Gauge-symmetry breaking and phase transitions in strongly and weakly interacting Bose systems

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(Received 3 June 1985)

A renormalizable parameter ξ ($0 \leqq \xi \leqq 1$) is introduced into the interacting Bose-system Hamiltonian such that the zero-mode amplitudes satisfy the commutation relation $[b_0, b_0^\dagger] = 1 - \xi$. The different phases of the system correspond to specific values of ξ. The normal component (phase I) is obtained when $\xi = 0$ and there are two ordered states according to the strength of the interaction: Phase II′ ($0 < \xi \ll 1$) is found in the strongly interacting system and is characterized by a broken gauge symmetry without Bose-Einstein condensation, while phase II″ ($\xi = 1$), in the weakly interacting system, displays Bose-Einstein condensation. The continuity equation for the particle density holds in both ordered phases to within an infinitesimal term. A gauge transformation on the field operators shows that ξ "twists" the phase angle related to the zero mode, and a response to this twisting field is defined in terms of the generalized force $\chi(T)$ associated with ξ. For the ideal Bose gas $\chi(T)$ equals the condensate density and hence is simply related to the helicity modulus introduced by Fisher *et al.* A renormalization-group transformation shows that the dense system transition (I-II′) gives rise to a two-thirds correlation-length critical exponent, whereas a unity crossover exponent appears in the dilute system transition (I-II″), in agreement with the experimental findings.

I. INTRODUCTION

The renormalization-group approach to critical phenomena of classical systems introduced by Wilson[1] was extended by Singh to the study of the normal[2] and condensed[3] phases of a Bose system. Singh found that both phases display a transition analogous to that of a classical two-component Heisenberg model, with a correlation length critical exponent $\nu = 0.6$ ($d = 3$) to order $\epsilon = 4 - d$, d being the system dimensionality. This "two-thirds" power law is a consequence of the complex nature of the boson field as demonstrated by Fisher, Barber, and Jasnow.[4] These authors have also shown that this exponent becomes equal to unity in the case of the ideal Bose gas. The two-thirds and the linear power laws have been confirmed by experiments performed with dense[5] and dilute[6] ^{4}He systems, respectively.

It has been shown recently[7] that Singh's treatment of the normal phase does not lead to Bose-Einstein condensation as the transition temperature T_c is approached. For condensation to occur, a crossover exponent $\nu_0 = 1$ appears which is associated with the condensate correlation length. Moreover, a critical exponent $\nu_1 = 0.6$ is seen to be associated with the noncondensate. It looks as if two mechanisms compete in the transition of the normal phase: the complex nature of the boson field characterized by the two-thirds power law and the Bose-Einstein condensation connected with the linear power law. The former prevails in dense, strongly interacting systems whereas the latter in dilute, weakly interacting systems.

The natural subsequent investigation should concern the condensed side of the transition, the critical behavior as the temperature T approaches T_c from below. This problem poses a preliminary question on how the condensed phase is described. The two usual procedures to treat condensed bosons are due to Bogoliubov. Firstly, the well-known Bogoliubov prescription[8] that amounts to the replacement of the $k = 0$ Bose amplitudes, a_0 and $a_0^\dagger$, by the c-number $(N_0)^{1/2}$, N_0 being the number of particles in the $k = 0$ state. Secondly, the introduction of a gauge-symmetry breaking term into the Hamiltonian[9] by means of a fictitious infinitesimal field coupled to the amplitudes a_0 and $a_0^\dagger$. Slight variations of these methods and even a combination of both are found in the literature.[10–15] The Bogoliubov prescription neglects fluctuations in the condensate occupation number altogether, and hence it is appropriate for the description of dilute systems only. However, the validity of such conservation laws as, e.g., the continuity equation, is difficult to ensure.[15] In contrast, Bogoliubov's second method has the advantage of preserving the operator nature of the zero-mode amplitudes, thus taking into account microscopic fluctuations of the condensate. Furthermore, the continuity equation is satisfied to within an infinitesimal term.[15] Based on an analogy with magnetic systems, this scheme has, nevertheless, the drawback concerning a physical interpretation of the external field.

A common feature in both Bogoliubov's approaches resides in the breaking of the gauge symmetry which is the fundamental distinction between the normal and condensed phases. In Sec. II a new Hamiltonian $H(\xi)$ is defined in terms of a real external parameter ξ. The normal phase corresponds to $\xi = 0$ and the phase with broken gauge symmetry to $\xi > 0$. It is shown that the continuity equation is satisfied (to within an infinitesimal term) and that the parameter ξ produces a phase "twist"[4] of the $k = 0$ mode. We also show that the limit $\xi \to 1$ corresponds to the Bogoliubov prescription. In Sec. III the renormalization-group (RG) transformation is used to determine the critical value of ξ. The ξ ensemble is worked out in Sec. IV for the ideal Bose gas. We conclude with a discussion in Sec. V.

II. THE MODEL

Consider a system of bosons of mass m enclosed in a volume $V=L^d$ and interacting via a two-body potential $U(x)$. The model Hamiltonian is given by

$$H=-\frac{\hbar^2}{2m}\int d^dx\,\psi^\dagger\nabla^2\psi+\frac{u_0}{4}\int d^dx\,|\psi|^4\,, \tag{2.1}$$

where $u_0=2U(k=0)$ is the interaction constant and the field operators are

$$\begin{aligned}\psi&=\psi_0+\psi_1\,,\\ \psi_0&=V^{-1/2}b_0\,,\\ \psi_1&=V^{-1/2}\sum_{k\neq 0}a_k e^{ikx}\,,\end{aligned} \tag{2.2}$$

where the zero-mode amplitude is defined by

$$b_0=fa_0+g(N_0)^{1/2}\,, \tag{2.3}$$

together with the following condition on the ensemble averages,

$$\langle a_0\rangle=\langle a_0^\dagger\rangle=0\,. \tag{2.4}$$

The operators a_k and $a_k^\dagger$, including the case $k=0$, are the usual Bose annihilation and creation operators, and the parameters f and g are real and non-negative. The normal phase corresponds to $f=1$ and $g=0$, and the Bogoliubov prescription to $f=0$ and $g=1$. To circumvent the zero f value, which amounts to neglecting the dynamical behavior of the $k=0$ particles, Szépfalusy[14] defines a canonical transformation in terms of $f=g=1$.

However, from the observation that (2.3) and (2.4) yield the relation

$$\langle b_0^\dagger b_0\rangle=f^2\langle a_0^\dagger a_0\rangle+g^2N_0\,, \tag{2.5}$$

we shall interpret the parameters f and g as being probability amplitudes so that

$$f^2+g^2=1\,. \tag{2.6}$$

Accordingly, we introduce a real parameter ξ that can take on values in the interval [0,1], and define

$$f=\sqrt{1-\xi},\quad g=\sqrt{\xi}\,. \tag{2.7}$$

In this context the normal phase and the Bogoliubov prescription correspond to the extreme values of the range. On the other hand, Szépfalusy's transformation does not fit into this scheme.

From (2.2), (2.3), and (2.7), the basic commutation relations of the Bose amplitudes and fields are modified according to

$$[b_0,b_0^\dagger]=1-\xi\,, \tag{2.8}$$

$$[\psi(x,t;\xi),\psi^\dagger(x',t;\xi)]=\delta^d(x-x')-\xi V^{-1}\,. \tag{2.9}$$

Now we proceed to determine the continuity equation of the system. The number-density and the total-number operators are given, respectively, by

$$\begin{aligned}\hat{n}(x,t;\xi)&=\psi^\dagger(x,t;\xi)\psi(x,t;\xi)\,,\\ \hat{N}(\xi)&=\int d^dx\,\hat{n}(x,t;\xi)\,.\end{aligned} \tag{2.10}$$

From (2.1), (2.9), and (2.10) one can determine the commutator between the Hamiltonian and the total-number operator,

$$[H(\xi),\hat{N}(\xi)]=\xi\mu_0\int d^dx\,I(x,t;\xi)\,, \tag{2.11}$$

with

$$\begin{aligned}I(x,t;\xi)=\frac{1}{2V}\int d^dx'[&\psi^\dagger(x,t;\xi)\,|\psi(x',t;\xi)|^2\psi(x',t;\xi)\\ &-\psi^\dagger(x',t;\xi)\,|\psi(x',t;\xi)|^2\\ &\times\psi(x,t;\xi)]\,,\end{aligned} \tag{2.12}$$

and, likewise, the equation of motion for the number-density operator,

$$\partial_t\hat{n}(x,t;\xi)+\nabla\cdot\hat{J}(x,t;\xi)=\frac{i}{\hbar}\xi\mu_0 I(x,t;\xi)\,, \tag{2.13}$$

where $\hat{J}(x,t;\xi)$ is the current operator. By assuming, in general, that $I(x,t;\xi)\neq 0$ in V, Eq. (2.13) reveals that the continuity equation will be satisfied to within an infinitesimal term so long as ξu_0 is infinitesimal. Hence, in dense systems where u_0 is supposedly large, ξ must be an infinitesimal parameter $(0<\xi\ll 1)$. On the other hand, for dilute systems ξ need not be infinitesimal. In particular, the ideal Bose gas satisfies the continuity equation whatever the magnitude of ξ.

Now, we consider a gauge transformation on the field (2.2), namely,

$$e^{-i\theta\hat{N}(\xi)}\psi(\xi)e^{i\theta\hat{N}(\xi)}=\psi_0(\xi)e^{i(1-\xi)\theta}+\psi_1 e^{i\theta}\,. \tag{2.14}$$

It is well known that the motion of a quantal fluid may be described by a gauge transformation, $\psi\to\psi e^{i\theta}$, where the phase angle θ is related to the velocity of the fluid, i.e., $v=(\hbar/m)\nabla\theta$. From (2.14) we find that the velocity of the $k=0$ particles, v_0, differs from the velocity of the $k\neq 0$ particles, v_1, that is

$$\begin{aligned}v_1&=(\hbar/m)\nabla\theta\,,\\ v_0&=(1-\xi)v_1\,.\end{aligned} \tag{2.15}$$

Fisher *et al.*[4] have introduced the concept of a helicity modulus, $\Upsilon(T)$, which measures the incremental free energy resulting from an imposed phase "twist" of the order parameter. In the same spirit, one may interpret $-\xi\theta$ in (2.14) as a "twisting" phase angle with respect to the overall phase. As $\Upsilon(T)$ is a response to an associated twisting field, one may similarly introduce a response in terms of the generalized force that is coupled to the parameter ξ. Thus, let $K(\xi)$ be the dimensionless grand-canonical Hamiltonian

$$K(\xi)=\beta[H(\xi)-\mu\hat{N}(\xi)] , \tag{2.16}$$

where μ is the chemical potential and $\beta=1/k_BT$ is the inverse temperature.

From (2.1), (2.2), and (2.16), the generalized force per unit volume, $\chi(T,V,\mu,\xi)$, is equal to

$$\begin{aligned}\chi(T,V,\mu,\xi)&=\frac{-1}{V}\left\langle\frac{\partial K(\xi)}{\partial\xi}\right\rangle\\&=\frac{\beta\mu}{V}(N_0-\langle a_0^\dagger a_0\rangle)+\frac{\beta u_0}{4V}\int d^dx\left[(1-\xi)^{-1/2}\left\langle\frac{a_0^\dagger}{\sqrt{V}}|\psi|^2\psi+\psi^\dagger|\psi|^2\frac{a_0}{\sqrt{V}}\right\rangle\right.\\&\qquad\left.-\xi^{-1/2}(n_0)^{1/2}(n_0)^{1/2}\langle|\psi|^2\psi+\psi^\dagger|\psi|^2\rangle\right] ,\end{aligned} \tag{2.17}$$

where the ensemble averages refer to the grand-canonical ensemble specified by (2.16). In Sec. IV we will find a simple relation between $\Upsilon(T)$ and $\chi(T)$ for the ideal Bose gas.

We finally remark that although a more general expression for χ can be easily deduced in terms of $f(\xi)$ and $g(\xi)$, together with their derivatives $f'(\xi)$ and $g'(\xi)$, Eq. (2.17) depends on the particular choice (2.7). In contrast, the parameter ξ that appears explicitly in the continuity equation (2.13) and, also, in the gauge transformation (2.14), comes solely from the operator part of (2.3), viz., from the replacement $1-f^2\to\xi$. Equations similar in from to (2.13) and (2.14) would have resulted if we let $0<f<1$ and $g=0$ at the outset, a fact that emphasizes the relevance of allowing for microscopic fluctuations of the zero-mode particles.

III. RENORMALIZATION-GROUP TRANSFORMATION

To determine the critical behavior of the system it is more convenient to perform a RG transformation[1] on the Hamiltonian (2.16) expressed in terms of the Bose amplitudes a_k and $a_k^\dagger$.[2,3] Elimination of the high-frequency modes is accomplished by factorizing the Hilbert space as $h_0\otimes h_1$, where h_0 and h_1 are the subspaces on which the Bose amplitudes operate within the ranges $0\leqq|k|\leq\Lambda/\zeta$ and $\Lambda/\zeta<|k|\leqq\Lambda$, respectively. The parameter ζ is a scale factor such that $\zeta>1$, and Λ denotes the intrinsic momentum cutoff of the order of the thermal momentum

$$\lambda_T^{-1}=(2\pi\hbar^2/mk_BT)^{-1/2} .$$

As $b_0(\xi)$ and $b_0^\dagger(\xi)$ operate on h_0, the effective low-frequency Hamiltonian is identical to the one found previously by Singh,[2] upon the replacement $a_0\to b_0(\xi)$ and $a_0^\dagger\to b_0^\dagger(\xi)$. Next, in order to take into account the ordered states, we apply the scale changes introduced in an earlier paper,[7] namely,

$$k'=\zeta k,\quad x'=\zeta^{-1}x , \tag{3.1}$$

$$a_k=a'_{k'},\quad a_k^\dagger=(a'_{k'})^\dagger , \tag{3.2}$$

$$b_0(\xi)=\zeta^{y/2}b'_0(\xi'),\quad b_0^\dagger(\xi)=\zeta^{y/2}(b'_0)^\dagger(\xi') . \tag{3.3}$$

where y is a non-negative real parameter and

$$b'_0(\xi')=(1-\xi')^{1/2}a_0+(\xi'N'_0)^{1/2} .$$

It has been shown[7] that (3.1)–(3.3) reproduce the same recursion relations obtained by Singh,[2] except that the renormalized mass m'_0 of the $k=0$ particles now differs from the mass of the $k\neq0$ particles, m'_1, i.e.,

$$m'_0=\zeta^{-y+2-\eta}m,\quad m'_1=\zeta^{2-\eta}m , \tag{3.4}$$

with $\eta=O(u_0^2)$. The parameter y can take on two values only, according to whether or not the transition exhibits Bose-Einstein condensation,[7]

$$y=\begin{cases}0 & \text{without Bose-Einstein condensation,}\\ d>2 & \text{with Bose-Einstein condensation .}\end{cases} \tag{3.5}$$

According to (3.4) and (3.5), Bose-Einstein condensation implies fixed-point masses $m_0^*=0$ and $m_1^*=\infty$.

The renormalized version of (2.8) obviously reads

$$[b'_0(\xi'),(b'_0)^\dagger(\xi')]=1-\xi' , \tag{3.6}$$

and from (2.2) and (3.3) we obtain the central result

$$\xi'=1-(1-\xi)\zeta^{-y} . \tag{3.7}$$

The fixed-point parameter ξ^* follows from (3.5) and (3.7),

$$\xi^*=\begin{cases}\xi & \text{without Bose-Einstein condensation} \quad (3.8)\\ 1 & \text{with Bose-Einstein condensation .} \quad (3.9)\end{cases}$$

We notice that (3.8) and (3.9) hold for both phases, $\xi=0$ and $\xi>0$.

Finally, we show next that the arguments given in Sec. III of Ref. 7, concerning the Bose-Einstein condensation as the critical temperature is approached from above ($\xi=0$), may be extended to the case when T_c is reached from below ($\xi>0$). After the scale change [Eqs. (3.1)–(3.3)] the total particle density $n=\langle\hat{N}\rangle/V$, and the order parameter $\langle\psi\rangle$ become

$$n=\zeta^{y-d}\left[(1-\xi')\frac{\langle a_0^\dagger a_0\rangle}{V'}+\xi'n'_0\right]+\zeta^{-d}\sum_{k\neq0}\frac{\langle a_k^\dagger a_k\rangle}{V'}e^{ik'x'},$$
$$\langle\psi\rangle=\zeta^{(y-d)/2}\langle\psi'\rangle,\quad\langle\psi'\rangle=(\xi'n'_0)^{1/2} . \tag{3.10}$$

At the vicinity of the Bose-Einstein transition point one has $\zeta \gg 1$, $y=d$, and $\xi' \cong 1$, so that Eq. (3.10) gives

$$n \cong n_0', \quad \langle \psi \rangle = (n_0')^{1/2} . \tag{3.11}$$

This result represents a generalization of the previous treatment[7] because it holds whether $T \to T_c^+$ or $T \to T_c^-$. Moreover, the expression of the order parameter in (3.11) was derived without the usual recourse $\langle a_0 \rangle \to (N_0)^{1/2}$.

IV. IDEAL BOSE GAS

It is instructive to work out the model described in Sec. II for the ideal Bose gas. From (2.1)–(2.3) the corresponding dimensionless grand-canonical Hamiltonian reads

$$K_0(\xi) = -\beta\mu b_0^\dagger b_0 + \sum_{k\neq 0} \beta(\epsilon_k^0 - \mu) a_k^\dagger a_k \tag{4.1}$$

$$= K_0(\xi=0) - \xi\beta\mu(N_0 - a_0^\dagger a_0)$$

$$-\beta\mu[(1-\xi)\xi N_0]^{1/2}(a_0 + a_0^\dagger) , \tag{4.2}$$

where $\epsilon_k^0 = \hbar^2 k^2/2m$ and $K_0(\xi=0)$ is the usual ideal-gas Hamiltonian. Of special interest is the generalized force (2.17), which by taking into account Eqs. (2.5) and (2.7), becomes

$$\chi_0(T,V,\mu,\xi) = \beta\mu(N_0 - \langle b_0^\dagger b_0 \rangle)/(1-\xi)V . \tag{4.3}$$

It is straightforwad to calculate the contraction in (4.3) from the Hamiltonian (4.1), namely,

$$\langle b_0^\dagger b_0 \rangle = (1-\xi)(e^{-(1-\xi)\beta\mu} - 1)^{-1} . \tag{4.4}$$

In order to see the consistency of this result we take the limits $\xi \to 0$ and $\xi \to 1$ in both Eqs. (2.5) and (4.4), which give, respectively,

$$\langle a_0^\dagger a_0 \rangle = (e^{-\beta\mu} - 1)^{-1} \quad (\xi \to 0) , \tag{4.5}$$

$$N_0 = -(\beta\mu)^{-1} \quad (\xi \to 1) . \tag{4.6}$$

Equation (4.5) is the familiar distribution function of the zero-mode particles, and (4.6) is the occupation of this mode when $T \to T_c$ ($\mu \to 0$). Equation (4.6) might have been also obtained if, instead of $\xi \to 1$, we let $\mu \to 0$, and this observation is fundamental to the consistency of the present formalism. In fact, for $T > T_c$ one has $\mu < 0$ and $\xi = 0$ and, for $T < T_c$, $\mu = 0$ and $\xi > 0$. Inside the critical region ($T \cong T_c$), ξ may not vanish but μ must be close to zero. Hence, the limit $\xi \to 1$ in Eq. (4.6) implies that $\mu \cong 0$, and this equation is meaningful only to the extent that N_0 diverges. Consequently, in what follows the limit $\mu \to 0$ must come first.

We now substitute (4.4) in (4.3) and obtain an expression for the generalized force,

$$\chi_0(T,V,\mu,\xi) = V^{-1}\beta\mu[(1-\xi)^{-1}N_0 - (e^{-(1-\xi)\beta\mu} - 1)^{-1}] \tag{4.7}$$

$$= 1/(1-\xi)V \quad (\mu \to 0) . \tag{4.8}$$

In the limit of infinite volume, Eq. (4.7) shows that χ_0 vanishes when $T > T_c$ ($\xi = 0$), while Eq. (4.8) reveals that χ_0 is finite for $T \leqq T_c$ if, and only if, $\xi \to 1$. Therefore, we write

$$\chi_0(T) \equiv \lim_{V\to\infty} \lim_{\xi\to 1} \chi_0(T,V,\mu=0,\xi) . \tag{4.9}$$

We proceed now to determine N_0 as a function of ξ. The basic equation is (2.5), and if we are able to determine the contraction $\langle a_0^\dagger a_0 \rangle$, then N_0 will follow. The difficulty in calculating this contraction resides in the symmetry-breaking term of the Hamiltonian (4.2). However, following a well-known argument due to Bogoliubov,[8] the term proportional to $(N_0)^{1/2}$ in the Hamiltonian may be neglected in dilute systems. This is particularly true in the present case for $(N_0)^{1/2}$ in (4.2) is multiplied by $[(1-\xi)\xi]^{1/2}$, which is small in the interval $0 \leqq \xi \leqq 1$, and $(a_0 + a_0^\dagger)$ vanishes on the average according to (2.4).

Therefore omitting the symmetry-breaking term of (4.2) it is easy to show that the partition function is given by

$$Z_0(T,V,\mu,\xi) = Z_0(T,V,\mu,\xi=0)e^{\xi\beta\mu N_0}\langle e^{-\xi\beta\mu a_0^\dagger a_0} \rangle_0 . \tag{4.10}$$

The zero subscript in the ensemble average means that the average is taken with respect to the $\xi=0$ ensemble. Equations (4.2) and (4.10) yield the ensemble average of an operator $\hat{A}$, i.e.,

$$\langle \hat{A} \rangle = \langle e^{-\xi\beta\mu a_0^\dagger a_0} \hat{A} \rangle_0 \langle e^{-\xi\beta\mu a_0^\dagger a_0} \rangle_0^{-1} . \tag{4.11}$$

Clearly, $\langle \hat{A} \rangle = \langle \hat{A} \rangle_0$ if $\hat{A}$ is uncorrelated to $a_0^\dagger a_0$. Equations (4.10) and (4.11) can be simplified either by a cumulant expansion or, alternatively, using Wick's theorem on the exponential expansion of the ensemble average common to both equations. The result is

$$\langle e^{-\xi\beta\mu a_0^\dagger a_0} \rangle_0 = (1 + \xi\beta\mu \langle a_0^\dagger a_0 \rangle_0)^{-1} , \tag{4.12}$$

subject to the condition

$$|(\xi z \ln z)/(1-z)| < 1 , \tag{4.13}$$

where, for convenience, we have introduced the fugacity $z = e^{\beta\mu}$, and[16]

$$\langle a_0^\dagger a_0 \rangle_0 \equiv M = z/(1-z) . \tag{4.14}$$

Condition (4.13) obviously holds for z and ξ less than unity. As $T \to T_c$ from above, Eq. (4.14) reveals that the fugacity approaches unity as $z = 1 - M^{-1}$, $M \gg 1$. In this limiting case the inequality (4.13) still is satisfied because then $\xi < 1/z = 1 + M^{-1}$.

We now let $\hat{A} = a_0^\dagger a_0$ in (4.11). From (4.12) and using Wick's theorem on the numerator, we obtain finally

$$\langle a_0^\dagger a_0 \rangle = z/(1 - z + \xi z \ln z) . \tag{4.15}$$

An equation for N_0 can then be obtained by substituting Eqs. (2.7), (4.4), and (4.15) in (2.5), i.e.,

$$(1-\xi)z(z^\xi - z)^{-1} = (1-\xi)z(1 - z + \xi z \ln z)^{-1} + \xi N_0 . \tag{4.16}$$

As $z \cong 1$ we can make the approximations $\ln z \cong z - 1$ and $z^\xi \simeq 1 - \xi(1-z)$, and (4.16) amounts to

$$N_0 = 1/(1-\xi) \quad (z=1) . \tag{4.17}$$

Equation (4.17) implies that $\xi \to 1$ $(T \leqq T_c)$ in order to be consistent with Eq. (4.6): Both equations must provide a diverging N_0, which is compensated in the limit of infinite volume to give a finite density. Hence, dividing (4.17) by the volume, taking the limits $V \to \infty$ and $\xi \to 1$, and considering Eqs. (4.8) and (4.9), we finally conclude that in the thermodynamic limit,

$$\begin{aligned} \chi_0(T) &= n_0(T) = n(T) - n_1(T) \\ &= n(T)[1-(T/T_c)^{d/2}] \quad (T \leqq T_c) , \end{aligned} \tag{4.18}$$

where the latter equality comes from a known property of the ideal Bose gas,[16] $n(T)$ is the overall particle density, and $n_0(T)$ and $n_1(T)$ are the densities of the condensate and noncondensate, respectively. For a given temperature $T \leqq T_c$, $n_1(T) = n_1(T, z=1)$ is the saturated density of the excited particles for that temperature. At the critical temperature $n_1 = n$, so that χ_0 vanishes and, for temperatures below T_c, χ_0 equals the condensate density.

We remark that $\chi_0(T)$ is simply related to the helicity modulus $\Upsilon(T)$ mentioned in Sec. II. Fisher *et al.*[4] found that the helicity modulus is proportional to the superfluid density ρ_s, i.e., $\Upsilon(T) = (\hbar/m)^2 \rho_s$. For the ideal Bose gas one has $\rho_s = m n_0$,[4] and (4.18) gives

$$\Upsilon(T) = (\hbar^2/m)\chi_0(T) . \tag{4.19}$$

We conclude this discussion by showing that the thermodynamic properties of the present $(\xi > 0)$ system are the same as those of the customary $(\xi = 0)$ ideal Bose gas treatment. This is obviously the case when $T > T_c$, for the normal phase is given by $\xi = 0$. For $T < T_c$, we calculate the pressure from the partition function (4.10),

$$P(\xi) = P(\xi=0) + \Delta P(\xi) , \tag{4.20}$$

$$\Delta P(\xi) = V^{-1} k_B T \{ \ln[(1-z)/(1-z+\xi z \ln z)] + \xi N_0 \ln z \} \tag{4.21}$$

$$\to -V^{-1} k_B T \ln(1-\xi) \quad \text{as } z \to 1 \ (T \leqq T_c) . \tag{4.22}$$

In the thermodynamic limit the contribution (4.22) vanishes. The internal energy increment, $\Delta E(\xi) = E(\xi) - E(\xi=0)$, whether determined directly from (4.11) or from the usual derivatives of the partition function (4.10), is also found to vanish. Therefore, both $\xi > 0$ and $\xi = 0$ systems have the same equations of state.

V. DISCUSSION

From the preceding sections we may define the phases of a Bose system in terms of the parameter ξ as follows.

Phase I: $\xi = 0$, gauge-symmetry invariant.

Phase II′: $0 < \xi \ll 1$, gauge-symmetry broken without Bose-Einstein condensation.

Phase II″: $\xi = 1$, gauge-symmetry broken with Bose-Einstein condensation.

Phase I is the normal component while the ordered state corresponds to either phase II′ or II″.

According to our formulation, a natural criterion for the existence of phases II′ and II″ far from the critical point consists of a finite χ in the thermodynamic limit, for $0 < \xi \ll 1$ and $\xi = 1$, respectively. In Sec. IV it was explicitly shown that the ideal Bose gas is of type II″ $(T < T_c)$. Although Eq. (2.17) allows a calculation for interacting systems, the dense case remains a very difficult problem.

However, the existence of both phases, II′ and II″, may be inferred from their behavior within the critical region. The phase transition I-II′ is characterized by a correlation-length critical exponent equal to two-thirds, whereas a unity crossover exponent appears in the transition I-II″.[7] From theoretical[2,7] and experimental[5,6] results, phase II′ is typical of strongly interacting systems, whose transition is governed by the complex nature of the order parameter, while phase II″ is found in weakly interacting systems where the order-parameter fluctuations are negligible.

As indicated in Sec. II, the different ranges of ξ values related to the ordered phases are defined upon the requirement that the continuity equation (2.12) is valid to within an infinitesimal term. It is a feature of the present formulation that the infinitesimal term depends on the interaction constant, in contrast to an analogous term resulting from Bogoliubov's approach of an external field coupled to the zero-mode amplitudes.[9,15] A physical interpretation for this latter field is difficult to ensure. On the other hand, the parameter ξ may be regarded as the fraction of the $k=0$ particles that are not described by the operators a_0 and $a_0^\dagger$, or, alternatively, the fraction whose fluctuations are negligible. Hence, phase II″ corresponds to an entire condensation of the zero-mode particles, whereas phase II′ corresponds to an infinitesimal condensation.

Finally, it is worth mentioning the recent conjecture that Bose-Einstein condensation is not a necessary condition for the appearance of superfluidity.[17,18] Further investigation on the possibility of superfluid behavior in phase II′ is of interest in this context.

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