PHYSICAL REVIEW B VOLUME 33, NUMBER 3 1 FEBRUARY 1986

Exponents for $1/f$ noise near a continuum percolation threshold

A.-M. S. Tremblay
Département de Physique et Centre de Recherche en Physique du Solide, Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1

Shechao Feng
Department of Physics, Harvard University, Cambridge, Massachusetts 02138 and Schlumberger-Doll Research, Old Quarry Road, Ridgefield, Connecticut 06877-4108

P. Breton
Département de Physique et Centre de Recherche en Physique du Solide, Université de Sherbrooke, Sherbrooke, Québec, Canada J1K 2R1

(Received 9 September 1985)

New bounds for the exponent characterizing the amplitude of the resistance noise near the percolation threshold of discrete random networks are found. The difference between the lower and upper bounds is very small, so that an accurate estimate of the noise exponent can be obtained in all dimensions. Continuum corrections to these exponents for the random-void class of systems are then calculated within the nodes-links-blobs model of percolating networks.

It has recently been shown[1,2] that resistance noise[3] should diverge near the percolation threshold with a new characteristic exponent which cannot be related to the previously known percolation exponents. This new exponent is a member of a hierarchy which can be naturally defined in terms of the moments of the current distribution for finite-size samples with a unit injected current. Members of that hierarchy include the fractal dimension of the percolation backbone $\bar{d}_B$, the conductivity exponent t, the resistance noise exponent b, and the inverse of the correlation length exponent ν. In that context, it is important to verify experimentally the value of the noise exponent b.

Experiments on the resistance noise near percolation have been performed[4-9] by many different groups. The value of the exponent b which was obtained from a standard discrete percolation network model cannot, however, be extracted directly from any of these experiments because the systems used are *continuum* percolation systems, and it has been pointed out[10-13] that percolation transport exponents for a continuum system and for a discrete lattice model can differ significantly. This is the case also for resistance noise, but the continuum corrections in this case must be calculated in a slightly different way. We should also stress that for continuum percolation it is difficult to estimate $p-p_c$ experimentally; hence measurements of the noise magnitude as a function of resistance, two quantities which in principle depend on $p-p_c$, allow one to eliminate that unknown, giving at the same time a unique opportunity to check theories of continuum percolation. Previous experimental checks[14] have been limited to the case where the continuum corrections vanish.

The purpose of this paper is twofold. First, we derive a new upper bound for b within the framework of the "nodes-links-blobs" (NLB) picture of percolating networks, which, in conjunction with the known lower bound, gives a rather narrow range of possible values for the noise exponent in all dimensions. As a by-product, we find a new inequality between percolation-backbone exponents. Second, we investigate the corrections to the noise exponent within the frameworks of both the effective-medium approximation and the nodes-links-blobs scaling analysis. Results of numerical simulations are also presented.

Let us recall that[1,15] the relative resistance noise S_R for any network at frequency ω is given by $S_R=\langle\delta R^2\rangle_\omega/R^2$, where

$$\langle\delta R^2\rangle_\omega=\sum_\alpha i_\alpha^4\langle\delta r_\alpha^2\rangle_\omega/I^4 \tag{1}$$

and

$$R=\left(\sum_\alpha i_\alpha^2 r_\alpha\right)\Big/ I^2 \ , \tag{2}$$

where i_α, r_α, and $\langle\delta r_\alpha^2\rangle$ are, respectively, the average current, resistance, and resistance-fluctuation correlation function in branch α of the network, and I is the total injected current. It is assumed that resistance fluctuations in different branches are uncorrelated. We first consider the standard *discrete* network model, i.e., a model in which all bonds occupied with probability p have *identical* branch resistances and resistance fluctuations (noise). The noise of the entire network thus obeys the following inequality (from now on all quantities are understood to be measured at a given fixed frequency):

$$S_R=\frac{\langle\delta r^2\rangle}{R^2}\sum_\alpha i_\alpha^4\geq\frac{\langle\delta r^2\rangle}{R^2}\sum_{\alpha'}1 \ , \tag{3}$$

where i_α is measured with input current $I=1$ and the sum over α' runs only over the branches that carry a unit current. For a sample of size L in the fractal regime $L<<\xi$, where $\xi\sim(p-p_c)^\nu$ is the percolation correlation length, the average resistance depends on size L like $L^{-\beta_L}$, with an exponent $-\beta_L$ related to the usual conductivity exponent by $-\beta_L=t/\nu+(2-d)$, where d is the Euclidean dimension. Furthermore, the branches that carry a unit current are the singly connected bonds whose number[16] L_1 scales as $L^{1/\nu}$. This implies the following inequality:

$$S_R(L<<\xi)\sim L^{-b}\geq L^{-2\beta_L}$$

or $b\leq-2\beta_L-1/\nu$. This inequality can be seen as a consequence of the general inequality $x_n \geqslant x_{n-1}$ derived in Ref. 1 for exponents x_n which are members of the hierarchy. However, the above derivation clarifies the physics of this inequality. Using the known lower bound,[1] we obtain

$$-\beta_L \leqslant b \leqslant -2\beta_L - 1/\nu \quad . \tag{4}$$

Numerical values for the bounds are listed in Table I. Clearly these bounds determine the value of b with a reasonable accuracy. The close agreement between the Monte Carlo result[2,17] $b = 1.16 \pm 0.02$ in two dimensions and the upper bound suggests that indeed most of the noise comes from the singly connected bonds. Wright, Bergman, and Kantor[18] have independently come to a similar conclusion. Detailed histograms of the current distribution[19] also confirm the dominance of the singly connected bonds in producing the noise. In dimensions above six where mean-field theory holds, the upper bound in Eq. (4) becomes an equality. In passing, we note that in the scaling regime $L >> \xi$, the noise is described[1] by $S_R \sim (p - p_c)^{-\kappa}$, where $\kappa = \nu(d - b)$.

We have used the last column of Table I to list the fractal dimension of the backbone, which was derived earlier[1] as an upper bound for b. The numerical values clearly suggest that the following new inequality is satisfied:

$$-2\beta_L - 1/\nu \leqslant \bar{d}_B \quad . \tag{5}$$

We have not found a proof of Eq. (5).

Finally, we calculate the percolative noise exponent for the random-void class of *continuum* percolation models. It was shown recently[10] that percolation exponents for transport properties of this class of models can differ considerably from the corresponding ones for the standard discrete-lattice models. This is a consequence of the fact that the conducting bonds in the continuum models can have a singular power-law distribution. It is known[11,12,20] that such a distribution gives rise to modifications of transport exponents. The mapping of the random-void model to a random resistor network was described by Elam, Kerstein, and Rehr.[21] Conducting resistances are characterized by a channel width ϵ. It is assumed that the probability distribution for having a channel of width ϵ is continuous and takes on a finite limit at $\epsilon = 0$.

The relative noise in a single bond α can be computed by assuming that the material in a conducting channel is homogeneous and that the resistivity fluctuations are uncorrelated in space. Noting that it is a good approximation to assume flat equipotentials in the neck region, we find that the relative noise of a bond behaves as ϵ^{-v} and the resistance as ϵ^{-u} with the values of v and u listed in Table II, for both random-void and inverted random-void models.[10]

Halperin, Feng, and Sen[10] have found continuum corrections to the transport exponents by applying the NLB picture[16,22] of percolation clusters. The use of "typical" transport quantities for a "link" in their work is based on the possibility of finding an upper bound to the overall resistance by removing all the links that have atypically high resistances. No such variational principle exists for the noise problem.[23] This means in particular that when the average noise of a link diverges, we cannot in general argue, as for the resistance,[10] that it suffices to compute the typical value of the noise of a link instead of the average. Since the contribution of a link to the total noise depends on the current which flows through the link and since that current is determined by the resistance, it is the interplay between the relative noise and the resistance which determines whether a link contributes appreciably or not to the overall noise.

This has profound implications for simulations. While continuum corrections for the resistance could be obtained by computing the average conductivity within a correlation length[13] (which is finite despite the fact that the average resistance diverges), continuum corrections to the noise must be obtained from self-averaging samples, i.e., samples large enough to contain many correlation lengths. Coupled with the requirement that every correlation length contains many singly connected bonds, this means that it is almost impossible to compute reliable exponents with the standard simulation methods of percolation theory which almost always use finite-size scaling to compute exponents. We circumvent this problem by directly simulating the NLB model with only singly connected bonds. This is already a good approximation for the lattice model discussed earlier, and is even better for continuum systems, where the noise is even more dominated by singly connected bonds within a correlation length.

Using the methods described earlier,[2,24] we thus numerically computed the noise on $n \times n$ square lattices whose individual bonds are taken as the "links" of the NLB model.

TABLE I. Upper and lower bounds to the resistance noise exponents for the standard lattice percolation model. The numerical values with no superscript are extracted from various references given in Table I of A. B. Harris, Phys. Rev. B **28**, 2614 (1983).

d	ν	$-\beta_L = t/\nu + (2-d)$	$-2\beta_L - 1/\nu$	$\bar{d}_B$
2	$\frac{4}{3}$	0.97 ± 0.01[a]	1.19 ± 0.01	1.62 ± 0.02[b]
3	0.88 ± 0.01	1.2 ± 0.1[c]	1.26 ± 0.1	1.74 ± 0.04
4	0.66 ± 0.03	1.59 ± 0.07	1.66 ± 0.07	2.33 ± 0.1
5	0.57 ± 0.02	1.79 ± 0.04	1.82 ± 0.04	?
6	$\frac{1}{2}$	2	2	2

[a]J. G. Zabolitzky, Phys. Rev. B **30**, 4077 (1984); H. J. Herrmann, B. Derrida, and J. Vannimenus, *ibid.* **30**, 4080 (1984); D. C. Hong, S. Havlin, H. J. Herrmann, and H. E. Stanley, *ibid.* **30**, 4083 (1984); R. Rammal, J. C. A. d'Auriac, and A. Benoit, *ibid.* **30**, 4087 (1984); C. J. Lobb and D. J. Frank, *ibid.* **30**, 4090 (1984).

[b]H. J. Hermann and H. E. Stanley, Phys. Rev. Lett. **59**, 1121 (1984).

[c]B. Derrida, D. Stauffer, H. J. Herrmann, and J. Vannimenus, J. Phys. (Paris) Lett. **44**, L701 (1984).

TABLE II. If ϵ is the neck width parameter, the conductance of a bond scales as ϵ^u, while its relative noise scales as ϵ^{-v}. The overall relative noise of the sample S_R scales as R^w. Uncertainty on w is at least as large as that for κ/t, which can be deduced from the data in Table I. Additional uncertainties arise when one extrapolates from the NLB model to real percolation clusters.

Model	u	v	$w = \bar{\kappa}/\bar{t}$
2D random-void	$\frac{1}{2}$	$\frac{3}{2}$	3.2
2D inverted random-void	0	1	0.87
3D random-void	$\frac{3}{2}$	$\frac{5}{2}$	2.1
3D inverted random-void	$\frac{1}{2}$	$\frac{3}{2}$	2.4

In other words, every bond (link) is occupied, but its resistance R_l and its contribution to the resistance fluctuations $\langle R_l^2\rangle$ are determined, in dimensionless units, from, respectively,

$$R_l=\sum_{j=1}^{L_1}\frac{1}{\epsilon_j^u}\ , \tag{6a}$$

$$\langle\delta R_l^2\rangle=\sum_{j=1}^{L_1}\frac{1}{\epsilon_j^{v+2u}}\ , \tag{6b}$$

where ϵ_j are random numbers taken from a uniform distribution on the interval (0,1). From Eq. (1), the overall noise is given then by (assuming $I=1$)

$$\langle\delta R^2\rangle=\sum_l\langle\delta R_l^2\rangle i_l^4\ , \tag{7}$$

where i_l is the current in link l. Calculations are done for various values of v and u. For each of these values, we consider various L_1 and various lattice sizes $n\times n$. We have checked that for $n=32$, averaging over a few samples, typically 6 to 15, gives the same result as that calculated from a sample large enough that it is completely self-averaging.[25] Most of the following results are for $n=32$. Our results reproduce the NLB prediction for the resistance. $R_l\approx L_1^x$ with $x=1$ for $u<1$ and $x=u$ for $u>1$. This is the first numerical check that the current indeed distributes itself in such a way that it is correct to use the typical values for the resistance. Let us define the continuum corrected noise exponent $\bar\kappa$ as in $S_R(L>>\xi)\sim(p-p_c)^{-\bar\kappa}$. To facilitate comparisons with experiments, we can eliminate the variable $(p-p_c)$ in terms of the sample resistance $R\sim(p-p_c)^{-\bar t}$; i.e., we write $S_R\sim R^w$ for the scaling regime $L>>\xi$, with $w=\bar\kappa/\bar t$. (Note that Q in Ref. 6 equals $w+2$.)

The simplest analytical approach to our problem which seems to take into account the important interplay between noise and resistance is the effective-medium approximation[26] (EMA). However, the results obtained from the EMA for the continuum noise problem are not in agreement with those of computer simulations. Since the EMA calculation is closely related to that of Ref. 11, we quote the results here without any further details.[27] For $u<1$, and $v<1$, the discrete network result $w=1$ is recovered. For $u>1$ and $v<1$, $w=1/u$. For $1<v<1+2u$, $w=1+(v-1)/u$ when $u<1$ and $w=v/u$ when $u>1$.

We now discuss the predictions of the NLB model by generalizing the arguments of Halperin, Feng, and Sen.[10] The essence of the analysis was first given by Halperin[28] and was contained in Ref. 6. The numerical results agree surprisingly well for all three cases identified theoretically below. The essence of the analysis is to look for the extra dependence of the overall resistance noise on L_1 (or $p-p_c$) due to the continuous distribution of the bond resistances and resistance fluctuations. We examine in turn the three regimes of interest in the u-v parameter space.

(a) *$u<1$ and $v+2u<1$.* Here the averages of the sums in Eqs. (6) converge, so the average resistance and resistance noise of a link exist and the exponents are those of the standard discrete network, namely,[2] $\kappa=1.12\pm0.02$ and (Table I) $t=1.29\pm0.01$ in $d=2$ and $\kappa=1.56\pm0.1$ [calculated from Table I and Eq. (4)] and $t=1.94\pm0.1$ (Table I) in $d=3$.

(b) *$u<1$ and $1-2u<v<1+2u$.* The average resistance of a link [Eq. (6a)] is well defined, but the average of the sum appearing in Eq. (6b) for the resistance noise diverges. However, in summing over noises of all links, one must take into account the fact that the currents differ in each link. The current in a link may be calculated by assuming that the potential drop in every link is about the same, this being a consequence of the fact that the link network is well connected, as suggested by Halperin.[28] (Hence even when an atypically resistive link is connected to a node, there are other ones present which allow the potential drop to adjust to the average.) Let V_d be the average potential drop in a link. Then the current in a link is $V_d/\sum_j\epsilon_j^{-u}$. We consider two types of links: First, n links are such that none of the resistances forming the link is larger than the average link resistance $R_l\approx L_1$. This means that the minimum value of ϵ for any of the resistances forming the n links is $\epsilon_m\approx L_1^{-1/u}$. n links thus occur with probability $P_>=1-L_1\epsilon_m$. From Eq. (7), the average total resistance noise from the n links is thus

$$\langle\delta R^2\rangle_>=P_>L_1\int_{\epsilon_m}^1\frac{d\epsilon}{\epsilon^{v+2u}}\left(\frac{V_d}{L_1}\right)^4\sim\langle\delta R^2\rangle_{\text{lattice}}L_1^{(v+2u-1)/u}\ . \tag{8}$$

The second type of links, the a links, is such that one of the ϵ's (ϵ_a) in the link is smaller than ϵ_m. The current in such

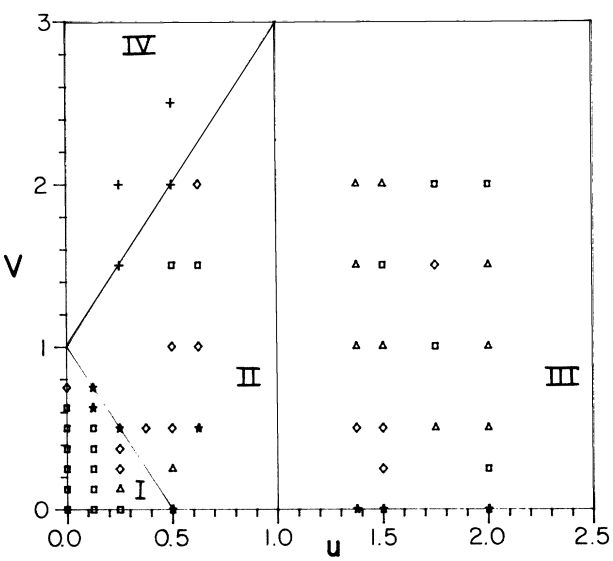

FIG. 1. Comparison between the numerically simulated NLB model noise exponent and the scaling results. For each point on the figure, the continuum correction to the noise exponent $\bar\kappa-\kappa$ was calculated by least-square fits on double logarithmic plots, L_1 taking the values 20, 40, 60, 80, and 100 at fixed n. The n^{-2} dependence of the noise was verified. Let N be the number of samples used for each L_1 and n. Let σ be the percentage disagreement between the average measured $\bar\kappa-\kappa$ and the predictions in Eqs. (10) and (11). For $n=50$ we found for the case $(u,v)=(0.5,1.5)$ and $N=3$, $\sigma<6\%$; for the cases $(u,v)=(1.5,1.0)$ and (0.25, 0.25) and $N=1$, $\sigma<1\%$. For $n=32$ we took $N=15$ in region (II) and $N=6$ in region (III). There, $\sigma<1\%$ for □, $\sigma<3\%$ for Δ, $\sigma<6\%$ for ◇ and $\sigma<10\%$ for $*$. In region (I), $N=6$ and the absolute error s (since $\bar\kappa=\kappa$ in this regime) is $s<0.01$ for □, $s<0.03$ for Δ, $s<0.1$ for ◇, and $s<0.3$ for $*$. In region (IV), $N=20$, and at points + there is no convergence.

a link is thus $i_l \approx V_d/(L_1+\epsilon_a^{-u}) \approx V_d\epsilon_a^u$. The contribution of an a link to the overall noise is dominated by that single resistance and takes the value $\langle \delta R^2 \rangle i_l^4 \approx V_d^4 \epsilon_a^{-v+2u}$. Since $L_1\, d\epsilon_a$ is the probability of having an a link whose width ϵ_a is between 0 and ϵ_m, we can integrate over ϵ_a to obtain the contribution to the resistance noise from all the a links

$$\langle \delta R^2 \rangle_< = L_1 \int_0^{\epsilon_m} \frac{d\epsilon_a}{\epsilon_a^{v+2u}} V_d^4 \epsilon_a^{4u} \sim \langle \delta R^2 \rangle_{\text{lattice}} L_1^{(v+2u-1)/u} \tag{9}$$

Note that both $\langle \delta R^2 \rangle_<$ and $\langle \delta R^2 \rangle_>$ scale the same way. Given that there are no continuum corrections to the overall resistance exponent, we get for this case

$$\bar{\kappa} - \kappa = \frac{v+2u-1}{u} \quad . \tag{10}$$

This result is well verified numerically (see Fig. 1). Note that when $v > 1+2u$, the integral in Eq. (9) diverges. This is in agreement with the numerical results shown in region (iv) of Fig. 1, where the calculated noise magnitude changes wildly from sample to sample without a well-defined average.

(c) *$u > 1$ and $v+2u > 1$.* This case can be analyzed in a similar way to case (b). Since the link resistance here scales as $R \approx L_1^u$ (this implies a continuum correction to the resistance exponent[10] $u-1$); division between n links and a links is given by $\epsilon_m \approx L_1^{-1}$. Thus the noise exponent has a continuum correction,

$$\bar{\kappa} - \kappa = v+1 \quad , \tag{11}$$

a result which is also well verified by the numerical calculations presented in Fig. 1.

To conclude, we have computed continuum corrections to the exponent describing the divergence of the magnitude of resistance noise close to the percolation threshold. The special cases of the random-void and inverted random-void models in $d=2$ and 3 are summarized in Table II. Full simulations beyond the NLB model are still needed, but the difference with the results obtained here should be smaller than the accuracy of presently available experiments. The special case of the two-dimensional random-void model, ($v=\frac{3}{2}, u=\frac{1}{2}$) has been used by Garfunkel and Weissman[6] in explaining their experimental results.

We would like especially to thank B. I. Halperin and M. B. Weissman for very useful discussions. We are also indebted to R. Koch, J. Garfunkel, R. Laibowitz, and J. Mantese for discussing and explaining their experimental results to us prior to publication. We also thank R. Rammal and P. N. Sen for useful conversations. A.-M.S.T. acknowledges the Natural Sciences and Engineering Research Council of Canada (NSERC) for support through an individual operating grant and through its program "Attache de recherche." S.F. acknowledges partial support by the NSF through the Harvard Materials Science Research Laboratory and Grant No. DMR-82-07431. Some computing time was provided by NSERC and the Atmospheric Environment Service (Dorval).

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