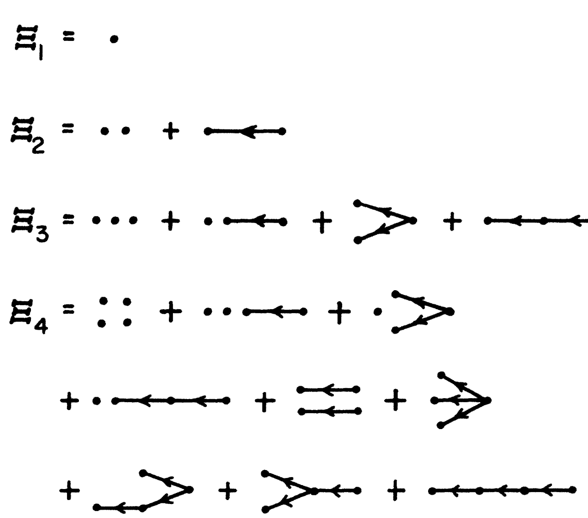


FIG. 1. Diagrammatic representation of $\Xi_N$.

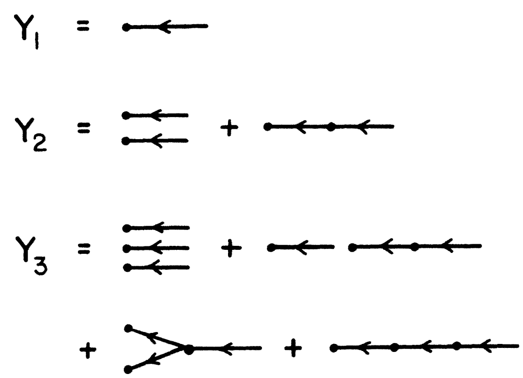


FIG. 2. Diagrammatic representation of $Y_N(\alpha)$.

$$\beta\tilde{\epsilon}(\alpha)=\int d\gamma'\Delta(\alpha',\alpha)\sum_{N=0}^{\infty}Y_N(\alpha')\ . \tag{B7}$$

By substituting Eq. (B6) into the right-hand side of (B7), $\tilde{\epsilon}(\alpha)$ is found to satisfy the integral equation

$$\beta\tilde{\epsilon}(\alpha)=\int d\gamma'\Delta(\alpha',\alpha)\exp[-\beta\tilde{\epsilon}(\alpha')]\ . \tag{B8}$$

From Eqs. (B4), (B6), and (B7) we obtain

$$\ln\Xi=L\int d\gamma\exp[-\beta\tilde{\epsilon}(\alpha)]\ . \tag{B9}$$

A set of equations (B8) and (B9) describes thermodynamics of a simple soliton gas.

The above analysis is easily extended to the soliton-breather gas, Eq. (27), and one obtains Eqs. (29)–(31), which are simple generalization of Eqs. (B8) and (B9).

## APPENDIX C: SOLVING EQ. (34)

It is convenient to introduce

$$\eta(x)=\xi_B^{-1}\exp[\beta\epsilon_B(\alpha,\theta)],\quad x=2M\theta/m \tag{C1}$$

and rewrite Eq. (34) as

$$\begin{aligned}\ln\eta(x)&=\beta mx\cosh\alpha-\beta\mu_B\\ &\quad+2\int_0^x dy\, y\eta^{-1}(y)+2x\int_x^{\pi M/m}dy\,\eta^{-1}(y)\ .\end{aligned} \tag{C2}$$

Differentiating Eq. (C2) with respect to $x$, we have

$$\eta'/\eta=\beta m\cosh\alpha+2\int_x^{\pi M/m}dy\,\eta^{-1}(y)\ , \tag{C3}$$

where the prime denotes derivative with respect to $x$. Differentiation of (C3) yields

$$(\eta'/\eta)'=-2/\eta\ . \tag{C4}$$

A general solution to the second-order differential equation (C4) is found to be

$$\eta(x)=A^{-2}\sinh^2[A(x+B)]\ . \tag{C5}$$

The integration constants $A$ and $B$ are determined so that Eq. (C5) satisfies the original integral equation (C2). Substituting Eq. (C5) into (C3) and noting that we are concerned with the weak-coupling limit $(M/m\to\infty)$, we have

$$A=\tfrac{1}{2}\beta m\cosh\alpha\ . \tag{C6}$$

Substitution of Eq. (C5) into Eq. (C2) yields

$$\ln[A^{-1}\sinh(AB)]=-\tfrac{1}{2}\beta\mu_B\ , \tag{C7}$$

from which we have

$$B=\exp(-\tfrac{1}{2}\beta\mu_B)=\xi_B^{-1/2} \tag{C8}$$

for $\beta m \ll 1$. From Eqs. (C1), (C5), (C6), and (C8) we get Eq. (35), the solution of Eq. (34).

## APPENDIX D: ONE-DIMENSIONAL GAS OF HARD RODS

Let us consider a classical gas consisting of $N$ hard rods of length $a$ in a one-dimensional box of size $L$. The position shift in two-body collision is simply given by $a$. Therefore a set of equations (29)–(31) in the text can be applied to this system in the following form:

$$-\beta\Omega=e^{\beta\mu}\int\frac{dp}{2\pi}e^{-\beta\epsilon(p)}\ , \tag{D1}$$

$$\beta\epsilon(p)=\beta E(p)+e^{\beta\mu}\int\frac{dp'}{2\pi}a\,e^{-\beta\epsilon(p')}\ , \tag{D2}$$

where $E(p)=p^2/2m$ ($m$ is the mass) is the kinetic energy of the rod. From these equations we readily obtain

$$-\beta\Omega=\beta P=b(\beta,\mu)\ , \tag{D3}$$

where $P$ is the pressure of the gas and the function $b(\beta,\mu)$ is defined by

$$b(\beta,\mu)=(\beta/a)[\epsilon(p)-E(p)]\ . \tag{D4}$$

From this definition and Eq. (D2), we have

$$be^{ab}=(m/2\pi\beta)^{1/2}e^{\beta\mu}\ , \tag{D5}$$

which gives $b$ as a function of $\beta$ and $\mu$.

The density of rods $n=N/L$ is given by

$$n=-\left[\frac{\partial\Omega}{\partial\mu}\right]_\beta=\frac{1}{\beta}\left[\frac{\partial b}{\partial\mu}\right]_\beta=\frac{b}{1+ab}\ , \tag{D6}$$

where we have used Eq. (D5) to derive the last equality. Eliminating $b$ from Eqs. (D3) and (D6), we have the equation of state

$$P=nT/(1-na)\ , \tag{D7}$$

which agrees with the exact result by Tonks.[23]


[1] M. Fowler and X. Zotos, Phys. Rev. B **24**, 2634 (1981); **25**, 5806 (1982).

[2] M. Imada, K. Hida, and M. Ishikawa, Phys. Lett. **90A**, 79 (1982); J. Phys. C **16**, 35 (1983).

[3] A. Luther, Phys. Rev. B **14**, 2153 (1976).

[4] M. Takahashi and M. Suzuki, Prog. Theor. Phys. **48**, 2187 (1972).

[5] J. A. Krumhansl and J. R. Schrieffer, Phys. Rev. B **11**, 3535 (1975).

[6] J. F. Currie, J. A. Krumhansl, A. R. Bishop, and S. E. Trullinger, Phys. Rev. B **22**, 477 (1980).

[7] K. Maki and H. Takayama, Phys. Rev. B **22**, 3223 (1979).

[8] N. Gupta and B. Sutherland, Phys. Rev. A **14**, 1790 (1976).

[9] T. Schneider and E. Stoll, Phys. Rev. B **22**, 5317 (1980).

[10] For recent development of the path-integral approach, see T. Miyashita and K. Maki, Phys. Rev. B **28**, 6733 (1983); **31**, 1836 (1985).

[11] Formally our approach is similar to the *S*-matrix formulation of statistical mechanics by R. Dashen, S.-k. Ma, and H. J. Bernstein, Phys. Rev. **187**, 345 (1969).

[12] R. F. Dashen, B. Hasslacher, and A. Neveau, Phys. Rev. D **15**, 3424 (1975).

[13] N. Theodorakopoulos, Z. Phys. B **46**, 367 (1982); Phys. Rev. B **30**, 4071 (1984).

[14] H. Takayama and M. Ishikawa, Prog. Theor. Phys. **74**, 479 (1985).

[15] K. Hida, M. Imada, and M. Ishikawa, J. Phys. C **16**, 4945 (1983).

[16] M. Fowler, Phys. Rev. B **26**, 2514 (1982).

[17] L. A. Takhtadzhyan and L. D. Faddeev, Teor. Mat. Fiz. **21**, 160 (1974) [Theor. Math. Phys. **21**, 1046 (1974)].

[18] V. E. Korepin and L. D. Faddeev, Teor. Mat. Fiz. **25**, 147 (1975) [Theor. Math. Phys. **25**, 1039 (1975)].

[19] L. A. Takhtadzhyan, Zh. Eksp. Teor. Fiz. **66**, 476 (1974) [Sov. Phys.—JETP **39**, 228 (1974)].

[20] J. F. Currie, M. B. Fogel, and F. L. Palmer, Phys. Rev. A **16**, 796 (1977).

[21] L. D. Landau and E. M. Lifshitz, *Statistical Physics,* 3rd ed. (Pergamon, Oxford, 1980), Secs. 103 and 107.

[22] K. Maki, Phys. Rev. B **32**, 3075 (1985).

[23] L. Tonks, Phys. Rev. **50**, 955 (1936).

[24] K. Sasaki, Prog. Theor. Phys. **70**, 593 (1983); **71**, 1169 (1984).

[25] V. E. Korepin, Teor. Mat. Fiz. **34**, 3 (1978) [Theor. Math. Phys. **34**, 1 (1978)].

[26] R. Jackiw and G. Woo, Phys. Rev. D **12**, 1643 (1975).

[27] V. E. Korepin, Teor. Mat. Fiz. **41**, 169 (1979) [Theor. Math. Phys. **41**, 953 (1979)].

[28] A. A. Abrikosov, L. P. Gor'kov, and I. Ya. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics,* 2nd ed. (Pergamon, Oxford, 1965), Sec. 15.