Local magnetic field distributions. III. Disordered systems

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We conclude our investigations of the local magnetic field distributions $P(h)$ by examining disordered spin systems. Exact analytical results are given on the Bethe lattice with random bonds and the Sherrington-Kirkpatrick (SK) spin-glass model. Below the spin-glass transition temperature T_g of the SK model, the slope of $P(0)$ as a function of temperature is found to be almost linear, frozen to its value at T_g, in a manner reminiscent of other freezing phenomena found for this model. Simulations on finite-range models appear to show a correlated onset of certain features characteristic of the spin-glass phase of the SK (infinite-range) model but at a temperature higher than any associated with a phase transition.

I. INTRODUCTION

This is the final paper in a series of three concerned with the study of the local magnetic field distributions[1,2] and is devoted to disordered systems. Since relatively few exact results are known for these systems, in particular, with regard to the nature of spin-glass ordering, the study of $P(h)$ in the spirit developed in I[1] and II[2] offers useful potential for new information on their properties. As indicated earlier in I, for Ising spins $P(h)$ contains information both for statics and for dynamics. For vector spins it suffices only for certain static thermal properties.

In Sec. II we shall exploit a general formula for $P(h)$, developed in II, which has closed-form expressions on Bethe lattices. Finite-temperature results are given explicitly for chains.

Section III is concerned with the canonical mean field spin-glass model, the infinite-ranged Sherrington-Kirkpatrick (SK) model.[3,4] Analytic results for $P(h)$ for temperatures greater than the ordering temperature T_g are readily obtained by a variety of techniques. For $T<T_g$ the problem is much harder but we present a general analytic solution in terms of a function $P(x,y)$ introduced by Sommers and Dupont,[5] amenable to numerical solution and related to the Parisi function $q(x)$.[6] We also indicate the existence of more transparent limiting results. Monte Carlo simulations were used to supplement analytic studies and indicated the new observation that $P(0)$ is almost linear for $T<T_g$, a result analytically valid to leading orders, with $dP(0)/dT$ quasifrozen to its value at T_g.

Section IV is concerned with short-ranged spin-glass models, studied here only by Monte Carlo simulation. Our simulations suggest that, unlike the infinite-ranged model, there is little apparent correlation between a phase transition temperature[7-10] and a corresponding characteristic feature in $P(h)$. On the other hand, however, in all cases there does appear to be an approximately linear $P(0)$ behavior beneath a temperature which correlates approximately (at least) with that at which $P(h)$ flattens. In the SK model these correlations are also present *and* correlate with the phase transition temperature T_g.

II. GENERAL FORMULAS AND THE BETHE LATTICE

For any Ising Hamiltonian

$$\mathscr{H}=-\sum_{(ij)}J_{ij}\sigma_i\sigma_j-\sum_i b_i\sigma_i \;, \tag{2.1}$$

where (ij) denotes *pairs* of different sites i,j that are each counted once in the summation and the b_i are local external fields, the local-field distribution at any site i is defined as

$$P_i(h)=\langle\delta(h-h_i)\rangle,\;\; h_i=b_i+\sum_j J_{ij}\sigma_j \;, \tag{2.2}$$

where $\langle\cdots\rangle$ refers to a thermal average.

We shall normally be interested in the spatial average

$$P(h)=N^{-1}\sum_i P_i(h)\;. \tag{2.3}$$

Furthermore, in this paper we shall be concerned with disordered systems. Disorder averaging will be denoted by $\langle\cdots\rangle_d$. Explicitly we shall be concerned with J_{ij} disorder. As we shall indicate further below, $P(h)$ is a self-averaging quantity; that is, in the thermodynamic limit

$$P(h)=\langle P_i(h)\rangle_d \;, \tag{2.4}$$

so that a $\langle \cdots \rangle_d$ around $P(h)$ is in fact superfluous in the thermodynamic limit and therefore will normally be omitted.

In the rest of this section we shall concentrate on Bethe lattices, allowed to have random, but quenched, nearest-neighbor exchange bonds. For this problem a generalization of Eq. (6) of II is useful, namely,

$$P_i(h)=\langle \delta(h-h_i)\rangle' Z_1(\beta h)\frac{\mathrm{Tr}' e^{-\beta\mathscr{H}'}}{\mathrm{Tr} e^{-\beta\mathscr{H}}} \;, \tag{2.5}$$

where the primes denote the exclusion of the site i; Z_1 is the single spin paramagnetic partition function, $Z_1(\beta h)=2\cosh(\beta h)$; and the inverse temperature $\beta=1/k_BT$. Equation (2.5) is potentially useful for Bethe-lattice problems because of the absence of closed loops on such lattices. We shall consider explicitly two special cases of particular interest, branching ratio $z=2$ which corresponds to a bond-random chain and can be studied exactly,[11] and $z\to\infty$ with the variance of the exchange distribution scaling as z^{-1}, which relates to an infinite-ranged spin glass.[12] Further limited results will also be considered for general z.

The analogue of Eq. (7) of II for the self-consistent cluster Hamiltonian is

$$\mathscr{H}^i=-\sum_j J_{ij}\sigma_i\sigma_j-H_i\sigma_i-\sum_j H_j\sigma_j \;, \tag{2.6}$$

where i is any site and the j are its z nearest neighbors,

$$H_i=b_i+H^0 \;. \tag{2.7}$$

H^0 is a generating field, taken to be zero at the end, and

$$H_j=\sum_{l\neq i} J_{jl}\langle\sigma_l\rangle \;, \tag{2.8}$$

where the l are the $(z-1)$ neighbors of j not equal to i. The partition function associated with site i is

$$Z_i=e^{\beta H_i}C_{i+}+e^{-\beta H_i}C_{i-} \;, \tag{2.9}$$

where

$$C_{i\pm}=\prod_j \{2\cosh[\beta(J_{ij}\pm H_j)]\} \;, \tag{2.10}$$

leading to

$$\langle\sigma_i\rangle=\frac{e^{\beta H_i}C_{i+}-e^{-\beta H_i}C_{i-}}{e^{\beta H_i}C_{i+}+e^{-\beta H_i}C_{i-}} \tag{2.11}$$

and

$$\langle\sigma_j\rangle=\frac{e^{\beta H_i}C_{i+}\tanh[\beta(J_{ij}+H_j)]-e^{-\beta H_i}C_{i-}\tanh[\beta(J_{ij}-H_j)]}{e^{\beta H_i}C_{i+}+e^{-\beta H_i}C_{i-}} \;. \tag{2.12}$$

These equations, together with Eq. (2.8), form an infinite set of coupled transcendental simultaneous equations that determine sets of solutions $\{\langle\sigma_i\rangle\}$. In general their solution is not possible, although simplifications occur for pure systems (II) or for $z=2$, $b_i=0$. The limit $z\to\infty$, $J_{ij}\sim z^{-1/2}$ leads to the Thouless-Anderson-Palmer (TAP) equations[12] for the SK (Ref. 4) spin-glass model, to be discussed further in Sec. III. Given the solution to the transcendental equations, the average local-field distribution can be obtained from Eq. (2.5), giving

$$P(h)=2\cosh(\beta h)\int_{-\infty}^{\infty}\frac{dx}{2\pi}e^{-ihx}N^{-1}\sum_i\left\langle\frac{\prod_j[2\cosh(\beta H_j+ixJ_{ij})]}{e^{\beta H_i}C_{i+}+e^{-\beta H_i}C_{i-}}\right\rangle_d \;. \tag{2.13}$$

For $\{b_i\}=0$ and above any ordering temperature T_c

$$\{H_j\}=\{\langle\sigma_j\rangle\}=0 \tag{2.14}$$

and thus

$$P(h)=\cosh(\beta h)\int_{-\infty}^{\infty}\frac{dx}{2\pi}e^{-ihx}\times N^{-1}\sum_i\left\langle\prod_{j=1}^{z}\left[\frac{\cos(J_{ij}x)}{\cosh(\beta J_{ij})}\right]\right\rangle_d \;. \tag{2.15}$$

Clearly, from (2.15), if the J_{ij} are independently distributed, $P(h)$ is self-averaging for $T>T_c$.

For the special case of a one-dimensional (1D) chain with symmetric exchange distribution, $p(J_{ij})=p(-J_{ij})$, $P(h)$ may be written alternatively as a convolution:

$$[P(h)]_{\mathrm{1D\,symm}}=\int_{-\infty}^{\infty}dJ\,p(J)p(h+J)\times\{1-\tanh(\beta J)\tanh[\beta(h+J)]\} \;, \tag{2.16}$$

as obtained using a different method by Barma.[11,13]

We consider three specific examples of symmetric $p(J_{ij})$, each with variance $\frac{1}{2}J^2$. For a bond distribution consisting of two δ functions

$$p(J_{ij})=\tfrac{1}{2}[\delta(J_{ij}-J/\sqrt{2})+\delta(J_{ij}+J/\sqrt{2})] \;, \tag{2.17}$$

we obtain

$$P(h)=\frac{1}{2}\frac{\cosh(\sqrt{2}\beta J)}{1+\cosh(\sqrt{2}\beta J)}[\delta(h-\sqrt{2}J)+\delta(h+\sqrt{2}J)]+\frac{1}{1+\cosh(\sqrt{2}\beta J)}\delta(h) \;. \tag{2.18}$$

For the rectangular distribution

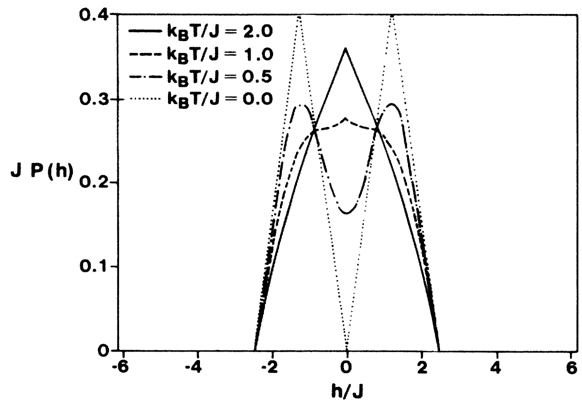

FIG. 1. Averaged local-field distribution for an Ising chain with nearest-neighbor bonds distributed with a symmetric rectangular distribution of standard deviation $J/\sqrt{2}$.

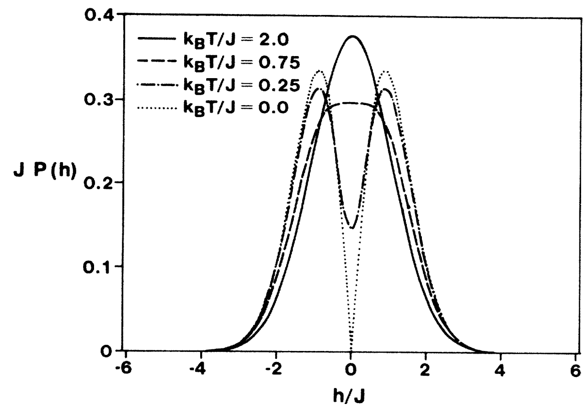

FIG. 2. Averaged local-field distribution for an Ising chain with nearest-neighbor bonds distributed with a symmetric Gaussian distribution of standard deviation $J/\sqrt{2}$.

$$p(J_{ij})=\frac{1}{\sqrt{6}J}, \quad -\sqrt{3/2}J \le J_{ij} \le \sqrt{3/2}J \;, \tag{2.19}$$

we obtain[11]

$$P(h)=\begin{cases} \dfrac{\coth(\beta|h|)}{3\beta J^2}\ln\left[\dfrac{\cosh(\sqrt{3/2}\beta J)}{\cosh[\beta(|h|-\sqrt{3/2}J)]}\right], & |h| \le \sqrt{6}J \\ 0, \text{ otherwise} . \end{cases} \tag{2.20}$$

This distribution (2.20) is exhibited for a variety of temperatures in Fig. 1, while Fig. 2 shows the corresponding results for a Gaussian of the same variance $\frac{1}{2}J^2$. In all cases $P(0)$ dips to zero at $T=0$. This is physically because at zero temperature the signs of $J_{i,i+1}$ and $\sigma_i\sigma_{i+1}$ are correlated so that

$$h_i = (|J_{i,i+1}| + |J_{i,i-1}|)\sigma_i \;, \tag{2.21}$$

giving no weight at $P(0)$.

As noted earlier, for general branching ratio z a general solution does not appear possible. However, it is straightforward to write down the results for $P(h)$ at $T=0$ and $T=\infty$; for symmetric $p(J)$, we obtain

$$P(h, T=0)=\tfrac{1}{2}\int_0^\infty d|J_1| \cdots \int_0^\infty d|J_Z|\, \tilde{p}(|J_1|)\tilde{p}(|J_2|)\times \cdots \times \tilde{p}(|J_Z|) \times \delta(|h|-|J_1|-|J_2|-\cdots-|J_Z|) \;, \tag{2.22}$$

where $\tilde{p}(|J_i|)$ is the probability of $|J_i|$, and

$$P(h, T=\infty)=\int_{-\infty}^{\infty} dJ_1 \int_{-\infty}^{\infty} dJ_2 \cdots \int_{-\infty}^{\infty} dJ_Z\, p(J_1)p(J_2)\times \cdots \times p(J_Z)\delta(h-J_1-J_2-\cdots-J_Z) \;. \tag{2.23}$$

It follows from (2.22) that at $T=0$, small h

$$P(h, T=0)=\frac{[\tilde{p}(0)]^z h^{z-1}}{2[(z-1)!]}+O(h^z) \;. \tag{2.24}$$

In particular, therefore, for the chain ($z=2$), one obtains the leading zero temperature behavior to be linear in h for small h. This is shown in the next section to be the case also for the SK spin glass which is effectively a $z\to\infty$ Bethe-lattice problem, but a rather special one with J_{ij} variance scaling as z^{-1}, thereby introducing frustration. However, in view of the result (2.24) for general *unfrustrated* Bethe lattices, this similarity can only be considered accidental.

Another novelty for symmetric chains is that just as the $T=0$, low-h behavior is linear with slope determined completely by $[p(0)]^2$, so too the $h=0$, low-T behavior is linear, again with slope determined completely by $[p(0)]^2$:

$$P(h=0, T)=2k_BT[p(0)]^2 \;, \tag{2.25}$$

$$P(h, T=0)=2h[p(0)]^2 \;. \tag{2.26}$$

Note that the second moment of $P(h, T=\infty)$ is z times the second moment of $p(J)$. We have fixed the second moment of $p(J)$ to be $\frac{1}{2}J^2$ in the one-dimensional examples so that the second moment of $P(h, T=\infty)$ is J^2, as in the SK model discussed in the next section.

III. INFINITE-RANGED ISING SPIN GLASS

A class of disordered spin systems of great current interest are the spin glasses.[14] The Hamiltonians of these systems are characterized by the combination of quenched spatial disorder and frustration.[15] Their low-temperature free energies are believed to be characterized by the existence of many stochastic, fractal-like, distributed metastable states. These manifest themselves in extremely long relaxation times and (quasi)nonergodicity. The lower critical dimension for a true phase transition remains an open question both for short-ranged bond-disordered model systems, such as the Edwards-Anderson[16] model

$$\mathscr{H} = -\sum_{(ij)} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j , \tag{3.1}$$

where the J_{ij} are finite-range quenched Gaussian random bonds of either sign, and for real site-disordered systems (such as *Cu*Mn or $Eu_{1-x}Sr_xS$). However, one model which is known to have a phase transition to a spin-glass state exhibiting all of the novel features of metastability, ultrametricity,[17] etc., is the infinite-ranged model of Sherrington and Kirkpatrick[4]

$$\mathscr{H} = -\sum_{(ij)} J_{ij}\sigma_i\sigma_j - b\sum_i \sigma_i , \tag{3.2}$$

where the sum is over *all* pairs (ij) and the J_{ij} are a quenched set of random bonds drawn independently from a distribution with mean J_0/N and variance J^2/N, where N is the number of sites. The bond distribution $p(J_{ij})$ is usually taken as Gaussian,

$$p(J_{ij}) = \left[\frac{N}{2\pi J^2}\right]^{1/2} \exp\left[-\frac{(NJ_{ij}-J_0)^2}{2J^2}\right] , \tag{3.3}$$

although in the thermodynamic limit only the first two moments are relevant (at least above T_g and perhaps at all temperatures). As before, b is an external field. We shall now discuss the averaged local-field distribution for this model, concentrating initially on $J_0 = b = 0$.

As commented earlier, the site-averaged local-field distribution

$$P(h) = N^{-1}\sum_j \left\langle \delta\left[h - \sum_i J_{ij}\sigma_j\right]\right\rangle \tag{3.4}$$

is expected to be self-averaging;[18] i.e., in the thermodynamic limit, an average over all sites for a particular $\{J_{ij}\}$ is equal to an average over $\{J_{ij}\}$ for any single site k,

$$N^{-1}\sum_j \langle \delta(h-h_j)\rangle = \int \left[\prod_{(ij)} [dJ_{ij}\, p(J_{ij})]\right] \langle \delta(h-h_k)\rangle . \tag{3.5}$$

We shall later demonstrate this to be true.

The most powerful method to discuss bond-averaged properties of the SK model is the replica method.[16,4] This we discuss below in Sec. III C. However, to tie in with the discussion of Sec. II, we first consider an alternative procedure related to the Bethe lattice.

A. TAP theory

Thouless, Anderson, and Palmer[12] compared the unaveraged SK model to a set of spins on a Bethe lattice with bonds randomly distributed as in (3.3) with N replaced by z and argued that in the thermodynamic limit the Bethe solution with $z = N \to \infty$ correctly describes the states of the SK model. The SK scaling introduces frustration even in the Bethe model as $z = N \to \infty$. The TAP equations follow from Eqs. (2.9)–(2.12):

$$\langle \sigma_i \rangle = \tanh\left[\beta \sum_j J_{ij} \tanh(\beta H_j)\right] + O(N^{-1}) , \tag{3.6}$$

$$\langle \sigma_j \rangle = \tanh(\beta H_j) - \langle \sigma_i \rangle \beta J_{ij}[1-\tanh^2(\beta H_j)] + O(N^{-1}) , \tag{3.7}$$

or, eliminating H_j,

$$\langle \sigma_i \rangle = \tanh\left[\beta\sum_j J_{ij}\langle\sigma_j\rangle - \langle\sigma_i\rangle\beta\sum_j J_{ij}^2(1-\langle\sigma_j\rangle^2)\right] + O(N^{-1}) , \tag{3.8}$$

the second term in the large parentheses being the famous Onsager correction. As shown by TAP these equations lead to the spin-glass ordering temperature $T_g = J/k_B$, as found earlier from the replica method.[4]

Since a conventional Bethe-lattice problem is unfrustrated, even with random bonds, whereas the SK spin glass is fundamentally frustrated, it should be noted that the Bethe-lattice equivalence holds only in the limit of J_{ij} variance scaling as z^{-1} with $z \to \infty$, the frustration then lying in the Onsager term. Short-range spin glasses cannot be adequately[18,19] represented by Bethe lattices even in a mean field sense, although TAP analogues exist on the conventional lattice.

For $T > T_g$ all of the $\langle\sigma\rangle$ are zero and Eq. (2.13) may be used to yield

$$P(h) = \langle P(h)\rangle_d = \frac{1}{\sqrt{2\pi}}\frac{1}{J}\cosh(\beta h)\exp(-h^2/2J^2)\exp(-\beta^2J^2/2) \tag{3.9}$$

$$= \frac{1}{2J\sqrt{2\pi}}\left[\exp\left[-\frac{(h-\beta J^2)^2}{2J^2}\right] + \exp\left[-\frac{(h+\beta J^2)^2}{2J^2}\right]\right] . \tag{3.10}$$

That is, $P(h)$ is given by the sum of two Gaussians each of width J centered at $h = \pm\beta J^2$. This expression is exhibited for a variety of temperatures $T \geq T_g$ in Fig. 3. It is interesting to note that exactly at the transition temperature $P(h)$ is flat at $h=0$ (while it is peaked for $T > T_g$), although we have already cautioned in I against converse deductions.

Furthermore, considering the temperature dependence of $P(0)$,

$$P(0) = \frac{1}{\sqrt{2\pi}}\frac{1}{J}\exp\left[-\frac{J^2}{2(k_BT)^2}\right], \quad T \geq T_g \tag{3.11}$$

we note that

$$\left.\frac{dP(0)}{dT}\right|_{T=T_g} = \frac{[P(0)]_{T=T_g}}{T_g} = \frac{1}{\sqrt{2\pi}}\frac{k_B}{J^2}\exp(-\tfrac{1}{2}) , \tag{3.12}$$

that is, at T_g a tangent to the curve of $P(0)$ against T

passes through the origin. We shall later argue that this observation may be relevant to the low-temperature behavior.

Before passing to $T < T_g$ we might note that the result (3.9) for $T > T_g$ can be obtained in a number of equivalent ways. As well as that used above, another is the replica procedure discussed in Sec. III C while a third employs a series expansion with resummation using the N scaling.[12,4,20] Each of the last two of these employs the characteristic function $G(k)$ defined by

$$P(h) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} G(k) \exp(-ikh) , \quad (3.13)$$

so that

$$G(k) = N^{-1} \sum_i \left\langle \exp\left[ik \sum_j J_{ij}\sigma_j \right] \right\rangle . \quad (3.14)$$

The series expansion proceeds as follows. Rewriting

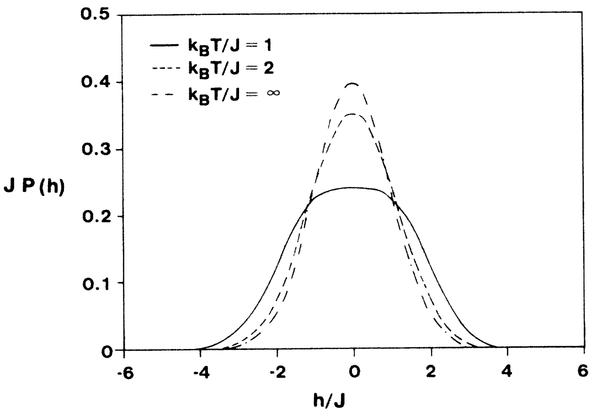

FIG. 3. Local-field distribution function of the symmetric Sherrington-Kirkpatrick spin glass for $T \geq T_g$ for zero external field [from the exact expression (3.9)]. The transition temperature is $T_g = J/k_B$.

$$G(k) = N^{-1} \sum_i \left\langle \prod_j \{\cos(kJ_{ij})[1 + i\sigma_j \tan(kJ_{ij})]\} \right\rangle = N^{-1} \sum_i \left[\left[\prod_l \cos(kJ_{il}) \right] \left\langle \prod_j [1 + i\sigma_j \tan(kJ_{ij})] \right\rangle \right]$$

$$= N^{-1} \sum_i \left[\prod_l \cos(kJ_{il}) \right] \left[1 + i \sum_j \langle \sigma_j \rangle \tan(kJ_{ij}) - \tfrac{1}{2} \sum_{j,m}{}' [\langle \sigma_j \sigma_m \rangle \tan(kJ_{ij}) \tan(kJ_{im})] + \cdots \right] , \quad (3.15)$$

where the prime indicates the exclusion of $j = m$.

In the paramagnetic phase all odd spin averages are zero. Further simplifications result from SK scaling,

$$\langle J_{ij} \rangle_d = J_0/N, \quad \langle J_{ij}^2 \rangle_d = J^2/N . \quad (3.16)$$

For example, in the paramagnetic average,

$$\langle \sigma_j \sigma_m \rangle_{\text{para}} = -\frac{\beta}{2} J_{jm} + \beta^2 \sum_l J_{jl} J_{ml} + \cdots , \quad (3.17)$$

the only thermodynamically relevant term for use in (3.15) is the second with $l = i$, while similarly in higher even order paramagnetic averages the relevant term is

$$\langle \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \times \cdots \times \sigma_{i_m} \rangle_{\text{para}}^{\text{Rel}} = \beta^m J_{i_1 i} J_{i_2 i} \times \cdots \times J_{i_m i} , \quad \text{for } m \text{ even} . \quad (3.18)$$

Thus, noting also that $\tan(kJ_{ij})$ can be replaced by kJ_{ij} in the limit $N \to \infty$, we obtain

$$G(k) = N^{-1} \sum_i \left[\prod_l \cos(kJ_{il}) \right] \left[1 - \frac{(\beta k)^2}{2} \sum_{j,m} J_{ij}^2 J_{im}^2 + \frac{(\beta k)^4}{4!} \sum_{j,m,n,p} J_{ij}^2 J_{im}^2 J_{in}^2 J_{ip}^2 + \cdots \right] \quad (3.19)$$

$$= \exp(-k^2 J^2/2) \cos(\beta J k) . \quad (3.20)$$

Substituting into (3.13), we obtain for $T > T_g$

$$P(h) = \frac{1}{\sqrt{2\pi} J} \exp(-\beta^2 J^2/2 - h^2/2J^2) \cosh(\beta h) , \quad (3.9')$$

as earlier. Note that there is no J_0 dependence, just as the pure mean-field result for $P(h)$ has no width above T_c.

For $T < T_g$ the solution to the TAP equations is not known exactly, although some information is available on mappings of integrated quantities to the replica theory[22] and on the density of metastable states,[23] as well as certain perturbative and heuristic results.[12] Therefore, we turn, in this section, just to the low temperature, small-h limit. In this limit TAP speculated on the form of a related local-field distribution

$$P_{\text{TAP}}(h) = N^{-1} \sum_i \delta(h - \langle h_i \rangle) \quad (3.21a)$$

$$= N^{-1} \sum_i \delta \left[h - \sum_j J_{ij} \langle \sigma_j \rangle \right] . \quad (3.21b)$$

It should be noted that this is *not* generally the same as our local-field distribution

$$P(h) = N^{-1} \sum_i \langle \delta(h - h_i) \rangle , \quad (2.3')$$

indeed P_{TAP} does *not* correctly provide the thermodynamic quantities through relations such as Eq. (21) of I. However, in the limit $T \to 0$ for a system that orders, that is, when $P_i(h)$ is of the form

$$P_i(h) = \delta(h - h_i^0) , \tag{3.22}$$

these two definitions yield essentially the same result. In the $T=0$ limit, TAP argued that $P_{\text{TAP}}(h)$ should be linear in h at small h and speculated on the coefficient of linearity,

$$P_{\text{TAP}}(h) = 0.307h + O(h^2) . \tag{3.23}$$

Hence, we might expect that at $T=0$, if their speculation is correct, also

$$P(h) = 0.307h + O(h^2) \tag{3.24}$$

on the basis of a $T=0$ identification of the two definitions. A more precise discussion of the relation between the definitions is to be found in Sec. III C, via another distribution function[5] which further permits a numerical evaluation of the coefficient of linearity.

B. Computer simulation

To complement the analytic study we have performed Monte Carlo simulations for an SK model with bond probability distribution

$$p(J_{ij}) = \tfrac{1}{2}(1 + J_0/\sqrt{N}J)\delta(J_{ij} - J/\sqrt{N}) + \tfrac{1}{2}(1 - J_0/\sqrt{N}J)\delta(J_{ij} + J/\sqrt{N}) . \tag{3.25}$$

As pointed out earlier, we expect (3.3) and (3.25) to give the same results in the thermodynamic limit in which only the first two moments of $p(J_{ij})$ are relevant, but (3.25) is advantageous for computer simulation since J_{ij} can be represented by a single binary digit, thereby reducing the memory storage required and increasing the execution speed of the simulation.

The results reported here are for $J_0 = 0$ and for averages over four different $N = 1020$ systems. Details of the simulation procedure are to be found in Ref. 20. Briefly, however, in all cases the systems were first run to approximately equilibrate the system as measured by its energy, with $P(h)$ measured over the subsequent 200 Monte Carlo steps per spin (MCS's). The initial equilibration involved at least 50 MCS's, with measurements *below* T_g involving an initial equilibration at T_g and, in some cases, at lower intermediate temperatures. Significantly below T_g (at least) these run times are *not* sufficiently long to explore *all* the metastable states believed to characterize the free energy phase space[21] but rather are likely to be representative of a typical low-lying metastable state. Indeed, for any one sample (specific set of $\{J_{ij}\}$) repeated simulations lead to little variation in the observed $P(h)$, while one would expect the actual metastable state reached to be different in each case, thereby suggesting that the states reached are typical. Nor is there much variation at any one temperature from sample to sample, thereby indicating self-averaging.

For $T > T_g$ good agreement with (3.9) is found. Figure 4 shows results for four different temperatures, including two with $T < T_g$, exhibiting the growth of a zero field dip for $T < T_g$.

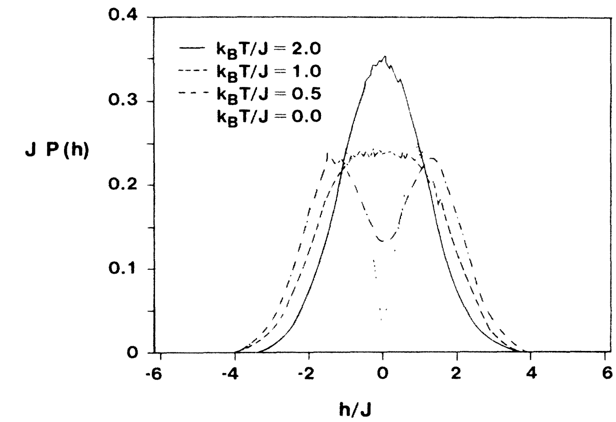

FIG. 4. Computer-simulation results for the local-field distribution of the symmetric SK spin glass, based on averages over four different $N = 1020$ systems (see the text).

The data at $T=0$ were accumulated by taking 40 different $\{J_{ij}\}$ configurations and searching for one state in each that was stable against single spin flips. Kirkpatrick and Sherrington[4] and Palmer and Pond[24] have already studied the zero-temperature problem in detail, so no attempt was made to be more careful in searching for local minima. Those authors, in fact, report lower values for $P(h=0, T=0)$, and it is believed that for an infinite system, $P(0,0)=0$. However, even using states stable against only single spin flips, the small h slope obtained from this simulation, $0.31/J^2$, is in good agreement with the TAP[12] prediction of $0.307/J^2$ discussed in the previous chapter. Hence, the slope appears to be less sensitive than the intercept to the particular local minimum one is in.

As the temperature is reduced from $T_g/2$ to 0, most of the change in $P(h)$ occurs for small h. The large-h regions are relatively stable. This suggests that the formation of the $h=0$ dip may be an important characteristic of the spin-glass phase, as indeed was suggested long ago.[4,25]

In view of the analytic observation (3.12) and the belief that $P(0)=0$ at zero temperature we have compared the experimental $P(0,T)$ with the linear extrapolation of Eq. (3.11); viz.,

$$\tilde{P}(0, T < T_g) = \frac{1}{\sqrt{2\pi}} \frac{1}{J} \exp(-\tfrac{1}{2}) \frac{k_B T}{J} . \tag{3.26}$$

As indicated in Fig. 5, the fit is remarkably good, deviating only for $T \sim 0$ where, as we have indicated above, we believe the numerical results to be overestimates. The phenomenon of observables in the SK spin-glass freezing-in to their values at T_g is not new, cf. the constancy of the zero-field susceptibility below T_g (Refs. 6 and 26) and the Parisi-Thoulouse (PaT) hypothesis,[27] but to the best of our knowledge the suggestion that $dP(0)/dT$ should freeze is new. In the next section we provide an analytic demonstration that the freeze-in of $dP(0)/dT$ is exact to the two leading orders in $\tau = (T_g - T)/T_g$, corrections not arising until $O(\tau^3)$. This incompleteness of the approximate freeze-in is again reminiscent of the PaT hypothesis.[28,29]

C. Replica theory

In this section we use the replica trick to study the local-field distribution of the SK spin glass. The method used is an analogue of that used by Parisi[30] to calculate the average overlap distribution function and as extended for other observables by Elderfield.[31] Our starting point is to express $P(h)$ in terms of the characteristic function $G(k)$ as defined in (3.13) and (3.14).

The replica procedure is introduced via the identity (for any operator $\hat{\mathbf{O}}$)

$$\langle \hat{\mathbf{O}} \rangle = \frac{\mathrm{Tr}(\hat{\mathbf{O}} e^{-\beta H})}{\mathrm{Tr}\, e^{-\beta H}} = \frac{\mathrm{Tr}(\hat{\mathbf{O}} e^{-\beta H})(\mathrm{Tr}\, e^{-\beta H})^m}{(\mathrm{Tr}\, e^{-\beta H})^{m+1}} , \tag{3.27}$$

whence, introducing replica labels $\alpha = 1,2,\ldots,n = m+1$ and taking the limit $n \to 0$, one obtains

$$\langle \hat{\mathbf{O}} \rangle = \lim_{n\to 0} \left\{ \mathrm{Tr}_n \left[\hat{\mathbf{O}}^1 \exp\left[-\beta \sum_{\alpha=1}^{n} H^\alpha \right] \right] \right\} , \tag{3.28}$$

the trace being over the n replicas indicated by the superscript labels α. For the case under explicit consideration here,

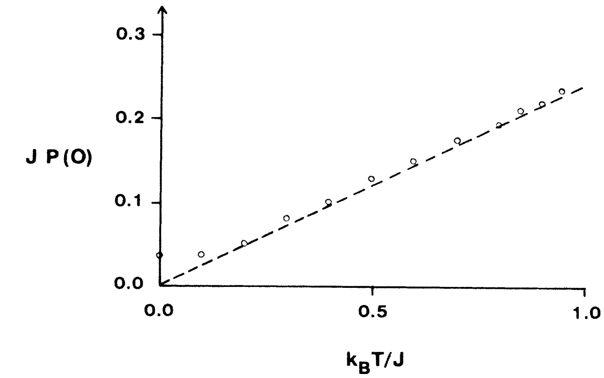

FIG. 5. $P(0)$ for the symmetric SK model as a function of temperature for $T \le T_g$ as obtained by computer simulation (see the text), compared with the linear extrapolation Eq. (3.26).

$$\mathscr{H} = -\sum_{(ij)} J_{ij} \sigma_i \sigma_j , \tag{3.2$'$}$$

$G(k)$ can thus be expressed as

$$G(k) = \lim_{n\to 0} \left[\frac{1}{N} \sum_i \mathrm{Tr}\exp\left[+ik \sum_j J_{ij}\sigma_j^1 + \frac{1}{k_B T} \sum_{(jl)} \sum_{\alpha=1}^{n} J_{jl}\sigma_j^\alpha \sigma_l^\alpha \right] \right] . \tag{3.29}$$

Averaging over the $\{J_{ij}\}$ distribution with the usual symmetric SK scaling, Eq. (3.3), results in

$$\langle G(k) \rangle_d = \lim_{n\to 0} \left[\frac{1}{N} \sum_i \mathrm{Tr}_n \exp\left[\frac{J^2}{2Nk_B^2T^2} \sum_{(lm)} \sum_{\alpha,\beta=1}^{n} \sigma_l^\alpha \sigma_l^\beta \sigma_m^\alpha \sigma_m^\beta + \frac{ikJ^2}{Nk_BT} \sum_l \sum_{\alpha=1}^{n} \sigma_l^1 \sigma_i^\alpha \sigma_l^\alpha - \frac{k^2J^2}{2} \right] \right] , \tag{3.30}$$

where $\langle \cdots \rangle_d$ denotes the average. It is immediately clear that there is nothing special about site i in the exponent, so that self-averaging is valid, as stated earlier. Equally, there is nothing special about the labeling of one replica as 1. Rearranging,

$$G(k) = \lim_{n\to 0} \left\{ \frac{1}{N} \sum_i \mathrm{Tr}_n \exp\left[\frac{J^2nN}{4k_B^2T^2} - \frac{k^2J^2}{2} + \frac{ikJ^2}{k_BT}\sigma_i^1 + \frac{J^2}{2Nk_B^2T^2} \sum_{(\alpha\beta)\neq(\alpha 1)} \left[\sum_l \sigma_l^\alpha \sigma_l^\beta \right]^2 \right.\right.$$

$$\left.\left. + \frac{J^2}{2Nk_B^2T^2} \sum_{(\alpha 1)} \left[\sum_l \sigma_l^\alpha \sigma_l^1 + ikk_BT\sigma_i^1 \right]^2 + \frac{k^2J^2}{N} \right] \right\} , \tag{3.31}$$

where $(\alpha\beta)$ denotes a pair of different indices. Applying a Hubbard-Stratonovich transformation[32] leads to

$$G(k) = \lim_{n\to 0} \left[\frac{1}{N} \sum_i \exp\left[\frac{J^2Nn}{4k_B^2T^2} - \frac{k^2J^2}{2} \left[1 - \frac{2}{N} \right] \right] \right.$$

$$\times \int \left\{ \prod_{(\alpha\beta)} \left[\frac{J}{2k_BT} \left[\frac{N}{\pi} \right]^{1/2} dq^{(\alpha\beta)} \right] \right\} \exp\left[-\frac{N^2J^2}{2k_B^2T^2} \sum_{(\alpha\beta)} (q^{(\alpha\beta)})^2 \right]$$

$$\left. \times \mathrm{Tr}_n \exp\left[\frac{J^2}{k_B^2T^2} \sum_{(\alpha\beta)\neq(\alpha 1)} q^{(\alpha\beta)} \sum_l \sigma_l^\alpha \sigma_l^\beta + \frac{J^2}{k_B^2T^2} \sum_{(\alpha 1)} q^{(\alpha 1)} \left[\sum_l \sigma_l^\alpha \sigma_l^1 + ikk_BT\sigma_i^n \right] + \frac{ikJ^2}{k_BT}\sigma_i^1 \right] \right] . \tag{3.32}$$

Since, for any function $f(\sigma)$,

$$\mathrm{Tr}\exp\left[\sum_l f(\sigma_l)\right]=\exp\{N\ln\mathrm{Tr}\exp[f(\sigma)]\}\ , \tag{3.33}$$

where the first trace is over N sites and the second is single-site, the integrals in (3.32) are extremally dominated in the limit $N\to\infty$. In this limit the q are precisely those of standard spin-glass theory, as determined from extremal domination of the free-energy functional; i.e., from the $n\to 0$ limit of

$$q^{(\alpha\beta)}=\frac{\mathrm{Tr}\left[\sigma^\alpha\sigma^\beta\exp\left[\frac{J^2}{k_B^2T^2}\sum_{(\alpha\beta)}q^{(\alpha\beta)}\sigma^\alpha\sigma^\beta\right]\right]}{\mathrm{Tr}\exp\left[\frac{J^2}{k_B^2T^2}\sum_{(\alpha\beta)}q^{(\alpha\beta)}\sigma^\alpha\sigma^\beta\right]}\ , \tag{3.34}$$

the traces being single-site. $G(k)$ is thus given by

$$G(k)=\exp\left[-\frac{k^2J^2}{2}\right]\lim_{n\to 0}\left[\frac{1}{2^n}\mathrm{Tr}_n\exp\left[\frac{J^2}{k_B^2T^2}\sum_{(\alpha\beta)}q^{(\alpha\beta)}\sigma^\alpha\sigma^\beta+\frac{ikJ^2}{k_BT}\left[\sigma^1+\sum_{\alpha\neq 1}q^{(\alpha 1)}\sigma^\alpha\right]\right]\right]\ , \tag{3.35}$$

where $q^{(\alpha\beta)}$ is given by (3.34) and where the identities

$$G(0)=1 \tag{3.36}$$

and

$$\lim_{n\to 0}\left[\frac{1}{2^n}\mathrm{Tr}\exp\left[\frac{J^2}{k_B^2T^2}\sum_{(\alpha\beta)}q^{(\alpha\beta)}\sigma^\alpha\sigma^\beta\right]\right]=1 \tag{3.37}$$

have been used to identify the constant of proportionality.

A constant magnetic field is incorporated by using

$$P(h)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp[-ik(h-b)]G(k)\ , \tag{3.38}$$

with the expression for $G(k)$ modified by the inclusion of an additional term $\beta b\sum_\alpha\sigma^\alpha$ in the argument of the exponential of Eq. (3.35) and similarly in the exponentials of Eq. (3.34).

Before turning to the general case, we consider first, two special cases; (i) $T>T_g$ and (ii) $P(0)$.

(i) For $T>T_g$ and $b=0$, $q^{(\alpha\beta)}$ is zero and (3.35) simplifies to

$$G(k)=\exp(-k^2J^2/2)\cos(kJ^2/k_BT)$$

and hence

$$P(h)=\frac{1}{2\sqrt{2\pi}}\frac{1}{J}\left[\exp\left[-\frac{(h-J^2/k_BT)^2}{2J^2}\right]+\exp\left[-\frac{(h+J^2/k_BT)^2}{2J^2}\right]\right]\ ,$$

as obtained earlier in Sec. III A.

For $b\neq 0$, $q^{(\alpha\beta)}$ is nonzero at all temperatures. For $T>T_{\mathrm{AT}}$, the de Almeida–Thouless[33] temperature, $q^{(\alpha\beta)}$ is replica symmetric, that is

$$q^{(\alpha\beta)}=q,\quad \text{for all } (\alpha\beta)\ , \tag{3.39}$$

so that q satisfies the self-consistency equation

$$q=\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\exp(-z^2/2)\tanh^2[\beta(Jz\sqrt{q}+b)]\ . \tag{3.40}$$

The de Almeida–Thouless temperature is given by the simultaneous solution of Eq. (3.40) with $\beta=\beta_{\mathrm{AT}}=(k_BT_{\mathrm{AT}})^{-1}$ and

$$(\beta_{\mathrm{AT}}J)^2\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\exp(-z^2/2)\times\mathrm{sech}^4[\beta_{\mathrm{AT}}(Jz\sqrt{q}+b)]=1\ . \tag{3.41}$$

Replica symmetry leads to a simplification of (3.35),

$$G(k)=\exp(-k^2J^2/2)\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]\lim_{n\to 0}\left[\frac{1}{2^n}\times\mathrm{Tr}_n\exp\left[\beta(b+Jz\sqrt{q}+ikJ^2q)\times\sum_{\alpha\neq 1}\sigma^\alpha+\beta(b+Jz\sqrt{q}+ikJ^2)\sigma^1\right]\right]\ , \tag{3.42}$$

$$=\exp\left[-\frac{k^2J^2(1-q)}{2}\right]\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]\left[\frac{\sum_{\sigma=\pm 1}\exp\{\beta\sigma[b+Jz\sqrt{q}+ikJ^2(1-q)]\}}{2\cosh[\beta(b+Jz\sqrt{q}\,)]}\right]\ , \tag{3.43}$$

whence

$$P(h)=\frac{\exp[-\beta^2J^2(1-q)/2]}{2\pi J(1-q)^{1/2}}\int_{-\infty}^{\infty}dz\,e^{-z^2/2}\frac{\cosh(\beta h)}{\cosh[\beta(b+zJ\sqrt{q})]}\exp\left[\frac{-(h-b-zJ\sqrt{q})^2}{2J^2(1-q)}\right] \tag{3.44}$$

$$=\frac{1}{2\pi J}\exp\left[-\frac{(h-b)^2}{2J^2}-\frac{\beta^2J^2}{2}(1-q)\right]\int_{-\infty}^{\infty}dz\,e^{-z^2/2}\frac{\cosh(\beta h)}{\cosh\{\beta[b(1-q)+hq+Jz\sqrt{q(1-q)}]\}} \tag{3.45}$$

One may readily check that this $P(h)$ yields conventional replica-symmetric results,[3,4] such as, the magnetization per spin,

$$m=\int_{-\infty}^{\infty}dh\,P(h)\tanh(\beta h)=\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\tanh[\beta(b+zJ\sqrt{q})]\ , \tag{3.46}$$

the internal energy

$$\frac{E}{N}=-\tfrac{1}{2}\int_{-\infty}^{\infty}dh\,P(h)(h+b)\tanh(\beta h)=-\frac{\beta J^2}{2}(1-q^2)-bm\ , \tag{3.47}$$

and the average local exchange field

$$\left\langle\sum_{i,j}J_{ij}\sigma_j\right\rangle=\int_{-\infty}^{\infty}dh\,P(h)(h-b)=\beta J^2(1-q)m\ . \tag{3.48}$$

Since replica symmetry is unbroken only for $T>T_{\rm AT}$ these results are valid only in that temperature region. We note, however, that if one (incorrectly) extrapolates (3.44) or (3.45) to zero temperature, one obtains the $T=0$ local mean-field distribution obtained by Schowalter and Klein[34]

$$P_{\text{Schowalter-Klein}}(h)=\begin{cases}(2\pi J^2)^{-1/2}\exp[-\frac{1}{2}(h-b+\Delta)^2/J^2], & h<-\Delta\\ 0, & |h|<\Delta\\ (2\pi J^2)^{-1/2}\exp[-\frac{1}{2}(h-b-\Delta)^2/J^2], & h>\Delta\end{cases} \tag{3.49}$$

where

$$\Delta=(2/\pi)^{1/2}J\exp[-\tfrac{1}{2}(b/J)^2]\ . \tag{3.50}$$

In view of the observation (3.12) it is tempting to wonder if $dP(0)/dT$ is equal to $P(0)/T$ at $T_{\rm AT}$ for all b. Explicit calculation demonstrates, however, that this is not the case.

(ii) Returning now to the case of $b=0$, but without restriction to replica symmetric $q^{(\alpha\beta)}$, $P(0)$ is given by

$$P(0)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}G(k)$$

$$=\frac{1}{(2\pi)^{1/2}J}\exp\left[-\frac{\beta^2J^2}{2}\right]\lim_{n\to0}\left[\frac{1}{2^n}\mathrm{Tr}_n\exp\left[(\beta J)^2\sum_{(\alpha\beta)}q^{(\alpha\beta)}\sigma^\alpha\sigma^\beta-(J/k_BT)^2(A+A^2/2)\right]\right]\ , \tag{3.51}$$

where

$$A=\sum_{\alpha\neq1}q^{(\alpha1)}\sigma^\alpha\sigma^1\ . \tag{3.52}$$

For $T>T_g$, $q^{(\alpha\beta)}=0$ and this immediately reproduces

$$P(0)=\frac{1}{\sqrt{2\pi}}\frac{1}{J}\exp(-\beta^2J^2/2)\ . \tag{3.11'}$$

For $T<T_g$ a careful treatment of the $q^{(\alpha\beta)}$ behavior is needed. For small $\tau=(T_g-T)/T_g$ perturbation analysis may be employed. Noting that to order τ, $q^{(\alpha\beta)}=\tau$, we obtain for $P(0)$ to order τ^4,

$$P(0)=\frac{1}{\sqrt{2\pi}}\frac{1}{J}\exp\left[-\frac{\beta^2J^2}{2}\right]\times\lim_{n\to0}\left[1-(\beta J)^4\left[\frac{3}{2}-\frac{(\beta J)^4}{2}\right]\sum_{\alpha\neq1}(q^{(\alpha1)})^2+\tfrac{13}{4}\tau^4+O(\tau^5)\right]\ . \tag{3.53}$$

To order τ^2 replica-symmetric theory suffices to give

$$P(0)=\frac{1}{\sqrt{2\pi}}\frac{1}{J}\exp(-\tfrac{1}{2})[1-\tau+O(\tau^3)]\ , \tag{3.54}$$

i.e., to demonstrate the absence of corrections to the linear extrapolation (3.26) to order τ^2. To proceed further we use the Parisi ansatz[6] within which

$$\lim_{n\to0}\left[\sum_{\alpha\neq\beta}f(q^{(\alpha\beta)})\right]=-\int_0^1dx\,f(q(x))\ , \tag{3.55}$$

where $q(x)$; $0\le x\le1$ is the Parisi function and $f(q)$ is any function.

To the order needed to get $P(0)$ correct to order τ^4, $q(x)$ is given by[35,36]

$$q(x)=\tfrac{1}{2}(1+3\tau)x+O(\tau^3),\quad x<x_1$$
$$=q(1),\quad x>x_1 \tag{3.56}$$

where

$$q(1)=\tau+\tau^2-\tau^3+O(\tau^4)\ , \tag{3.57}$$

$$x_1=2\tau-4\tau^2+O(\tau^3)\ , \tag{3.58}$$

so that

$$\lim_{n\to 0}\left[\sum_{\alpha\neq 1}(q^{(\alpha 1)})^2\right]=-\int_0^1 q^2(x)dx$$
$$=-\tau^2-\tfrac{2}{3}\tau^3+\tau^4+O(\tau^5)\ , \tag{3.59}$$

giving

$$P(0)=\frac{1}{\sqrt{2\pi}}\frac{1}{J}e^{-1/2}\left[1-\tau-\frac{\tau^4}{3}+O(\tau^5)\right]. \tag{3.60}$$

Thus, a Parisi ansatz removes corrections to (3.26) to order τ^3 but deviations from the linear extrapolation appear to order τ^4. We shall not attempt here to go to higher orders but it is clear from the Monte Carlo simulation that they must be compensatory to the τ^4 term to obtain the apparent linearity over the temperature range observed and to extrapolate to $P(0)=0$ at $T=0$.[37] We also note at this stage that below we demonstrate that the leading low-temperature behavior of $P(0)$ is linear in T.

(iii) We now turn to the general case of $h=0$. For greater completeness we allow for the inclusion of a constant external field b, but continue to restrict discussion to $\langle J_{ij}\rangle=0$ (although extension is straightforward). Thus, we use the generalized definition

$$P(h)=N^{-1}\sum_i\left\langle\delta\left[h-b-\sum_j J_{ij}\sigma_j\right]\right\rangle\ , \tag{3.61}$$

whence, as before,

$$P(h)=\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\exp[-ik(h-b)]G(k)\ , \tag{3.62}$$

where now,[38] averaging over the $\{J_{ij}\}$ distribution,

$$G(k)=\exp\left[-\frac{k^2J^2}{2}\right]\lim_{n\to 0}\left[\frac{1}{n}\frac{1}{2^n}\sum_{\gamma=1}^{n}\mathrm{Tr}_n\exp\left[k_B^2T^2J^2\sum_{(\alpha\beta)}q^{(\alpha\beta)}\sigma^\alpha\sigma^\beta+\frac{ikJ^2}{k_BT}\sum_{\alpha\neq\gamma}q^{(\alpha\gamma)}\sigma^\alpha-\frac{ikJ^2}{k_BT}\sigma^\gamma+\frac{b}{k_BT}\sum_\alpha\sigma^\alpha\right]\right]. \tag{3.63}$$

γ has been introduced to indicate the fact that the "test" replica is arbitrary.

Our further analysis is based on the Parisi ansatz,[6,39] in which the $n\times n$ matrix $q^{(\alpha\beta)}$ ($q^{(\alpha\alpha)}=0$) is first subdivided into $(n/m_1)^2$ $m_1\times m_1$ submatrices with $q^{(\alpha\beta)}=q_0$ in all but the n/m_1 submatrices along the principal diagonal of the original matrix, with then each of the $m_1\times m_1$ "diagonal" submatrices split further into $(m_1/m_2)^2$ $m_2\times m_2$ smaller matrices with $q^{(\alpha\beta)}=q_1$ in all but those on the diagonals of these $m_1\times m_1$ submatrices, and the subdivision procedure continued indefinitely in an analogous fashion with finally the limits

$$n\to 0\ ,$$
$$\frac{m_k}{m_{k+1}}\to 1-\frac{dx}{x}\ , \tag{3.64}$$

and

$$q_k\to q(x),\ \ 0\le x\le 1$$

being taken.[6]

In the present problem particular attention must be directed at each stage at the submatrix containing the $\gamma\gamma$ element. To illustrate this, consider first the lower level Parisi *ansatz* in which a single subdivision into $m_1\times m_1$ matrices is made. We start by explicitly removing the effects of the off-diagonal submatrices, writing

$$G(k)=\exp\left[-\frac{k^2J^2}{2}\right]\lim_{n\to 0}\left[\frac{1}{n}\frac{1}{2^n}\sum_\gamma\mathrm{Tr}\exp\left\{\frac{J^2}{2k_B^2T^2}q_0\left[\left[\sum_\alpha\sigma_\alpha\right]^2-n\right]+\frac{J^2}{k_B^2T^2}\sum_{(\alpha\beta)}(q^{(\alpha\beta)}-q_0)\sigma^\alpha\sigma^\beta\right.\right.$$
$$\left.\left.+\frac{ikJ^2}{k_BT}\sum_{\alpha\neq\gamma}q^{(\alpha\gamma)}\sigma^\alpha+\frac{ikJ^2}{k_BT}\sigma^\gamma+\frac{b}{k_BT}\sum_\alpha\sigma^\alpha\right\}\right] \tag{3.65}$$

so that now the second term in the exponential is nonzero only in the diagonal submatrices. Within the restricted ansatz being employed in this illustration, $q^{(\alpha\beta)}=q_1$ everywhere within these submatrices, so that

$$\sum_{(\alpha\beta)}(q^{(\alpha\beta)}-q^0)\sigma^\alpha\sigma^\beta\to\tfrac{1}{2}(q_1-q_0)\sum_D\left[\left[\sum_\alpha\sigma^\alpha\right]^2-m_1\right]\ , \tag{3.66}$$

where $\sum_D$ refers to a summation over the diagonal submatrices. Hence, using a Hubbard-Stratonovich transformation,[32]

$$G(k)=\exp\left[-\frac{k^2J^2}{2}\right]\int_{-\infty}^{\infty}\frac{dz_0}{\sqrt{2\pi}}\exp\left[-\frac{z_0^2}{2}\right]$$
$$\times\lim_{n\to 0}\left[\frac{1}{2^n}\left\{\mathrm{Tr}\exp\left[\frac{\beta^2J^2}{2}(q_1-q_0)\left[\sum_{\alpha=1}^{m_1}\sigma^\alpha\right]^2+\beta(b+z_0J\sqrt{q_0}+ikJ^2q_0)\sum_{\alpha=1}^{m_1}\sigma^\alpha\right]\right\}^{n/m_1-1}\right.$$
$$\times\left\{\mathrm{Tr}\exp\left[\frac{\beta^2J^2}{2}(q_1-q_0)\left[\sum_{\alpha=1}^{m_1}\sigma^\alpha\right]^2+\beta(b+z_0J\sqrt{q_0}+ikJ^2q_1)\sum_{\alpha=1}^{m_1}\sigma^\alpha\right.\right.$$
$$\left.\left.\left.+ik\beta J^2(1-q_1)\sigma^\gamma\right]\right\}\right], \tag{3.67}$$

where we have indicated explicitly the special character of the submatrix containing the $\gamma\gamma$ element of $q^{(\alpha\beta)}$. With further Hubbard-Stratonovich transformations there results

$$G(k)=\lim_{n\to 0}\left\{\exp\left[-\frac{k^2J^2}{2}\right]\int_{-\infty}^{\infty}\frac{dz_0}{\sqrt{2\pi}}\exp\left[-\frac{z_0^2}{2}\right]\right.$$
$$\times\left[\int_{-\infty}^{\infty}\frac{dz_2}{\sqrt{2\pi}}\exp\left[-\frac{z_2^2}{2}\right]\cosh^{m_1}\{\beta[b+z_0J\sqrt{q_0}+z_2J(q_1-q_0)^{1/2}+ikJ^2q_0]\}\right]^{n/m_1-1}$$
$$\times\left[\int_{-\infty}^{\infty}\frac{dz_1}{\sqrt{2\pi}}\exp\left[-\frac{z_1^2}{2}\right]\cosh^{m_1-1}\{\beta[b+z_0J\sqrt{q_0}+z_1J(q_1-q_0)^{1/2}+ikJ^2q_1)]\}\right.$$
$$\left.\left.\times\cosh\{\beta[b+z_0J\sqrt{q_0}+z_1J(q_1-q_0)^{1/2}+ikJ^2]\right]\right\}. \tag{3.68}$$

To obtain $P(h)$, variable shifts $z_0\to z_0-ikJ\sqrt{q_0}$, $z_1\to z_1-ikJ(q_1-q_0)^{1/2}$ are performed, followed by integration over k and the limit $n\to 0$, giving

$$P(h)=\int_{-\infty}^{\infty}dy\,P^1(y)\sum_{S=\pm 1}\frac{e^{S\beta y}}{2\cosh(\beta y)}\frac{1}{[2\pi J^2(1-q_1)]^{1/2}}\exp\left[-\frac{[h-b-S\beta J^2(1-q_1)-y]^2}{2J^2(1-q_1)}\right], \tag{3.69}$$

where

$$P^1(y)=\int_{-\infty}^{\infty}\frac{dz_0}{\sqrt{2\pi}}e^{-z_0^2/2}\int_{-\infty}^{\infty}\frac{dz_1}{\sqrt{2\pi}}e^{-z_1^2/2}\delta(y-b-z_0J\sqrt{q_0}-z_1J(q_1-q_0)^{1/2})$$
$$\times\cosh^{m_1}(\beta y)\left[\int_{-\infty}^{\infty}\frac{dz_2}{\sqrt{2\pi}}e^{-z_2^2/2}\cosh^{m_1}\{\beta[y+(z_2-z_1)(q_1-q_0)^{1/2}]\}\right]^{-1}. \tag{3.70}$$

The full Parisi ansatz[6] requires infinite subdivision. K subdivisions yield the approximation

$$G(k)=\lim_{n\to 0}\exp\left[-\frac{k^2J^2}{2}(1-q_K)\right]\int_{-\infty}^{\infty}\frac{dz_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dz_1}{\sqrt{2\pi}}\cdots\frac{dz_K}{\sqrt{2\pi}}$$
$$\times\exp\left[\sum_{i=0}^{K}\left[-\frac{z_i^2}{2}+ikJz_i(q_i-q_{i-1})^{1/2}-(m_{i+1}-m_i)\phi_i\right]\right]$$
$$\times 2\cosh\left[\beta\left[b+ikJ^2(1-q_K)+\sum_{i=0}^{K}z_iJ(q_i-q_{i-1})^{1/2}\right]\right], \tag{3.71}$$

where

$$q_{-1}=0\,, \tag{3.72}$$

$$m_{-1}=n,\quad m_{K+1}=1 \tag{3.73}$$

and

$$e^{m_i\phi_i}=\mathrm{Tr}_{m_{i+1}}\left\{\exp\left[(\beta J)^2\sum_{(\alpha\beta)}(q^{(\alpha\beta)}-q_i)S^\alpha S^\beta+\beta\left[b+\sum_{k=0}^{i}z_kJ(q_k-q_{k-1})^{1/2}\right]\sum_\alpha S^\alpha\right]\right\}. \tag{3.74}$$

In the continuous limit, therefore,

$$\begin{aligned}P(h)=&\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp\left[-ik(h-b)-\frac{(kJ)^2}{2}[1-q(1)]\right]\\&\times\int_{-\infty}^{\infty}\mathscr{D}z(x)\int_{-\infty}^{\infty}\frac{dz_0}{\sqrt{2\pi}}\exp\left[-\int_0^1dx\frac{[z(x)]^2}{2}-\frac{z_0^2}{2}+ikJz_0\sqrt{q(0)}+ikJ\int_0^1dx\,z(x)\sqrt{q'(x)}\right.\\&\left.\qquad-\int_0^1dx\,\phi\left[x,b+z_0J\sqrt{q_0}+J\int_0^x d\omega\,z(\omega)\sqrt{q'(\omega)}\right]\right]\\&\times 2\cosh\left[\beta\left[b+ikJ^2[1-q(1)]+z_0J\sqrt{q(0)}+J\int_0^1dx\,z(x)\sqrt{q'(x)}\right]\right],\end{aligned} \tag{3.75}$$

where $\phi(x,y)$ satisfies the differential equation[6]

$$\frac{\partial}{\partial x}\phi(x,y)=-\frac{J^2}{2}q'(x)\left[\frac{\partial^2}{\partial y^2}\phi(x,y)+x\left[\frac{\partial}{\partial y}\phi(x,y)\right]^2\right] \tag{3.76}$$

with the boundary condition

$$\phi(1,y)=\ln[2\cosh(\beta y)]\,. \tag{3.77}$$

$\mathscr{D}z(x)$ denotes the functional differential product

$$\mathscr{D}z(x)=\prod_x[dz(x)/\sqrt{2\pi}] \tag{3.78}$$

and the prime denotes differentiation with respect to the argument.

Equation (3.75) can be further transformed to express $P(h)$ simply in terms of the function $P(x,y)$ introduced by Sommers and Dupont.[5] To this end we first integrate by parts,

$$\begin{aligned}\int_0^1dx\,\phi\left[x,b+z_0J\sqrt{q_0}+J\int_0^x d\omega\,z(\omega)\sqrt{q'(\omega)}\right]=&\ln\left\{2\cosh\left[\beta\left[b+z_0J\sqrt{q_0}+J\int_0^x d\omega\,z(\omega)\sqrt{q'(\omega)}\right]\right]\right\}\\&-\int_0^1dx\,x\frac{\partial}{\partial x}\phi(x,y)\Big|_{y=b+z_0J\sqrt{q(0)}+J\int_0^x d\omega z(\omega)\sqrt{q'(\omega)}}\\&-J\int_0^1dx\,xz(x)\sqrt{q'(x)}\frac{\partial\phi(x,y)}{\partial y}\Big|_{y=b+z_0J\sqrt{q_0}+J\int_0^x d\omega z(\omega)\sqrt{q'(\omega)}}\end{aligned} \tag{3.79}$$

Substituting into (3.73), using the differential equation (3.76), and changing the functional integration variable to

$$\zeta(x)=z(x)-\beta Jx\sqrt{q'(x)}\frac{\partial\phi}{\partial y}(x,y)\Big|_{y=b+z_0J\sqrt{q_0}+J\int_0^x d\omega z(\omega)\sqrt{q'(\omega)}}, \tag{3.80}$$

$P(h)$ can be re-expressed as

$$P(h)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp\{-ik(h-b)-(kJ)^2[1-q(1)]/2\}$$
$$\times\int_{-\infty}^{\infty}\mathscr{D}\zeta(x)\int_{-\infty}^{\infty}\frac{dz_0}{\sqrt{2\pi}}\exp\left[-\int_0^1 dx\,[\zeta(x)]^2/2-z_0^2/2+ikJz_0\sqrt{q(0)}+ikJ\int dx\,z(x)\sqrt{q'(x)}\right]$$
$$\times\frac{\cosh\left[\beta(b+z_0J\sqrt{q(0)}+J\int_0^1 dx\,z(x)\sqrt{q'(x)}+ikJ^2[1-q(1)]\right]}{\cosh\left[\beta\left[b+z_0J\sqrt{q(0)}+J\int_0^1 dx\,z(x)\sqrt{q'(x)}\right]\right]}, \tag{3.81}$$

which by comparison with Ref. 5 leads to

$$P(h)=\int_{-\infty}^{\infty}dy\,P_{\rm SD}(1,y)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\exp\{-ik(h-b)-(kJ)^2[1-q(1)]/2\}\frac{\cosh(\beta\{y+ikJ^2[1-q(1)]\})}{\cosh(\beta y)}. \tag{3.82}$$

Here $P_{\rm SD}(x,y)$ is the Sommers-Dupont function[5,40] which satisfies

$$\frac{\partial}{\partial x}P_{\rm SD}(x,y)=\frac{J^2}{2}q'(x)\left[\frac{\partial^2}{\partial y^2}P_{\rm SD}(x,y)-2x\beta\frac{\partial}{\partial y}m(x,y)P_{\rm SD}(x,y)\right], \tag{3.83}$$

where

$$\frac{\partial}{\partial x}m(x,y)=-\frac{J^2}{2}q'(x)\left[\frac{\partial^2}{\partial y^2}m(x,y)+2x\beta m(x,y)\frac{\partial}{\partial y}m(x,y)\right], \tag{3.84}$$

with boundary conditions

$$m(1,y)=\tanh(\beta y), \tag{3.85}$$

$$P_{\rm SD}(0,y)=\frac{\exp[-(y-b)^2/2q(0)J^2]}{[2\pi q(0)]^{1/2}J}. \tag{3.86}$$

$P_{\rm SD}(x,y)$ has recently been interpreted[41] as the probability density (in y) of finding the local magnetization $m(x,y)$ for states with mutual overlap $q(x)$.[30]

Performing the k integration in Eq. (3.82) finally yields

$$P(h)=\int_{-\infty}^{\infty}dy\,P_{\rm SD}(1,y)\frac{\cosh(\beta h)}{\cosh(\beta y)}\frac{\exp\{-(h-y)^2/2J^2[1-q(1)]\}}{\{2\pi J^2[1-q(1)]\}^{1/2}}\exp\{-(\beta J)^2[1-q(1)]/2\}. \tag{3.87}$$

In fact a simple extension gives the joint local-field, local-spin distribution

$$P(h,s)=N^{-1}\sum_l\left\langle\delta_{s,\sigma_l}\delta\left[h-b-\sum_j J_{lj}\sigma_j\right]\right\rangle \tag{3.88}$$

as

$$P(h,s)=\int_{-\infty}^{\infty}dy\,P_{\rm SD}(1,y)\frac{e^{\beta sy}}{\cosh(\beta y)}\frac{\exp(-\{h-y-s\beta J^2[1-q(1)]\}^2/2J^2[1-q(1)])}{\{2\pi J^2[1-q(1)]\}^{1/2}}. \tag{3.89}$$

$P(h)$ is related to $P(h,s)$ by

$$P(h)=\sum_{s=\pm1}P(h,s). \tag{3.90}$$

The established identities[5,6,26,41] for magnetization, internal energy, and mean local field follow immediately:

$$m=\int_{-\infty}^{\infty}dh\,P(h)\tanh(\beta h)=\int_{-\infty}^{\infty}dy\,P_{\rm SD}(1,y)\tanh(\beta y), \tag{3.91}$$

$$\frac{E}{N}=-\tfrac{1}{2}\int_{-\infty}^{\infty}dh\,P(h)(h+b)\tanh(\beta h)$$
$$=-\frac{\beta J^2}{2}\left[1-\int_0^1 dx\,[q(x)]^2\right]-bm, \tag{3.92}$$

$$N^{-1}\sum_{i,j}J_{ij}\langle\sigma_j\rangle=\int_{-\infty}^{\infty}dh\,P(h)(h-b)$$
$$=\beta J^2\left[1-\int_0^1 dx\,q(x)\right]m$$
$$=\beta J^2[1-q(1)]m$$
$$+\int_{-\infty}^{\infty}dy\,P_{\rm SD}(1,y)(y-b). \tag{3.93}$$

At low temperatures $\beta J^2[1-q(1)]$ is of $O(T)$. Hence, as $T\to0$, $P(h)\to P_{\rm SD}(1,h)$. At zero temperature $P_{\rm SD}(x,y)$ has been evaluated numerically at a number of x values, including $x=1$, by Sommers and Dupont.[5] They find that $P(1,y)$ at $T=0$ has a $y\to0$ slope of 0.31 ± 0.01, in reasonable accord with the simulations of Palmer and Pond[24] and ourselves and with the speculations of Thouless *et al.*[12]

Furthermore, it follows from (3.87) and other properties of $P_{\mathrm{SD}}(x,y)$ that

$$P(h=0,\,T)=\lambda T+O(T^2)\;, \tag{3.94}$$

a result which we stress is valid for arbitrary external field. To see this, we note that

$$\beta J^2[1-q(1)]=\alpha T+O(T^2) \tag{3.95}$$

leads to

$$P(0)=T\lim_{T\to 0}\left[\int_{-\infty}^{\infty}d\tilde{y}\frac{P(1,\tilde{y}T)}{T}\frac{\exp(-\tilde{y}^2/2\alpha)}{\sqrt{2\pi\alpha}}\times\frac{\exp(-\alpha/2)}{\cosh\tilde{y}}+O(T^2)\right]\;, \tag{3.96}$$

while, furthermore, the marginal stability condition[5,26,42]

$$1=(\beta J)^2\int_{-\infty}^{\infty}dy\,P(1,y)\,\mathrm{sech}^4(\beta y)\;, \tag{3.97}$$

yields

$$\lim_{T\to 0}\left[\int_{-\infty}^{\infty}d\tilde{y}\frac{P(1,\tilde{y}T)}{T}\mathrm{sech}^4\tilde{y}\right]=1\;. \tag{3.98}$$

Hence, $\lim_{T\to 0}[P(1,\tilde{y}T)/T]$ must be well defined in the limit $T\to 0$,

$$\alpha=\lim_{T\to\infty}\left[J^2\int d\tilde{y}\frac{P(1,\tilde{y}T)}{T}\mathrm{sech}^2\tilde{y}\right]>1\;, \tag{3.99}$$

and the coefficient λ of the linear T term in (3.94) must be a positive constant. We have not, however, determined the numerical value of λ.

The expression (3.87) also suffices to put bounds on $P(0)$ in the spin-glass phase in terms of lesser Parisi information. For example,[43]

$$P(0)=\int_{-\infty}^{\infty}dy\frac{P(1,y)}{\cosh^2(\beta y)}\left[\frac{1}{2}\sum_{s=\pm 1}\frac{\exp(-\{y-s\beta J^2[1-q(1)]^2\}/2J^2[1-q(1)])}{\{2\pi J^2[1-q(1)]\}^{1/2}}\right]\le\{[1-q(1)]/2\pi J^2\}^{1/2}\;. \tag{3.100}$$

(iv) Finally, in this section, we return briefly to the question of self-averaging in view of the current interest in non-self-averaging,[44] ultrametricity,[17] etc., in spin glasses. We have already indicated that $P(h)$ is self-averaging, in accord with the observation that quantities such as the internal energy and the magnetization are self-averaging even in spin glasses.[31]

Non-self-averaging field distributions involve replica overlaps, such as,

$$P(Q,\{J_{ij}\})=\left\langle\delta\left[Q-N^{-1}\sum_i h_i^1h_i^2\right]\right\rangle_{\{J_{ij}\}}\;, \tag{3.101}$$

where the 1,2 superscripts refer to two *real* replicas of the system with identical sets of exchange bonds $\{J_{ij}\}$. $\langle\cdots\rangle_{\{J_{ij}\}}$ refers to a thermodynamic average against

$$\mathscr{H}\{J_{ij}\}=\mathscr{H}^1\{J_{ij}\}+\mathscr{H}^2\{J_{ij}\}=-\sum_{(ij)}J_{ij}(\sigma_i^1\sigma_j^1+\sigma_i^2\sigma_j^2)-b\sum_i(\sigma_i^1+\sigma_i^2)\;, \tag{3.102}$$

where the $\{J_{ij}\}$ are a *particular set* of exchange bonds. $P(Q,\{J_{ij}\})$ is not self-averaging, that is

$$\langle[P(Q,\{J_{ij}\})]^k\rangle_{\{J_{ij}\}}\neq[\langle P(Q,\{J_{ij}\})\rangle_{\{J_{ij}\}}]^k\;. \tag{3.103}$$

The origin of this effect is the same as that for the Parisi overlap distribution function[30]

$$P(q,\{J_{ij}\})=\left\langle\delta\left[q-N^{-1}\sum_i\sigma_i^1\sigma_i^2\right]\right\rangle_{\{J_{ij}\}}\;, \tag{3.104}$$

namely, the *real-replica* coupling which occurs in the effective averaged Hamiltonian when $\{J_{ij}\}$ is averaged over. It may be evaluated by extension of the above and Refs. 17 and 30. Further field analogues of conventional magnetization overlap distributions can be envisaged readily, but will not be pursued here.

IV. FINITE-RANGED ISING SPIN-GLASS MODELS

In view of the results in the preceding section on the SK model, we were motivated to investigate $P(h)$ for some short-ranged models, with particular regard to any analogue of the linearity conjecture for $P(0)$ and the temperature for flattening of $P(h)$ near $h=0$. These studies used Monte Carlo simulations on simple-square and -cubic lattices, again taking a distribution $p(J_{ij})=\frac{1}{2}[\delta(J_{ij}-J)+\delta(J_{ij}+J)]$ because of its bit storage advantages. We note that it has been suggested that this model should exhibit a phase transition in 3D[7–10] at $T_g\sim 1.2$ J while in 2D, T_g is believed to be zero.[45]

Results for $P(h)$ in 2D and 3D are shown in Figs. 6 and 7 where there are plotted the weights[46] w_s of $P(h)$ where $P(h)=\sum_s w_s\delta(h-s)$; weights for $-s$ are the same as those for $+s$. There appeared to be no significant sample-to-sample variations, confirming our argument

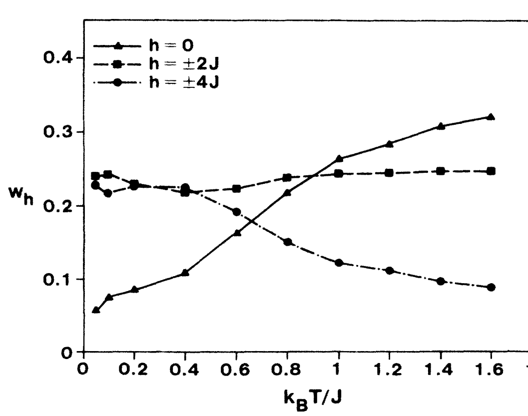

FIG. 6. Weights of $\delta(h-s)$ for the $\pm J$ nearest-neighbor Ising spin glass on a square net.

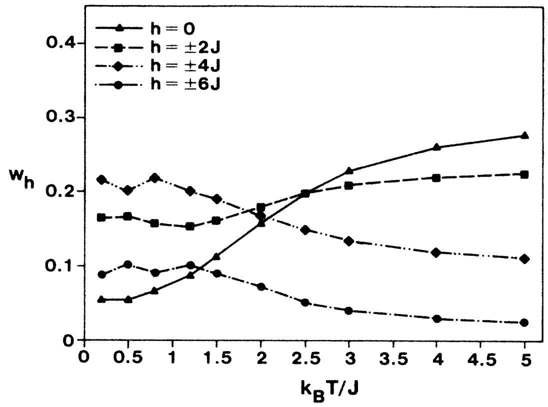

FIG. 7. Weights of $\delta(h-s)$ for the $\pm J$ nearest-neighbor Ising spin glass on a simple-cubic lattice.

that $P(h)$ should be self-averaging in the thermodynamic limit.[47]

Several points are of note. First, there is much qualitative similarity between the $P(h)$ results for 2D and 3D, despite their different transition characters. Second, both curves show regions of roughly linear $P(0)$ versus T behavior, although not extending down to $T=0$ (Ref. 48) and with their upper "limits" beyond the believed transition temperatures. Third, although in 3D $P(0)$ and $P(6)$ cross tantalizingly close to the probable transition temperature, $P(0)$, $P(2)$, and $P(4)$ become approximately equal at a significantly higher temperature[49] while a similar "flattening" occurs in 2D at a finite temperature of order 0.8 J.

It would thus appear that the correlation in the SK model of the flattening of $P(h)$ around $h=0$, the onset of an apparent freezing of $dP(0)/dT$ and linear extrapolation of $P(0)$, and the onset of ultrametricity with the phase transition temperature T_g may be special features of this infinite-ranged model. On the other hand, however, it is interesting to note that both in 2D and in 3D the temperatures at which quasilinearity of $P(0)$ commences and at which $P(0)$ and $P(2)$ become equal are comparable with one another.[49]

V. CONCLUSIONS

In this paper we have extended our studies of the local-field distribution $P(h)$ to disordered spin systems, restricted in the main text to Ising models but extendable to classical vector spins (see the Appendix), concentrating analytically on systems on Bethe lattices and the Sherrington-Kirkpatrick infinite-ranged spin glass. Further Monte Carlo simulation studies were performed on short-ranged spin glasses.

In all of the cases studied $P(h)$ is Gaussian at high temperatures and has $P(0)\to 0$ as $T\to 0$. In the case of the SK model in zero applied field we find that at $T=T_g$, the spin-glass ordering temperature, $P(h)$ becomes flat around $h=0$, $dP(0)/dT=P(0)/T$, and to a good approximation, $dP(0)/dT$ freezes to its $T=T_g$ value for all temperatures below T_g.[50] $P(h)$ is obtained analytically in terms of the Sommers-Dupont function $P_{SD}(x,y)$ but to date only numerical solution of that function has proven possible.

The Monte Carlo studies of the short-ranged spin-glass systems suggest that quasifreezing of $dP(0)/dT$ to a value leading to extrapolation of $P(0)$ to zero at $T=0$ is correlated with flattening of $P(h)$ around $h=0$ irrespective of whether or at what temperature a thermodynamic phase transition occurs.

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APPENDIX: VECTOR SPIN GLASS

In this appendix, we note analogous results to those of Sec. III for a classical m-vector SK spin glass, restricting discussion for simplicity to the case of zero external field and $T>T_g$. The model has Hamiltonian

$$\mathscr{H}=-\tfrac{1}{2}\sum_{i,j}J_{ij}\hat{\mathbf{S}}_i\cdot\hat{\mathbf{S}}_j \tag{A1}$$

where the $\hat{\mathbf{S}}$ are m-dimensional unit vectors and J_{ij} is distributed as in (3.3). The quickest way to proceed is to begin with the vector generalization of Eq. (2.5),

$$P_i(\mathbf{h})=\langle\delta(\mathbf{h}-\mathbf{h}_i)\rangle' Z_1(\beta h)\frac{\mathrm{Tr}'e^{-\beta H'}}{\mathrm{Tr}e^{-\beta H}}\ . \tag{A2}$$

Since $\langle\cdots\rangle'$ represents a thermal average with respect to

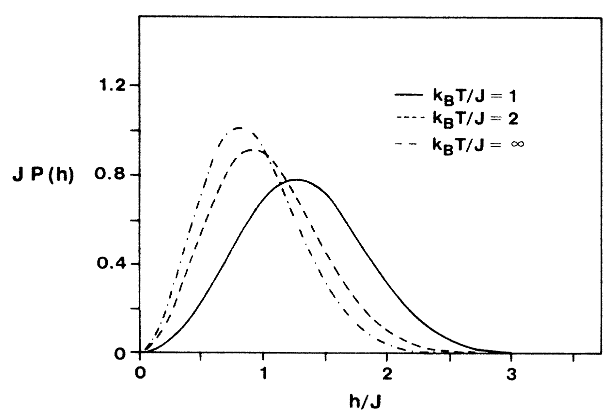

FIG. 8. Local-field distribution $P(h)=4\pi h^2P(\mathbf{h})$ for the symmetric Heisenberg Sherrington-Kirkpatrick spin glass for $T>T_g=J/3k_B$, as obtained from Eq. (A6) with $m=3$.

a Hamiltonian in which all the bonds into site i have been removed, then for $T > T_g$,

$$\langle \delta(\mathbf{h}-\mathbf{h}_i)\rangle' = P(\mathbf{h}, T=\infty) = \left[\frac{m}{2\pi J^2}\right]^{m/2} e^{-h^2 m/2J^2} \tag{A3}$$

The last step follows from considering the addition of N randomly oriented m-dimensional vectors. $Z_1(\beta h)$ is, as before, the paramagnetic partition function

$$Z_1(\beta h) = (2\pi)^{m/2}(\beta h)^{1-m/2} I_{m/2-1}(\beta h) \ . \tag{A4}$$

Finally, if we let F_N denote the free energy of an N-particle m-vector SK spin glass, then we can write

$$\mathrm{Tr}'[e^{-\beta H'}]/\mathrm{Tr}[e^{-\beta H}] = e^{\beta(F_N - F_{N-1})} = \frac{1}{2}\frac{\Gamma(\frac{1}{2}m)}{\pi^{m/2}} e^{-\beta^2 J^2/2m} \tag{A5}$$

with corrections of order $1/N$.

Equations (A3)—(A5) make up the components of (A2), and thus,

$$P(\mathbf{h}) = \frac{1}{2}\left[\frac{m}{\pi J^2}\right]^{m/2} \Gamma(\tfrac{1}{2}m)\exp(-\beta^2 J^2/2m) \times \exp(-h^2 m/2J^2)(\beta h)^{1-m/2} I_{m/2-1}(\beta h) \ . \tag{A6}$$

To illustrate this result, we show in Fig. 8, $P(h) = 4\pi h^2 P(\mathbf{h})$ for the Heisenberg ($m=3$) case.

We note that for small h,

$$P(\mathbf{h}) = \left[\frac{m}{2\pi J^2}\right]^{m/2} e^{-\beta^2 J^2/2m} \times \left[1 - \frac{mh^2}{2J^2}\left[1 - \frac{\beta^2 J^2}{m^2}\right] + O(h^4)\right] \tag{A7}$$

and hence the coefficient of the h^2 term vanishes at $k_B T_g = J/m$ as in the SK Ising spin glass.

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q_1, x_1 oscillate in sign and have coefficients growing in magnitude suggesting that the expansion may only be asymptotic (see Ref. 35).

[38]We have replaced label 1 by γ and inserted $n^{-1}\sum_\gamma$ to indicate that the explicit label could be any of the set of n.

[39]See also C. de Dominicis, in *Proceedings of the Heidelberg Colloquium on Spin Glasses,* Ref. 14, p. 103.

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[48]As emphasized earlier, however, low-temperature results suffer from equilibration problems and for the SK model were found to be too high in our limited simulations.

[49]N. Sourlas (private communication) has also made this observation.

[50]This result and a preliminary account of some of the results of this paper are given in *Magnetic Excitations and Fluctuations,* edited by S. W. Lovesey, U. Balucani, F. Borsa, and V. Tognetti (Springer-Verlag, Berlin, 1984).