Analysis of extended series for bond percolation on the directed square lattice

J. W. Essam
Department of Mathematics, Royal Holloway College and Bedford College, University of London, Egham Hill, Egham, Surrey TW2 0EX, United Kingdom

K. De'Bell
Department of Physics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

J. Adler
Department of Physics and Astronomy, Tel Aviv University, 69978 Ramat Aviv, Israel

F. M. Bhatti*
Department of Mathematics, Royal Holloway College and Bedford College, University of London, Egham Hill, Egham, Surrey TW2 0EX, United Kingdom

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Series expansions for the mean size and moments of the pair connectedness for bond percolation on the directed square lattice have been extended to order p^{35}. Standard Padé analysis of the mean-size series leads to the critical probability $p_c = 0.644\,701 \pm 0.000\,002$. Allowance for corrections to scaling gives a leading correction exponent $\Delta_1 = 1.000 \pm 0.012$ and an increase of the error bar on p_c to 0.000 012. The increased accuracy in the determination of p_c allows a corresponding improvement in the estimates of the exponents γ, $\nu_\perp$, and $\nu_\parallel$.

I. INTRODUCTION

The accurate determination of the critical exponents for directed percolation is of particular interest since the universality class of this model includes a number of other models representing a diverse set of physical systems.[1,2] Here we report an analysis of the first 35 terms of the moments of the pair connectedness $C_i(p)$:

$$S = \mu_{0,0} = \sum_{\substack{i \\ \text{sites}}} C_i(p), \quad S_0 = \sum_{\substack{i \\ (\mathbf{x}_i = 0)}} C_i(p) , \tag{1}$$

$$\mu_2^{(x)} = \mu_{2,0} = \sum_i x_i^2 C_i(p) \quad \text{and} \quad \mu_2^{(t)} = \mu_{0,2} = \sum_i t_i^2 C_i(p) \tag{2}$$

for bond percolation on the directed square lattice (all parallel bonds directed in the same sense) (Table I). These series were obtained by supplementing the transfer-matrix method of Blease[3] with a weak subgraph expansion as described previously.[4] S and S_0 in (1) may be identified with the mean cluster size and mean diagonal cluster size, respectively, and in (2), $\mathbf{x}_i$ and $\mathbf{t}_i$ denote the position vectors of site i perpendicular to and parallel to the preferred (1,1) direction of fluid flow, respectively.

This extension of the known number of coefficients for these series has permitted a considerable reduction of the error in the estimate of the critical probability p_c for this problem, which in turn has lead to improved estimates of the leading critical exponents γ, ν_0, $\nu_\perp$, and $\nu_\parallel$ and correction to scaling exponent Δ_1.

II. ANALYSIS

A. Padé-approximant analysis

Our initial analysis consisted of forming Padé approximants to the derivatives of the logarithms of S, S/S_0, $\mu_2^{(x)}/S$, and $\mu_2^{(t)}/S$ and identifying the values of the residues at p_c on pole-residue plots as γ, ν_0, $2\nu_\perp$, and $2\nu_\parallel$, respectively, (Table II). The value of p_c used was determined by inspection of the Padé table for the series S and the Euler transform [in terms of $z = p/(1+p)$] of that series (Table III). (Previous analysis based on fewer terms had found the Euler transform to give better convergence.[3,4]) Both tables appear very well converged and consistent with our estimate of

$$p_c = 0.644\,701 \pm 0.000\,002 .$$

This represents an adjustment of the central estimate and considerable reduction in the error bounds when compared with earlier estimates.[4] The corresponding estimates of γ, ν_0, $\nu_\perp$, and $\nu_\parallel$ are shown in Table II.

Scaling arguments require that[4]

$$\nu_0 = \nu_\perp \tag{3}$$

for two-dimensional lattices. Our results are just consistent with Eq. (3) at the central estimate of p_c; however, points on the pole-residue plot for S/S_0 from higher-order Padé approximants tend to fall to one side of the central value of p_c and, therefore, our quoted value of ν_0 was obtained by linear extrapolation through p_c and the error bounds on ν_0 only represent reasonable variations in

TABLE I. Coefficients of p^m in the moments [Eqs. (1) and (2)] of the pair connectedness for bond percolation on the directed square lattice.

m	S	S_0	$\mu_2^{(x)}$	$\mu_2^{(t)}$
0	1	1	0	0
1	2	0	2	2
2	4	2	8	16
3	8	0	24	72
4	15	5	64	252
5	28	0	156	764
6	50	14	358	2 094
7	90	−4	786	5 362
8	156	42	1 664	12 968
9	274	−20	3 434	30 138
10	466	126	6 902	67 446
11	804	−100	13 656	147 048
12	1 348	400	26 464	311 940
13	2 300	−376	50 772	649 860
14	3 804	1 248	95 754	1 325 234
15	6 450	−1 556	179 442	2 668 130
16	10 547	4 231	331 294	5 278 066
17	17 784	−5 588	609 496	10 346 200
18	28 826	13 880	1 106 106	19 977 010
19	48 464	−21 912	2 004 852	38 329 556
20	77 689	48 985	3 586 874	72 546 986
21	130 868	−76 404	6 423 028	136 785 444
22	207 308	165 712	11 351 274	254 596 418
23	350 014	−295 660	20 126 538	473 093 498
24	548 271	602 237	35 191 190	868 060 738
25	931 584	−1 017 452	61 883 196	1 593 517 724
26	1 433 966	2 072 268	107 179 834	2 887 257 826
27	2 469 368	−3 935 956	187 216 848	5 246 647 808
28	3 725 257	7 665 833	321 395 596	9 400 175 212
29	6 510 384	−13 411 588	558 468 104	16 935 336 776
30	9 590 838	26 634 782	950 702 594	30 035 008 322
31	17 192 714	−52 362 292	1 645 491 278	53 731 142 846
32	24 357 702	99 567 378	2 778 049 248	94 373 684 636
33	45 428 434	−176 237 580	4 796 424 622	167 898 005 054
34	61 388 268	348 090 340	8 028 750 772	292 175 943 812
35	119 938 514	−699 582 108	13 848 760 938	517 568 220 986

this linear extrapolation. Since the pole-residue points for S/S_0 are still scattered about the straight line drawn, we must conclude that, despite the extension of the known number of terms in the series, the Padé approximants for this series are still not well converged and the estimate of $\nu_\perp$ must be regarded as more reliable than that of ν_0.[5]

B. Correction to scaling analysis

Recently several authors have demonstrated the importance of nonanalytic correction to scaling terms in the analysis of series expansions.[6] For example, in analyzing a moment of the pair connectedness we must allow for a function of the form

$$\mu_{l,m}(p)=\sum_i \langle \mathbf{x}_i^2\rangle^{l/2} t^m C_i(p)$$

$$\sim (p_c-p)^{-\gamma-m\nu_\parallel-l\nu_\perp}[1+a_1(p_c-p)^{\Delta_1}+b(p_c-p)+a_2(p_c-p)^{\Delta_2}+\cdots] \quad (l \text{ even}) . \tag{4}$$

Therefore, we have analyzed the series of Table I with the methods of Adler *et al.*[6,7] The former method involves minimizing the effect of the correction, due to the first nonanalytic term, on the evaluation of the dominant exponent and is a generalization of the transform of Roskies;[8] whereas the latter method gives us a corroborating estimate of Δ_1.

In the former method the series $\mu_{l,m}(p)$ in p is transformed to a series in

$$y=1-(1-p/p_c)^{\Delta} , \tag{5}$$

and different Padé approximants to the function

TABLE II. Estimates of the critical probability p_c and dominant exponents.

	Standard Padé analysis	Correction to scaling analysis
p_c	$0.644\,701 \pm 0.000\,002$	$0.644\,701 \pm 0.000\,012$
γ	$2.277\,21 \pm 0.000\,01 + 90\,\Delta p_c$	$2.277\,21 \pm 0.000\,3 + 156\,\Delta p_c$
ν_0	$1.094 \pm 0.002 + 33\,\Delta p_c$	
$\nu_\perp$	$1.097 \pm 0.001 + 64\,\Delta p_c$	$1.097\,2 \pm 0.000\,6 + 60\,\Delta p_c$
$\nu_\parallel$	$1.733\,2(5) \pm 0.000\,1 + 68\,\Delta p_c$	$1.733\,4 \pm 0.001 + 70\,\Delta p_c$

$$G_\Delta(y) = \Delta(y-1)\frac{d}{dy}[\ln \mu_{l,m}(y)] = h - K/(1+K) \quad (6)$$

evaluated at $y=1$ are inspected for convergence. Here, $h = \gamma + m\nu_\parallel + l\nu_\perp$ and

$$K = a_1 p_c^{\Delta_1} \Delta_1 (y-1)^{\Delta_1/\Delta} , \quad (7)$$

and we expect the Padé approximants to be best converged for the correct value of p_c and $\Delta = \Delta_1$, in which case K becomes analytic and an accurate estimate of the dominant exponent is obtained. For a model where neither p_c nor Δ_1 is known, one searches for the best convergence in the (p_c, h, Δ) space,[6] since Δ is the variable parameter, not Δ_1. Here we use the value of p_c from the D ln Padé analysis as an initial guess, since if Δ_1 is close or equal to unity, this should be accurate. We note that a very strong check of the validity of our results is if the same value of Δ_1 and p_c is obtained for all quantities studied.

The second method used is believed to be most reliable for Δ_1 close to 1.0.[7] It involves studying Padé approximants to the logarithmic derivative of $B(p)$, where

$$B(p) = h\mu_{l,m}(p) + (p_c - p)\frac{d\mu_{l,m}(p)}{dp} . \quad (8)$$

This logarithmic derivative has a pole at p_c with residue $h + \Delta_1$; thus, here the input into the calculation is p_c and h. Again a search for regions of convergence of the different Padé approximants is made in the (p_c, h, Δ_1) space and the Δ_1 estimates from this method should be consistent with those from the former one.

Plots of the (γ, Δ_1) plane for the S series have been made for a range of p_c values centering on the estimate from the Padé analysis. The plots for $p_c = 0.644\,701$ are presented in Fig. 1; that of Fig. 1(a) being obtained by the method of Ref. 7 and that of Fig. 1(b) from the generalized Roskies transform.

In Fig. 1(a) we observe an intersection region $0.975 \le \Delta \le 1.025$, $2.27 \le \gamma \le 2.28$. In Fig. 1(b) there is an intersection region consistent with that found in Fig. 1(a) and a second region of convergence a little below $\Delta = 1.4$. Data obtained for other values of p_c by the generalized Roskies transform of the mean-size series shows that the point ($p_c = 0.644\,701$, $\Delta = 1.00$) corresponds to a sharply defined local minimum in the rms deviation of the γ values obtained from a range of fourteen Padé approximants each of which uses at least twenty nine-series coefficients. However, this point is not unique and other local minima may be found close by (but with larger rms deviations) which lie on a well-defined line in the (p_c, Δ) plane and, depending on which on the values of Δ is chosen as

TABLE III. Padé approximants to the mean cluster size (S) series (a) and Euler transform of that series $[z = p/(1+p)]$ (b). D = interfering defect.

Padé	[N/N−2]		[N/N−1]		[N/N]		[N/N+1]		[N/N+2]	
N	p_c	Expt.	p_c	Expt.	p_c	Expt.	p_c	Expt.	p_c	Expt.
					(a)					
11									0.644 694	2.2767
12							0.644 709	2.2778	0.644 703	2.2774
13					0.644 699	2.2770	0.644 701	2.2772	0.644 700	2.2772
14			0.644 701	2.2773	0.644 701	2.2772	0.644 701^D	2.2772	0.644 700	2.2772
15	0.644 70	2.2771	0.644 700	2.2771	0.644 700	2.2772	0.644 701	2.2772	0.644 701	2.2772
16	0.644 70	2.2771	0.644 701	2.2772	0.644 701	2.2772	0.644 701	2.2772	0.644 706	2.2774
17	0.644 701	2.2772	0.644 701^D	2.2772	0.644 700	2.2772				
18	0.644 700	2.2770								
Padé	[N/N−2]		[N/N−1]		[N/N]		[N/N+1]		[N/N+2]	
N	z_c	Expt.	z_c	Expt.	z_c	Expt.	z_c	Expt.	z_c	Expt.
					(b)					
11									0.391 984	2.276 72
12							0.391 988	2.277 47	0.391 987(5)	2.277 37
13					0.391 987	2.277 33	0.391 986	2.277 03	0.391 986(5)	2.277 16
14			0.391 987	2.277 32	0.391 986(5)	2.277 16	0.391 986	2.277 14	0.391 986(5)	2.277 16
15	0.391 987	2.277 19	0.391 986	2.277 13	0.391 986(5)	2.277 15	0.391 981^D	2.276 22	0.391 987	2.277 19
16	0.391 987	2.277 17	0.391 987	2.277 20	0.391 987	2.277 18	0.391 986(5)	2.277 17	0.391 989	2.277 44
17	0.391 987	2.277 19	0.391 986	2.277 10	0.391 987	2.277 24				
18	0.391 987	2.277 25								

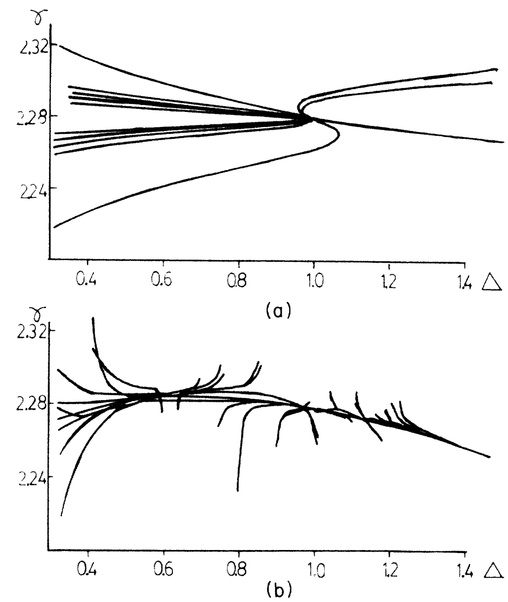

FIG. 1. Plots of the (γ,Δ) plane at the central estimate of p_c from the method of Adler *et al.* (Ref. 7) (a) and generalized Roskies transformation (b). We plot the [15,19], [16,18], [17,17], [18,16], [19,15], [15,18], [16,17], [17,16], and [18,15] Padé approximants.

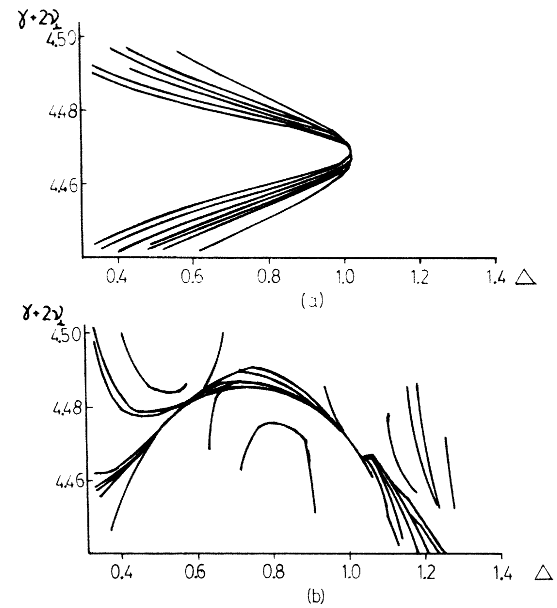

FIG. 3. Plots of the $(\gamma+2\nu_\perp,\ \Delta)$ plane at the central estimate of p_c from the method of Adler *et al.* (Ref. 7) (a) and generalized Roskies transformation (b). We plot the [15,18], [16,17], [17,16], [18,15], [15,17], [16,16], [17,15], [15,16], and [16,13] Padé approximants.

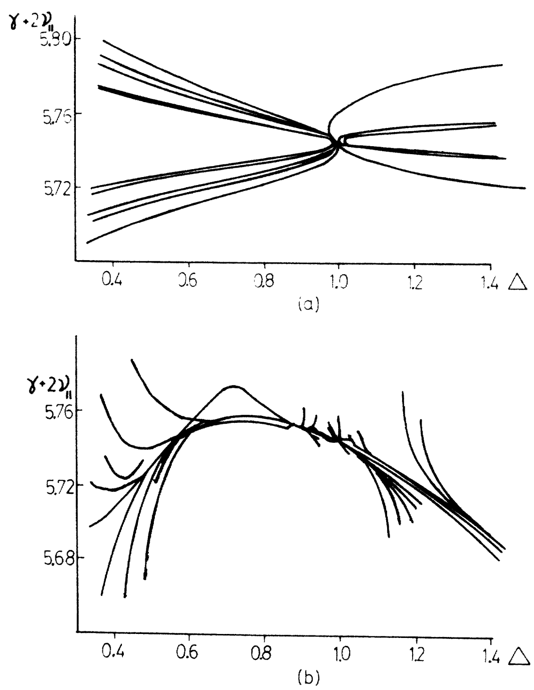

FIG. 2. Plots of the $(\gamma+2\nu_{\parallel},\Delta)$ plane at the central estimate of p_c from the method of Adler *et al.* (Ref. 7) (a) and generalized Roskies transformation (b). We plot the [15,18], [16,17], [17,16], [18,15], [15,17], [16,16], [17,15], [15,16], and [16,15] Padé approximants.

Δ_1, we find

$$p_c = 0.644\,701 - (\Delta_1 - 1)/1200 \pm 0.000\,002 \ . \tag{9}$$

Along this line

$$\gamma = 2.277\,21 - 0.13(\Delta_1 - 1) \pm 0.000\,01 \ . \tag{10}$$

Allowing for a reasonable range of rms deviations, we conclude that $|\Delta_1 - 1| \leq 0.012$ and, assuming Δ_1 to lie in this range, gives $p_c = 0.644\,701 \pm 0.000\,012$.

We have also looked at the plots for $\mu_2^{(t)}$ and $\mu_2^{(x)}$ obtained by the same two methods (Figs. 2 and 3) which enable estimates of $\gamma+2\nu_{\parallel}$ and $\gamma+2\nu_\perp$ to be made. In all cases we see converged regions near values of Δ consistent with the value of Δ_1 quoted above and, in the case of plots obtained by the Roskies method, resonances[9] near $\Delta = 0.5$. There are no well-defined lines in the Roskies data, as in the case of the mean-size expansion and in the region of the point (0.644 701, 1.00) the rms deviations are much higher and less rapidly varying. However, there is a systematic variation of the exponent estimates with choice of p_c and Δ_1. With $\Delta p_c = p_c - 0.644\,701$, we find for small Δp_c, and Δ_1 near 1, the estimates

$$\nu_{\parallel} = 1.7334 + 70 p_c - 0.04(\Delta_1 - 1) \pm 0.0005 \tag{11}$$

and

$$\nu_\perp = 1.0972 + 60 \Delta p_c - 0.02(\Delta_1 - 1) \pm 0.0004 \ , \tag{12}$$

which, when $\Delta_1 = 1$, are consistent with the results of Table II obtained from the standard unbiased $D\ln$ plots. Assuming $|\Delta_1 - 1| < 0.012$ gives the final estimates which allow for possible corrections to scaling shown in

Table II, we note that an estimate taken from $\mu_2^{(t)}$ and $\mu_2^{(x)}$ plots would give $|\Delta_1 - 1| < 0.08$.

III. DISCUSSION

Our estimate of the correction exponent Δ_1 is so close to unity that it might be supposed that only an analytic correction is observed [i.e., we are looking at the term with amplitude b in (2)]. If this is the case, then the results of standard Padé analysis (Table II) should be completely reliable (as in the case for the $d=2$, $S=\frac{1}{2}$ Ising model). If Δ_1 is merely close to 1.0, then the exponent values will deviate from standard Padé estimates in a manner determined by Eqs. (10)–(12). We have also analyzed the same series by the method of Baker and Hunter.[10] The results obtained are less well converged but are consistent with the above conclusions.

We may compare the Δ_1 estimates with those of Adler *et al.*[7] and those obtained from Reggeon field theory (RFT).[1,11,12] Adler *et al.* concluded $1.00 \le \Delta_1 \le 1.04$, using much shorter series and $p_c = 0.6446 \pm 0.0002$. The fact that the center of this range of Δ_1 is higher than the value given here is consistent with the observation that our estimate of Δ_1 increases as the assumed p_c decreases and that our revised estimate of p_c is higher. The value of $\Delta_1 = 1.04 \pm 0.02$ obtained from RFT and quoted by Adler *et al.*[7] is not consistent with $\Delta_1 = 1.000$ (analytic corrections only); however, this value was obtained from $\Delta_1 = \lambda \nu_{\parallel}$ with $\lambda = 0.60 \pm 0.01$ (Ref. 11) and $\nu_{\parallel} = 1.736 \pm 0.001$ (Ref. 12). If we use the value of $\lambda = 0.57 \pm 0.03$ given in Ref. 11, we obtain $\Delta_1 = 0.99 \pm 0.06$. Thus, any inconsistency appears to be within RFT and not between RFT and directed percolation.

The closeness of Δ_1 to unity suggests a tantalizing prospect. Most of the exactly soluble systems have analytic corrections. If directed bond percolation in two dimensions has indeed analytic corrections, the model may well turn out to be exactly soluble.

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*Permanent address: Department of Mathematics and Statistics, Shah Abdul Latif Campus, University of Sind, Khairpur Mirs (Sind), Pakistan.

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