Motion of damped sine-Gordon kinks in the presence of thermal fluctuations

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The motion of damped sine-Gordon kinks in the presence of thermal fluctuations is studied to first order in $k_B T$ by using a singular perturbation expansion. The change in the average shape of the kink is determined as well as its mobility. These results are compared with the earlier results of Kaup on the overdamped sine-Gordon equation, and it is found that the heavily damped limit of the damped sine-Gordon kink is *not* the same as that of the overdamped sine-Gordon kink. In particular, we find the change in the shape of a heavily damped sine-Gordon kink to be approximately one-half that of an overdamped sine-Gordon kink at comparable temperatures. We also verify that the singular perturbation method gives the correct results for the average kinetic energies of kinks and phonons. Lastly, we also evaluate the low-temperature correlation function for the damped sine-Gordon field and the diffusion coefficient for a soliton.

I. INTRODUCTION AND BACKGROUND

Recently, Kaup[1] studied the thermal effects on an overdamped sine-Gordon kink where the equation of motion is

$$\Gamma\partial_t u = \kappa\partial_x^2 u - \kappa\eta^2 \sin u + F + \epsilon\zeta \ , \tag{1}$$

where $u(x,t)$ is the field variable, Γ is the damping constant, κ is the torsion constant, η (which has units of inverse length) gives the strength of the retarding potential, F is a spatially independent externally applied torque, and ζ is a thermal random torque. In Eq. (1) ϵ is an expansion parameter which we shall use later. In the overdamped case, Kaup found that the mobility of an overdamped sine-Gordon kink in the static limit was the same as given by Büttiker and Landauer,[2] and he also found that the effect of an increase in the temperature is to increase this mobility by the factor $(1+1.075/\beta E_0 + \cdots)$, where E_0 is the rest energy of a kink and β is proportional to the inverse of the temperature. Kaup[1] also evaluated the change in the shape of an overdamped sine-Gordon kink caused by the thermal fluctuations, and he found that the kink somewhat flattens, increasing its width. This analytic work by Kaup[1] was based on a singular perturbation expansion wherein a comoving dynamical variable representing the center of the kink was introduced.

More recently, Salerno, Joergensen, and Samuelsen[3] studied the thermal sine-Gordon system which is

$$\partial_t^2 + \Gamma\partial_t u = \kappa\partial_x^2 u - \kappa\eta^2 \sin u + F + \epsilon\zeta \ . \tag{2}$$

They had used standard methods of stochastic processes.[4] Their work did not require a singular perturbation expansion because such is not required when one treats only first-order quantities. (But in order to go to any higher order, a singular perturbation expansion becomes necessary.) Among their results, they found that in the presence of a soliton, the average energy of the kth phonon mode was $k_B T$ while the average energy of the translational mode is $\frac{1}{2}k_B T$, where k_B is Boltzmann's constant and T is the temperature. They also studied the thermal sine-Gordon system in the absence of solitons and found that the average energy per mode is $k_B T$. They also studied the Brownian motion of a thermal soliton and found the average of the kinetic energy of the soliton to be $\frac{1}{2}k_B T$. Their results are in agreement with those obtained earlier by the classical statistical mechanics as derived by Bishop, Krumhansl, and Trullinger.[5]

Applications of kinks in dislocation theory can also be found in Seeger's work.[6] However, the main interests in dislocation theory are to develop the theory of kink-antikink formation and kink migration, since dislocation velocities seem to be more related to the kink-antikink formation, which is a more complicated problem than we are considering here. Also, the kink diffusivity in dislocation theory seems to be determined primarily by the interaction of kinks with foreign atoms and not the thermal fluctuations which we consider.

Since there is a variance in what is meant by the term "singular perturbation expansion," it is of value for us to pause briefly and to clarify what we mean by this term. A regular perturbation expansion is a power series in a small parameter expansion, while in a singular perturbation expansion one collects terms together in some other fashion. Sometimes a regular perturbation expansion has secular terms so that the expansion is not uniform or it breaks down at a certain level. In such a case we may be able to expand in a singular perturbation expansion so that the secular terms can be annihilated. This is the general idea in all singular perturbation expansions. However, there is a variety of definitions of exactly what is a singular perturbation problem.[7–12] One can classify the various definitions of what is a singular perturbation problem (see, for example, Refs. 7, 8, and 12) as follows.

(i) Sources of nonuniformities appear in relation to an infinite domain (for example, the appearance of secular terms in nonlinear oscillations). In the infinite domain case, the nonuniformity manifests itself as so-called secular terms, like $t^m \cos t$ and $t^m \sin t$. This type of singular perturbation problem has been classified as a "secular-type" problem.[7]

(ii) A small parameter multiplies the highest-order derivative term in a differential equation. In this case, the perturbation expansion cannot satisfy all the boundary and/or initial conditions, because the perturbed differential equations reduce their order in the perturbation expansions. Thus, the expansion ceases to be valid in some boundary and/or initial layers. This type of problem may be classified as being a "layer-type" problem.[8–10]

(iii) There is a change of type of partial differential equations. In this case, the classification type of the perturbed equations changes from that of the original equations, and nonuniformities might arise.[7,8]

(iv) Another case is the presence or occurrence of singularities. In this case, singularities that are not involved in the exact solution appear at a certain stage of the perturbation expansion.[7,8]

We remark that what we term to be singular perturbation theory here is as in (i) above. This method is also known in nonlinear dynamics as the method of Lindstedt and Poincaré.[7,11]

With this clarification of what is the method that we shall use, let us now describe what we shall do with it. We shall extend Kaup's study[8] of the thermal effects on the overdamped sine-Gordon kink motion to that of the damped sine-Gordon equation for any damping. As one can see upon comparing Eqs. (1) and (2), the only difference lies in the second time derivative which is present in Eq. (2) but not in Eq. (1). And, in the heavily damped limit ($\Gamma\to\infty$), except for short transients, solutions of the damped sine-Gordon equation, Eq. (2), become the same as those of the overdamped sine-Gordon equation, Eq. (1), for the same initial conditions. Therefore one would normally consider the overdamped sine-Gordon equation to be the heavily damped limit of the damped sine-Gordon equation. However, as we shall show here, this is *not* the case when one considers thermal effects. We shall show that although the thermal behavior of the overdamped sine-Gordon equation may be qualitatively the same as that of the heavily damped sine-Gordon equation, there still are definite quantitative differences in the results, sometimes even of order unity.

The manner by which this comes about is as follows. As we will show, the process of thermal averaging does not commute with the limit of $\Gamma\to\infty$ (except at the points $\Gamma=\infty$ and $\beta=0$). Why this is so, is that there are some terms which are small and vanish as $\Gamma\to\infty$, but still thermal average to a nonzero value. At the same time, there are other terms with larger rms values which do thermal average to zero. Thus, terms which usually may be neglected in the limit of $\Gamma\to\infty$ can make a significant contribution to thermal averages. This will be discussed more specifically in Sec. IV.

In Sec. II we shall extend Kaup's results[1] to the damped sine-Gordon kink. As was done earlier,[1] we shall use a singular perturbation expansion, expanding in both powers of the external torque, F, and in the random thermal torque, ζ. The detailed calculations are given in Appendix A. This expansion will be of the form[1,13] $u=u_0[x-x_0(t)]+\delta u(x,t)$, where u_0 will be a kink solution and $x_0(t)$ will be the position of the kink.[14] We now have two statistical variables, $x_0(t)$ and $\delta u(x,t)$, instead of $u(x,t)$. However, x_0 has only one degree of freedom. Thus δu must have exactly one less degree of freedom than u has, and this will be achieved by requiring δu to be orthogonal to the Goldstone mode.[3,15] This will impose one constraint on δu and thereby reduce the number of its degrees of freedom accordingly. Then δu and $x_0(t)$ together will have the same number of degrees of freedom as u originally had. For an interesting recent discussion of how the Goldstone mode is related to a universal localized relaxation mode, the reader is referred to Ref. 15.

This decomposition of u into the statistical variables δu and x_0 does have some problems. First, it is now not practical to define statistical averages as averages at constant x. To illustrate this, consider the ensemble average of $u(x,t)$. The first term to be averaged is $u_0[x-x_0(t)]$ which requires the average of a function of a statistical variable, not just the average of a statistical variable. Furthermore, since $x_0(t)$ will undergo a random walk, the ensemble average of $u_0[x-x_0(t)]$ will therefore be time dependent, complicating the physical interpretation considerably. The obvious way to bypass this is to define the ensemble average relative to the kink. Namely, first define a comoving coordinate, $X=x-x_0(t)$, so that $u(x,t)=u_0(X)+\delta u$. Now average the first term by keeping X the same for every element from the ensemble, and then it becomes clear the $\overline{u_0(X)}=u_0(X)$, since $u_0(X)$ is only a function and is not a statistical variable. All statistical variations have been swept into $\delta\mu$. For more discussion on this point, the reader is referred to Ref. 1.

In Sec. III we take the expansion in Sec. II and do the standard statistical averaging of various quantities. For this we use[3]

$$\overline{\zeta(x,t)}=0 \tag{3}$$

and for the second-order averages we take

$$\overline{\zeta(x,t)\zeta(x',t')}=\frac{2\Gamma}{\beta}\delta(x-x')\delta(t-t') . \tag{4}$$

The prefactor of $2\Gamma/\beta$ in (4) is determined by applying the fluctuation dissipation theorem,[3] where β is proportional to the inverse of the temperature, namely, $\beta=(k_BT)^{-1}$, where T is the temperature and k_B is Boltzmann's constant. Details of most of the calculations will be found in Appendix B.

The fact that kinks and antikinks do exist in the presence of thermal fluctuations is known from numerical studies.[16] It is only as the temperature becomes higher, namely, for $k_BT\gtrsim E_0$, where E_0 is the rest energy of a kink, that the kinks will be destroyed and/or created by the strong thermal fluctuations. Therefore, there is a finite range of temperatures ($0<k_BT\lesssim E_0$) wherein kinks and antikinks do have a stable existence.

In the last section we shall present our results and conclusions. We shall take the results from Sec. III and compare them with those of the overdamped case.[1] In particular, we find that our results do differ quantitatively from those for the overdamped sine-Gordon[1] kink sometimes by a factor of 2 or so. In particular, we find this to be true for the change in the shape of a kink or an antikink caused by the thermal fluctuations. What happens here is that statistical averages of small (in the limit of $\Gamma\to\infty$)

positive-definite quantities can end up being larger than the statistical average of a nominally large quantity (but which statistical averages to a small value). However, the temperature effect on the average velocity of a kink or an antikink does not have such a dramatic difference between the two cases. In addition, we shall also evaluate the average energy of a kink, the correlation function for the sine-Gordon field in the presence of a kink, and the diffusion coefficient for kinks. For these latter quantities, we shall show that one will get the expected results even if one uses the comoving coordinate[1,14,17] and this singular perturbation expansion.

II. SINGULAR PERTURBATION EXPANSION

First we expand in the thermal torques. We take

$$u(x,t)=u_0(X)+\epsilon u_1(X,\tau)+\epsilon^2 u_2(X,\tau)+\cdots , \quad (5)$$

where

$$X=X_0+\epsilon X_1(t)+\epsilon^2 X_2(t)+\cdots , \quad (6)$$

$$\tau=t , \quad (7)$$

and we find it adequate to choose

$$X_0=x-v_0 t , \quad (8)$$

where v_0 is the zeroth-order velocity of the kink.

Inserting the above into Eq. (2) and expanding then gives in zeroth order

$$(v_0^2-\kappa)u_{0XX}-\Gamma v_0 u_{0X}+U(u_0)=F , \quad (9)$$

in first order

$$u_{1\tau\tau}-2v_0 u_{1X\tau}+(v_0^2-\kappa)u_{1XX}-\Gamma v_0 u_{1X}+\Gamma u_{1\tau}+U_u(u_0)u_1=\zeta-2v_0 v_1 u_{0XX}+(\dot v_1+\Gamma v_1)u_{0X} , \quad (10)$$

and in second order

$$u_{2\tau\tau}-2v_0 u_{2X\tau}+(v_0^2-\kappa)u_{2XX}-\Gamma v_0 u_{2X}+\Gamma u_{2\tau}+U_u(u_0)u_2 =2v_1 u_{1X\tau}-2v_0 v_1 u_{1XX}-(2v_0 v_2+v_1^2)u_{0XX}+(\dot v_1+\Gamma v_1)u_{1X}+(\dot v_2+\Gamma v_2)u_{0X}-\tfrac{1}{2}u_1^2 U_{uu}(u_0) , \quad (11)$$

where the overdot indicates differentiation with respect to time. Also, we have used

$$v_i=-\dot X_i . \quad (12)$$

This follows from (5), (6), and (8), whereby we may define the center of the kink to be at $X=0$. Then from (6) and (8) it follows that the kink's velocity is

$$v=v_0+\epsilon v_1+\epsilon^2 v_2+\cdots \quad (13)$$

with v_i defined by (12). Thus v_i can be interpreted as the ith order of the kink's velocity.

In order to obtain analytical results, we now must treat the weak torque case, that is, for small F. We shall introduce superscripts to designate the order of the expansion in F. Expanding in a regular perturbation expansion in F, we have

$$u_i=u_i^{(0)}+Fu_i^{(1)}+\cdots , \quad i=0,1,2,\ldots \quad (14)$$

and

$$v_i=v_i^{(0)}+Fv_i^{(1)}+\cdots , \quad i=0,1,2,\ldots . \quad (15)$$

In the sine-Gordon case one can show that $v_0^{(0)}=0$. By expanding the zeroth-order equation (9), $u_0^{(0)}$ and $u_0^{(1)}$ satisfy the following respective equations:

$$-\kappa u_{0XX}^{(0)}+U(u_0^{(0)})=0 \quad (16)$$

and

$$-\kappa u_{0XX}^{(1)}+U_u(u_0^{(0)})u_0^{(1)}=1+v_0^{(1)}u_{0X}^{(0)} . \quad (17)$$

If we define the operator $L^{(0)}$ by

$$L^{(0)}=-\kappa\partial_X^2+U_u(u_0^{(0)}) , \quad (18)$$

then Eq. (17) becomes

$$L^{(0)}u_0^{(1)}=1+\Gamma v_0^{(1)}u_{0X}^{(0)} . \quad (19)$$

As is well known, (i) the operator $L^{(0)}$ has a zero eigenvalue, the Goldstone mode,[5] and (ii) $u_{0X}^{(0)}$ and ψ_b must be proportional where the nonzero constant of proportionality, c, may be defined by

$$u_{0X}^{(0)}=c\psi_b , \quad (20)$$

where ψ_b is the zero-eigenvalue bound state of $L^{(0)}$.

In the sine-Gordon case one takes

$$U(u)=\kappa\eta^2\sin u , \quad (21)$$

where the solution for the zeroth-ordered equation (16) is[5]

$$u_0(X)=4\tan^{-1}(se^z) , \quad (22)$$

where $s=+1$ for a kink, -1 for an antikink, and $z=\eta X$.

We obtain the continuous eigenvalues of $L^{(0)}$ from the eigenvalue problem

$$L^{(0)}\psi_l=\lambda_l\psi_l \quad (23)$$

by evaluating (23) at $X=\pm\infty$ and by using the fact that when very far away from the kink ($X\to\pm\infty$) the eigenfunction ψ_l must approach e^{ilX}. These eigenvalues are

$$\lambda_l=\kappa(l^2+\eta^2) . \quad (24)$$

Since the operator $L^{(0)}$ is self-adjoint, then for the sine-Gordon case, the eigenfunctions ψ_b,ψ_l form a complete set[1,18] which spans the space of functions of X. The orthogonality relations are[5]

$$\langle\psi_b\,|\,\psi_b\rangle=1 , \quad (25)$$

$$\langle \phi_l \mid \psi_{l'} \rangle = \delta(l - l') , \tag{26}$$

$$\langle \psi_b \mid \psi_l \rangle = 0 = \langle \phi_l \mid \psi_b \rangle , \tag{27}$$

where we use the angular brackets to indicate the inner product

$$\langle u \mid v \rangle \equiv \int_{-\infty}^{\infty} u(X) v(X) dX . \tag{28}$$

The completeness relation has the form

$$\delta(X - X') = \psi_b(X')\psi_b(X) + \int_{-\infty}^{\infty} dl\, \phi_l(X')\psi_l(X) . \tag{29}$$

In the above equation, ϕ_l denotes the adjoint eigenfunctions. But since $L^{(0)}$ is a self-adjoint operator, it then follows that the adjoint eigenfunctions are some linear combination of the eigenfunctions of $L^{(0)}$ having the same eigenvalue λ_l. Thus

$$\phi_l = a_l \psi_{-l} + b_l \psi_l . \tag{30}$$

Then it follows that[1]

$$\langle \phi_{l'} \mid \phi_l \rangle = a_l \delta(l' + l) + b_l \delta(l - l') . \tag{31}$$

A relation which shall be useful later is that when f_l is an even function in l, we find that[1]

$$\int_{-\infty}^{\infty} dl\, \psi_l f_l \langle \phi_{l'} \mid \phi_l \rangle = f_{l'} \phi_{l'} . \tag{32}$$

With the above preliminaries, the determination of the expansion quantities $v_j^{(n)}(\tau)$ and $u_j^{(n)}(X,\tau)$ as functions of $\zeta(X,\tau)$ is straightforward but tedious. Expressions for these quantities when $j \le 2$ and $n \le 1$ are given in Appendix A. Given these expansions, we may then proceed to perform the various ensemble averages, which we shall do in the next section.

III. SECOND-ORDER ENSEMBLE AVERAGES

The solution in the preceding section is for one specific thermal torque, $\zeta(x,t)$, from our ensemble. We now average over all elements in the ensemble to determine average values[19,20] using (3) and (4). We simply note that because the thermal correlation time is taken to be zero, one finds upon transforming from (x,t) coordinates into (X,τ) coordinates that[1]

$$\overline{\zeta(X,\tau)} = 0 \tag{33}$$

and

$$\overline{\zeta(X,\tau)\zeta(X',\tau')} = \frac{2\Gamma}{\beta} \delta(X - X')\delta(\tau - \tau') . \tag{34}$$

This means that statistically the driving term is exactly the same in both coordinates. From Eq. (33) we conclude that all first-order thermal averages vanish, i.e., $\overline{v_1^{(0)}} = 0 = \overline{v_1^{(1)}}$, etc. The details for evaluating the various second-order thermal averages is given in Appendix B. Next we shall discuss their consequences.

A. Change in the shape of a kink due to thermal fluctuations

In this first subsection we shall evaluate the change in the shape of a kink or antikink due to thermal fluctuations. For this calculation, it is sufficient to take the $F = 0$ limit to obtain a nonzero result. The required term (B14) is evaluated in Appendix B and to second order

$$u = 4 \tan^{-1}(s e^z) + \epsilon^2 \overline{u_2^{(0)}} + \cdots , \tag{35}$$

where

$$\overline{u_2^{(0)}} = -\frac{s\, \mathrm{sech} z}{8\beta\kappa\eta} [z - \tfrac{1}{2} \tanh z (1 + \tfrac{1}{3} \mathrm{sech}^2 z)] . \tag{B14}$$

Note that to this order, one can rewrite (35) as

$$u = 4 \tan^{-1}[s \exp(z + \delta z)] , \tag{36}$$

where

$$\delta z = \frac{-\epsilon^2}{16\beta\kappa\eta} [z - \tfrac{1}{2}(1 + \tfrac{1}{3} \mathrm{sech}^2 z) \tanh z] . \tag{37}$$

Since δz is opposite in sign to z, we see that the effect of the thermal fluctuations is to flatten out the kink, thus increasing its width. We also see from Eq. (37) that the change in the shape of a kink is independent of Γ and is dependent only on the temperature.

B. Average velocity of a kink or an antikink

In this subsection we shall evaluate the average velocity of a kink or an antikink for F small. The zeroth-order result is given by (B15) while the second-order result is (B29). Thus, to this order the average velocity of a kink or an antikink is

$$\begin{aligned} v &= F v_0^{(1)} + F \overline{v_2^{(1)}} + \cdots \\ &= -\frac{\pi s F}{4\Gamma\eta} \left[1 + \frac{c}{\beta E_0} + \cdots \right] + \cdots , \end{aligned} \tag{38}$$

where $E_0 = 8\kappa\eta$ is the rest energy of a kink[5] and $c = c(A)$ is given by (B22)–(B30). Also, $c(A)$ is a positive function for all positive values of A (see Fig. 1) where A is given by (B28). This result shows that the effect of an increase in the temperature is to increase the average velocity by the above amount.

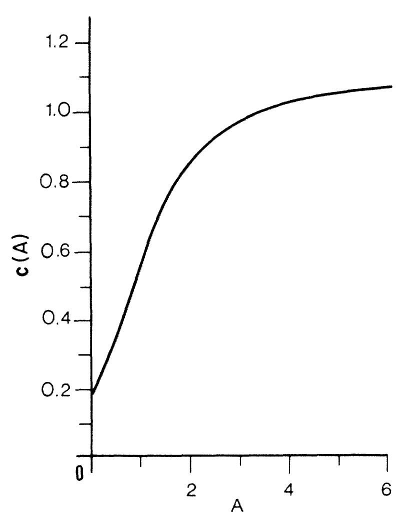

FIG. 1. Plot of $c(A)$ as a function of A.

IV. COMPARISON OF THE HEAVILY DAMPED CASE WITH THE OVERDAMPED CASE

In this section we shall compare the heavily damped results ($\Gamma \rightarrow \infty$) with the results of Kaup[1] in the overdamped case. The results are qualitatively the same but do differ quantitatively. An explanation of this will be given shortly.

Consider the change in the shape of a damped sine-Gordon kink as given by (B14). In the overdamped case, Kaup[1] found that the second-order change in the shape of a kink was given by

$$\overline{u_2^{(0)}} = -\frac{s}{8\beta\kappa\eta}(\operatorname{sech} z)[2z - \tfrac{1}{2}\tanh z(1+\tfrac{1}{3}\operatorname{sech}^2 z)] , \tag{39}$$

which differs only in the first term. Thus a heavily damped sine-Gordon kink is distorted by thermal fluctuations only about one-half as much as an overdamped sine-Gordon kink would be.

This dramatic difference between the heavily damped and the overdamped cases deserves an explanation, particularly since the heavily damped limit has the same solution as the overdamped case. One may readily verify the latter by taking the limit of the results in Appendix A. These results then become exactly the same as for the overdamped case.[1]

Therefore the difference must occur somewhere in Appendix B, and indeed that is the case. These results for the change in the shape are sufficiently simple that one can actually trace down the source of the difference. It is easy to see that in the damped case, Eqs. (B1)—(B4) give

$$\overline{u_2^{(0)}} = -\int_{-\infty}^{\infty} dl \frac{\psi_l(\mathcal{X})}{\lambda_l}[\overline{(v_1^{(0)})^2}\langle \phi_l \mid u_{0\mathcal{X}\mathcal{X}}^{(0)} \rangle + \tfrac{1}{2}\langle \phi_l \mid \overline{(u_1^{(0)})^2 U_{uu}} \rangle] , \tag{40}$$

while in the overdamped case, the first term would be absent. Thus the difference between the two results is due to this extra term which itself arises from the second time derivative in (2). This extra term in (40) is due to the kinetic motion of the center of the soliton, namely, the $v_1^2 u_{0\mathcal{X}\mathcal{X}}$ term in Eq. (11). This is the only part of the u_{tt} term which survives the averaging to second order.

In the limit of $\Gamma \rightarrow \infty$, this term does vanish. But after thermal averaging, although it does still vanish as $\Gamma \rightarrow \infty$, the thermal average is now found to be the same order of magnitude as the other term in Eq. (40). In other words, thermal averaging and the limit of $\Gamma \rightarrow \infty$ do not commute.

To clarify this further, let us compare the u_t term with the u_{tt} term. For the following argument, we shall assume that the kinetic motion of the center of the soliton dominates, and thus we take $u_{tt} \approx v_1^2 u_{0\mathcal{X}\mathcal{X}}$ and similarly $u_t \approx \Gamma v_1 u_{0\mathcal{X}}$. Clearly in this approximation, from the vanishing of all first-order averages and Eq. (B13), $\overline{\Gamma u_t} = 0$, $\overline{u_{tt}} = O(1/\beta\eta)$, and the latter does *not* vanish as $\Gamma \rightarrow \infty$. No matter how heavily damped a sine-Gordon kink is, the first term in (40) never does vanish. On the other hand, $\overline{(\Gamma u_t)^2} = O(\Gamma^2/\beta\eta)$, showing that the rms value of Γu_t is indeed larger than $\overline{u_{tt}}$. So for a given element from the ensemble, in general, we will have $|u_{tt}| < \Gamma |u_t|$ as $\Gamma \rightarrow \infty$ as was argued in Sec. I. But when we average over all ensemble elements, we find that Γu_t will average to zero while u_{tt} will average instead to a nonzero constant value (since $\overline{v_1^2} > 0$). Therefore for second-order thermal averages, the overdamped sine-Gordon kink is not the limit of the heavily damped sine-Gordon kink.

Next we shall compare the average velocity of a kink or an antikink in both cases. One simply evaluates the six integrals given by (B22)—(B27) in the limit of $A \rightarrow \infty$. One obtains[21]

$$c = c_0 + O(1/A^2) , \tag{41}$$

where

$$c_0 = 1.099\,711\,3 \ldots . \tag{42}$$

As $\Gamma \rightarrow \infty$, that is, as $A \rightarrow \infty$, $c \rightarrow c_0$, and the average velocity in (B30) reduces to

$$v = -\frac{\pi s F}{4\Gamma\eta}\left[1 + \frac{1.099\,711\,3}{\beta E_0} + \cdots \right] + \cdots . \tag{43}$$

In the overdamped case studied by Kaup,[1] the average velocity is given by

$$v = -\frac{\pi s F}{4\Gamma\eta}\left[1 + \frac{1.074\,704\,7}{\beta E_0} + \cdots \right] + \cdots . \tag{44}$$

The two results are almost the same and this time they differ only by a very small quantity. The reason for this difference again comes from the extra terms that arise due to the second time derivative in Eq. (2).

V. AVERAGE ENERGY OF KINKS AND PHONONS VIA THE SINGULAR PERTURBATION EXPANSION

In this section we shall describe the evaluation of the average energy of kinks and phonons in this singular perturbation expansion. These results are the same as by other methods since one only needs to calculate the average of first-order quantities. What we do here is simply to verify that the results obtained via the singular perturbation expansion will be the usual results as obtained by other methods.

As is usual,[3] we start with the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}(\kappa u_x^2 + u_t^2) + \kappa\eta^2(1-\cos u) , \tag{45}$$

which when transformed from (x,t) coordinates into $(\mathcal{X},\tau)$ coordinates becomes

$$\mathcal{H} = \tfrac{1}{2}[(\kappa \mathcal{X}_x^2 + \mathcal{X}_t^2)u_{\mathcal{X}}^2 + u_\tau^2 + 2\mathcal{X}_t u_{\mathcal{X}} u_\tau + \kappa\eta^2(1-\cos u)] , \tag{46}$$

Expanding in powers of ϵ, one has

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}_1 + \epsilon^2\mathcal{H}_2 + \cdots , \tag{47}$$

where

$$\mathcal{H}_0 = \tfrac{1}{2}(\kappa + v_0^2)u_{0\mathcal{X}}^2 + \kappa\eta^2(1-\cos u_0) , \tag{48}$$

$$\mathscr{H}_1=\kappa u_{0X}u_{1X}-v_0u_{0X}(u_{1\tau}-v_0u_{1X}-v_1u_{0X})+\kappa\eta^2u_1\sin u_0\ , \tag{49}$$

$$\mathscr{H}_2=\tfrac{1}{2}\kappa(u_{1X}^2+2u_{0X}u_{2X})+\tfrac{1}{2}(u_{1\tau}-v_0u_{1X}-v_1u_{0X})^2 -v_0u_{0X}(u_{2\tau}-v_0u_{2X}-v_1u_{1X}-v_2u_{0X}) +\kappa\eta^2(\tfrac{1}{2}u_1^2\cos u_0+u_2\sin u_0)\ . \tag{50}$$

In the weak-torque limit, the leading term is for zero torque. Thus we shall only consider the zero-torque case in this section.

The average kink energy is

$$E=\int_{-\infty}^{\infty}\overline{\mathscr{H}}dx\ , \tag{51}$$

and upon expanding, we have

$$E_n^{(0)}=\int_{-\infty}^{\infty}\overline{\mathscr{H}_n^{(0)}}dx,\quad n=0,1,2,\ldots\ , \tag{52}$$

where

$$E_0^{(0)}=8\kappa\eta\ , \tag{53}$$

$$E_1^{(0)}=0\ , \tag{54}$$

and

$$E_2^{(0)}=\tfrac{1}{2}\overline{v_1^{(0)})^2}\int_{-\infty}^{\infty}dX(u_{0X}^{(0)})^2 +\tfrac{1}{2}\int_{-\infty}^{\infty}dk(\lambda_k\overline{|f_{1k}^{(0)}|^2}+\overline{|f_{1k,\tau}^{(0)}|^2})\ . \tag{55}$$

Evaluating the first integral in (55) and the average quantities in the second integral of the same equation gives

$$\overline{(v_1^{(0)})^2}\int_{-\infty}^{\infty}dX(u_{0X}^{(0)})^2=\frac{1}{\beta} \tag{56}$$

and

$$\lambda_k\overline{|f_{1k}^{(0)}|^2}=\overline{|f_{1k,\tau}|^2}=\frac{\delta(0)}{\beta}\ . \tag{57}$$

The best way to interpret $\delta(0)$ in the above equation is to place the system in a box of length L, in which case one can replace $\delta(0)$ by $\delta(0)=L/2\pi$. Then one can rewrite Eq. (55) as

$$E_2^{(0)}=E_{2b}^{(0)}+\sum_n E_{2n}^{(0)}\ , \tag{58}$$

where $E_{2b}^{(0)}$ and $E_{2n}^{(0)}$ are the average energies of the translational mode (soliton)[22,23] and the nth mode of the continuous spectrum (phonons), respectively. Thus

$$E_{2b}^{(0)}=\frac{1}{2\beta}=\tfrac{1}{2}k_BT \tag{59}$$

and

$$E_{2n}^{(0)}=\frac{1}{\beta}=k_BT\ . \tag{60}$$

Therefore, the translational mode (soliton) of the sine-Gordon kink will have an energy of $\frac{1}{2}k_BT$, while the continuous modes (phonons) will have an energy average of k_BT per mode. These results are the same as the results obtained by using the standard methods of stochastic processes in Ref. 3 and also with the classical statistical mechanics derived in Ref. 5.

VI. CORRELATION FUNCTION AND DIFFUSION COEFFICIENT

In this section we shall evaluate the correlation of the thermal fluctuations in the region about a kink. Because of the presence of the kink, one cannot just simply calculate the correlations of $u(X,\tau)$ with $u(X',\tau')$ without singular terms entering. Instead, it is necessary to calculate the correlations relative to the average values. We define the general correlation function as

$$\sigma(X,\tau;X',\tau')=\overline{[u(X,\tau)-\overline{u(X,\tau)}][u(X',\tau')-\overline{u(X',\tau')}]}\ . \tag{61}$$

In the zeroth order in F, and second order in ϵ, one finds

$$\sigma(X,\tau;X',\tau')=\epsilon^2\overline{u_1^{(0)}(X,\tau)u_1^{(0)}(X',\tau')}\ . \tag{62}$$

With use of formula (A6) for $u_1^{(0)}$, Eq. (62) reduces to the integral form

$$\sigma(X,\tau;X',\tau')=\frac{1}{2\beta}\int_{-\infty}^{\infty}dl\frac{\psi_l(X)\phi_l(X')}{\lambda_l(\Gamma^2-4\lambda_l)^{1/2}}(\{\Gamma+(\Gamma^2-4\lambda_l)^{1/2}]\exp\{-\tfrac{1}{2}[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}]\,|\tau-\tau'|\,\} -[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}]\exp\{-\tfrac{1}{2}[\Gamma+(\Gamma^2-4\lambda_l)^{1/2}]\,|\tau-\tau'|\,\})\ . \tag{63}$$

For the static (equal-time) correlation function,[5] i.e., when $\tau=\tau'$, one can evaluate the integral in (63) and obtain the exact analytic result (first order in $1/\beta$ and zeroth order in F), $\sigma(X,\tau;X',\tau)=(1/\beta)g(z,z')$, where the function $g(z,z')$ is defined by (B10). The structure of this static correlation function can be seen by plotting the function $f(z,z')=4\kappa\eta g(z,z')$ versus $z-z'$ (see Figs. 2–4). We see in Fig. 2 that the correlation function is symmetric when $z'=0$ and at $z=z'=0$ it has the maximum value 1. The minimum value is negative when z is about ± 2, and the correlation function is effectively zero when z is larger than about ± 6. Figure 3 shows that f is not symmetric in z when $z'=1$. The maximum value is about 1.5 at $z=z'=1$, and it has a negative minimum value when $z\simeq -1$. In Fig. 4 the correlation function for z' large ($z'=10$) is again approaching a symmetric form but now is non-negative for all values of z. The maximum value is about 2 at $z=z'=10$. One could interpret what these figures show as follows. When the center of the kink is disturbed, the correlation function acts as if it had a certain resilience such that the average displacement was to be kept zero. However, as one moves away from the center of the soliton and disturbs the field well out onto the tail of the soliton, the average correlation goes positive, with the system responding on the average as if it were just a limp rope. Of course, the latter is exactly the same response that one would receive if no soliton were present in the first place.

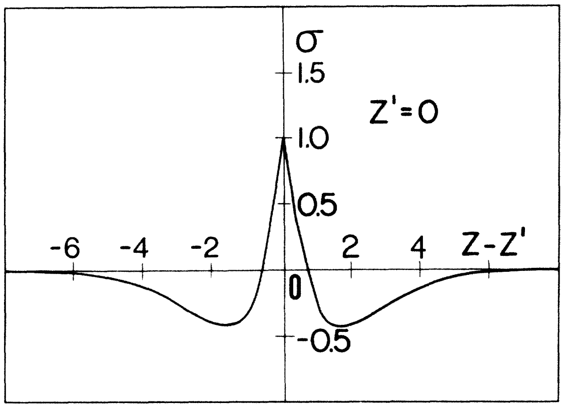

FIG. 2. The equal-time correlation function σ vs z for $z'=0$. This gives the correlation fluctuations relative to the center of the kink.

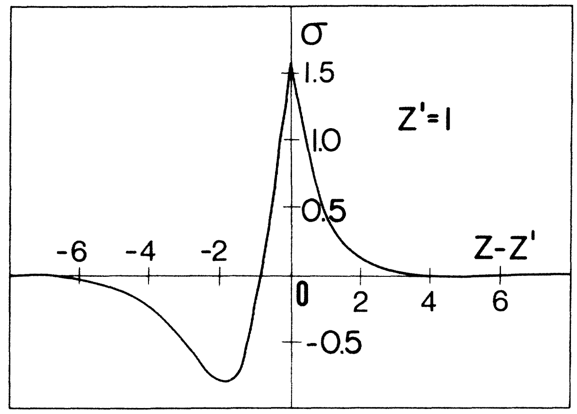

FIG. 3. The equal-time correlation function σ vs z for $z'=1$. This gives the correlation of fluctuations relative to $z'=1$ which is along the edge of a kink.

We shall now briefly comment on the case when $\mathcal{X}=\mathcal{X}'$, when (63) reduces to

$$\sigma(\mathcal{X},\tau;\mathcal{X},\tau')=\frac{2\eta}{\pi\beta E_0}\int_{-\infty}^{\infty}dl\frac{l^2+\eta^2\tanh^2 z}{(l^2+\eta^2)^2(\Gamma^2-4\lambda_l)^{1/2}}$$
$$\times([\Gamma+(\Gamma^2-4\lambda_l)^{1/2}]\exp\{-\tfrac{1}{2}[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}]\,|\tau-\tau'|\,\}$$
$$-[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}]\exp\{-\tfrac{1}{2}[\Gamma+(\Gamma^2-4\lambda_l)^{1/2}]\,|\tau-\tau'|\,\})\ . \tag{64}$$

These are two simple limits, $\Gamma\to 0$ and $\Gamma\to\infty$. In the first case, $\Gamma\to 0$, Eq. (64) can be reduced to

$$\sigma(\mathcal{X},\tau;\mathcal{X},\tau')=\frac{4}{\beta\pi E_0}\int_1^{\infty}dr\frac{r^2-\text{sech}^2 z}{r^3(r^2-1)^{1/2}}(e^{iyr}+e^{-iyr})\ , \tag{65}$$

where

$$y=\sqrt{\kappa}\eta\,|\tau-\tau'|\ , \tag{66}$$

which can be reduced to[21,24]

$$\sigma(\mathcal{X},\tau;\mathcal{X},\tau')=\frac{2}{\beta E_0}\left[2-2\int_0^y ds\,J_0(s)-(\text{sech}^2 z)\left[y^2\int_0^y ds\,J_0(s)-y^2J_1(y)+yJ_0(y)-\int_0^y ds\,J_0(s)-y^2+1\right]\right]\ . \tag{67}$$

In the other case, $\Gamma\to\infty$, Eq. (64) will reduce[21] to

$$\sigma(\mathcal{X},\tau;\mathcal{X},\tau')=\frac{4}{E_0}(1-\text{erf}(\eta\sqrt{Y})-(\text{sech}^2 z)\{-\eta^2Y[1-\text{erf}(\eta\sqrt{Y})]+\eta\sqrt{Y/\pi}\,e^{-\eta^2Y}-\tfrac{1}{2}\text{erf}(\eta\sqrt{Y})+\tfrac{1}{2}\})\ , \tag{68}$$

where now

$$Y=\frac{\kappa}{\Gamma}\,|\tau-\tau'| \tag{69}$$

and

$$\text{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\ . \tag{70}$$

As a final result we shall evaluate the diffusion coefficient.[25] At the center of a kink, $\mathcal{X}=0$, one has for kinks

$$x=\int_0^t v(s)ds\ , \tag{71}$$

where v is the velocity of a kink and x is the center of kink. To zeroth order in F and first order in $1/\beta$, the average of the quantity $\overline{x^2}$ is

$$\overline{x^2}=\frac{1}{(N^{(0)})^2}\int_0^t d\tau\int_0^t ds\int_{-\infty}^{\tau}d\tau'\int_{-\infty}^{\infty}d\mathcal{X}'\psi_b(\mathcal{X}')e^{-\Gamma(\tau-\tau')}\int_{-\infty}^{s}d\tau''\int_{-\infty}^{\infty}d\mathcal{X}''\psi_b(\mathcal{X}'')e^{-\Gamma(s-\tau'')}\overline{\xi(\mathcal{X}',\tau')\xi(\mathcal{X}'',\tau'')}\ . \tag{72}$$

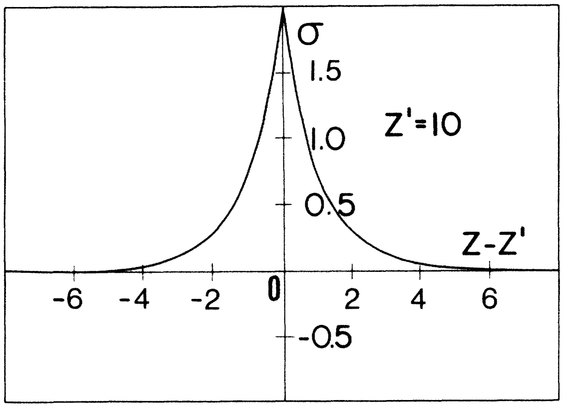

FIG. 4. The equal-time correlation function σ vs z for $z'=10$, which is well out on the tail of the kink, and is the same result as if the kink were absent.

With use of the second-order average of ζ [Eq. (57)], (72) reduces[21] to

$$\overline{x^2}=\frac{2\kappa k_B T}{\Gamma E_0}\left[t-\frac{1}{\Gamma}+\frac{1}{\Gamma}e^{-\Gamma t}\right] . \tag{73}$$

As $t\rightarrow\infty$, the first term dominates, and one has

$$(\overline{x^2})^{1/2}\simeq\sqrt{2Dt} \ , \tag{74}$$

where D is the diffusion coefficient,[26] and is given by

$$D=\frac{\kappa k_B T}{\Gamma E_0} . \tag{75}$$

Comparing this result with that for a free particle,[25]

$$D=\frac{k_B T}{m\eta} , \tag{76}$$

where $-\eta\dot{x}$ is the retarding force, demonstrates again that kinks do behave as particles.[27]

ACKNOWLEDGMENTS

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APPENDIX A

Using the results of Sec. II, we shall detail the expressions for the expansion coefficients $v_j^{(n)}(\tau)$ and $u_j^{(n)}(\chi,\tau)$ when $j=0,1,2$ and $\eta=0,1$. We start with (19) and expand it in our complete set of functions. To determine $v_0^{(1)}$, we take the inner product of (19) with ψ_b, obtaining

$$v_0^{(1)}=-\frac{\langle\psi_b\,|\,1\rangle}{\Gamma N^{(0)}} , \tag{A1}$$

where

$$N^{(0)}=\langle\psi_b\,|\,u_{0\chi}^{(0)}\rangle . \tag{A2}$$

Similarly, upon taking the inner product with χ_e, one obtains

$$u_0^{(1)}=\int_{-\infty}^{\infty}dl\frac{\psi_l\langle\phi_l\,|\,1\rangle}{\lambda_l} . \tag{A3}$$

Next we expand the first-order equation (10) by expanding u_1 and v_1 in regular perturbation expansions in F as in (14) and (15). We find that $u_1^{(0)}$ and $u_1^{(1)}$ satisfy

$$u_{1\tau\tau}^{(0)}+\Gamma u_{1\tau}^{(0)}+L^{(0)}u_1^{(0)}=\zeta+(\dot{v}_1^{(0)}+\Gamma v_1^{(0)})u_{0\chi}^{(0)} \tag{A4}$$

and

$$u_{1\tau\tau}^{(1)}+\Gamma u_{1\tau}^{(0)}+L^{(0)}u_1^{(1)}=2v_0^{(1)}u_{1\chi\tau}^{(0)}+\Gamma v_0^{(1)}u_{1\chi}^{(0)}-u_0^{(1)}u_1^{(0)}U_{uu}(u_0^{(0)})-2v_0^{(1)}v_1^{(0)}u_{0\chi\chi}^{(0)}+(\dot{v}_1^{(0)}+\Gamma v_1^{(0)})u_{0\chi}^{(1)}+(\dot{v}_1^{(1)}+\Gamma v_1^{(1)})u_{0\chi}^{(0)} . \tag{A5}$$

To solve these equations, we take

$$u_1^{(0)}(\chi,\tau)=\int_{-\infty}^{\infty}dl\,\psi_l(\chi)f_{1l}^{(0)}(\tau) , \tag{A6}$$

then inserting (A6) into (A4) and taking the inner product with ψ_b, one finds

$$v_1^{(0)}(t)=\int_{-\infty}^{t}d\tau g_1^{(0)}(\tau)e^{-\Gamma(t-\tau)} , \tag{A7}$$

where

$$g_1^{(0)}(\tau)=-\frac{1}{N^{(0)}}\langle\psi_b\,|\,\zeta\rangle . \tag{A8}$$

Similarly, one obtains

$$f_{1l}^{(0)}(\tau)=\frac{1}{(\Gamma^2-4\lambda_l)^{1/2}}\int_{-\infty}^{\tau}d\tau'\langle\phi_l\,|\,\zeta(\tau')\rangle(\exp\{-\tfrac{1}{2}[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\}-\exp\{-\tfrac{1}{2}[\Gamma+(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\}) . \tag{A9}$$

To determine $v_1^{(1)}$ and $u_1^{(1)}$, we take as before

$$u_1^{(1)}(\chi,\tau)=\int_{-\infty}^{\infty} dl\, \psi_l(\chi) f_{1l}^{(1)}(\tau)\ , \tag{A10}$$

and obtain

$$v_1^{(1)}(t)=\int_{-\infty}^{t} d\tau\, g_1^{(1)}(\tau) e^{-\Gamma(t-\tau)}\ , \tag{A11}$$

where

$$g_1^{(1)}(\tau)=-\frac{1}{N^{(0)}}[2v_0^{(1)}\langle\psi_b\,|\,u_{1\chi\tau}^{(0)}\rangle+\Gamma v_0^{(1)}\langle\psi_b\,|\,u_{1\chi}^{(0)}\rangle-\langle\psi_b\,|\,u_0^{(1)}u_1^{(0)}U_{uu}(u_0^{(0)})\rangle+(\dot v_1^{(0)}+\Gamma v_1^{(0)})\langle\psi_b\,|\,u_{0\chi}^{(1)}\rangle] \tag{A12}$$

and

$$f_{1l}^{(1)}(\tau)=\frac{1}{(\Gamma^2-4\lambda_l)^{1/2}}\int_{-\infty}^{\tau} d\tau' h_1^{(1)}(\tau')(\exp\{-\tfrac{1}{2}[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\}-\exp\{-\tfrac{1}{2}[\Gamma+(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\})\ , \tag{A13}$$

where

$$h_1^{(1)}(\tau)=2v_0^{(1)}\langle\phi_l\,|\,u_{1\chi\tau}^{(0)}\rangle+\Gamma v_0^{(1)}\langle\phi_l\,|\,u_{1\chi}^{(0)}\rangle-\langle\phi_l\,|\,u_0^{(1)}u_1^{(0)}U_{uu}(u_0^{(0)})\rangle-2v_0^{(1)}v_1^{(0)}\langle\phi_l\,|\,u_{0\chi\chi}^{(0)}\rangle+(\dot v_1^{(0)}+\Gamma v_1^{(0)})\langle\phi_l\,|\,u_{0\chi}^{(1)}\rangle\ . \tag{A14}$$

Lastly, we expand the second-order equation (11) by expanding u_2 and v_2 again in powers of F as in (14) and (15). Then $u_2^{(0)}$ and $u_2^{(1)}$ are found to satisfy the following equations, respectively:

$$u_{2\tau\tau}^{(0)}+\Gamma u_{2\tau}^{(0)}+L^{(0)}u_2^{(0)}=2v_1^{(0)}u_{1\chi\tau}^{(0)}-(v_1^{(0)})^2u_{0\chi\chi}^{(0)}+(\dot v_1^{(0)}+\Gamma v_1^{(0)})u_{1\chi}^{(0)}+(\dot v_2^{(0)}+\Gamma v_2^{(0)})u_{0\chi}^{(0)}-\tfrac{1}{2}(u_1^{(0)})^2U_{uu}(u_0^{(0)})\ , \tag{A15}$$

$$\begin{aligned}u_{2\tau\tau}^{(1)}+\Gamma u_{2\tau}^{(1)}+L^{(0)}u_2^{(1)}={}&2v_0^{(1)}u_{2\chi\tau}^{(0)}+\Gamma v_0^{(1)}u_{2\chi}^{(0)}-u_0^{(1)}u_2^{(0)}U_{uu}(u_0^{(0)})+2v_1^{(0)}u_{1\chi\tau}^{(1)}+2v_1^{(1)}u_{1\chi\tau}^{(0)}\\&-2v_0^{(1)}v_1^{(0)}u_{1\chi\chi}^{(0)}-2(v_0^{(1)}v_2^{(0)}+v_1^{(0)}v_1^{(1)})u_{0\chi\chi}^{(0)}-(v_1^{(0)})^2u_{0\chi\chi}^{(1)}\\&+(\dot v_1^{(0)}+\Gamma v_1^{(0)})u_{1\chi}^{(1)}+(\dot v_1^{(1)}+\Gamma v_1^{(1)})u_{1\chi}^{(0)}+(\dot v_2^{(0)}+\Gamma v_2^{(0)})u_{0\chi}^{(1)}\\&+(\dot v_2^{(1)}+\Gamma v_2^{(1)})u_{0\chi}^{(0)}-\tfrac{1}{2}(u_1^{(0)})^2U_{uuu}(u_0^{(0)})u_0^{(1)}-u_1^{(0)}u_1^{(1)}U_{uu}(u_0^{(0)})\ .\end{aligned} \tag{A16}$$

As before, we again take

$$u_2^{(i)}(\chi,\tau)=\int_{-\infty}^{\infty} dl\, \chi_l(\chi) f_{2l}^{(i)}(\tau),\quad i=0,1\ . \tag{A17}$$

Then one obtains

$$v_2^{(i)}(t)=\int_{-\infty}^{t} d\tau\, g_2^{(i)}(\tau)e^{-\Gamma(t-\tau)},\quad i=0,1 \tag{A18}$$

where

$$g_2^{(0)}(\tau)=-\frac{1}{N^{(0)}}[2v_1^{(0)}\langle\psi_b\,|\,u_{1\chi\tau}^{(0)}\rangle+(\dot v_1^{(0)}+\Gamma v_1^{(0)})\langle\psi_b\,|\,u_{1\chi}^{(0)}\rangle-\tfrac{1}{2}\langle\psi_b\,|\,(u_1^{(0)})^2U_{uu}(u_0^{(0)})\rangle]\ , \tag{A19}$$

$$\begin{aligned}g_2^{(1)}(\tau)=\frac{1}{N^{(0)}}[&2v_0^{(1)}\langle\psi_b\,|\,u_{2\chi\tau}^{(0)}\rangle+\Gamma v_0^{(1)}\langle\psi_b\,|\,u_{2\chi}^{(0)}\rangle-\langle\psi_b\,|\,u_0^{(1)}u_2^{(0)}U_{uu}(u_0^{(0)})\rangle+2v_1^{(0)}\langle\psi_b\,|\,u_{1\chi\tau}^{(1)}\rangle\\&+2v_1^{(1)}\langle\psi_b\,|\,u_{1\chi\tau}^{(0)}\rangle-2v_0^{(1)}v_1^{(0)}\langle\psi_b\,|\,u_{1\chi\chi}^{(0)}\rangle-(v_1^{(0)})^2\langle\psi_b\,|\,u_{0\chi\chi}^{(1)}\rangle\\&+(\dot v_1^{(0)}+\Gamma v_1^{(0)})\langle\psi_b\,|\,u_{1\chi}^{(1)}\rangle+(\dot v_1^{(1)}+\Gamma v_1^{(1)})\langle\psi_b\,|\,u_{1\chi}^{(0)}\rangle+(\dot v_2^{(0)}+\Gamma v_2^{(0)})\langle\psi_b\,|\,u_{0\chi}^{(1)}\rangle\\&-\tfrac{1}{2}\langle\psi_b\,|\,(u_1^{(0)})^2u_0^{(1)}U_{uuu}(u_0^{(0)})\rangle-\langle\psi_b\,|\,u_1^{(0)}u_1^{(1)}U_{uu}(u_0^{(0)})\rangle]\ ,\end{aligned} \tag{A20}$$

$$\begin{aligned}f_{2l}^{(i)}(\tau)=\frac{1}{(\Gamma^2-4\lambda_l)^{1/2}}\int_{-\infty}^{\tau} d\tau' h_2^{(i)}(\tau')(&\exp\{-\tfrac{1}{2}[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\}\\&-\exp\{-\tfrac{1}{2}[\Gamma+(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\}),\quad i=0,1\end{aligned} \tag{A21}$$

where

$$h_2^{(0)}(\tau)=2v_1^{(0)}\langle\phi_l\,|\,u_{1\chi\tau}^{(0)}\rangle-(v_1^{(0)})^2\langle\phi_l\,|\,u_{0\chi\chi}^{(0)}\rangle+(\dot v_1^{(0)}+\Gamma v_1^{(0)})\langle\phi_l\,|\,u_{1\chi}^{(0)}\rangle-\tfrac{1}{2}\langle\phi_l\,|\,(u_1^{(0)})^2U_{uu}(u_0^{(0)})\rangle\ , \tag{A22}$$

and

$$h_2^{(1)}(\tau)=2v_0^{(1)}\langle\phi_l\,|\,u_{2X\tau}^{(0)}\rangle+\Gamma v_0^{(1)}\langle\phi_l\,|\,u_{2X}^{(0)}\rangle-\langle\phi_l\,|\,u_0^{(1)}u_2^{(0)}U_{uu}(u_0^{(0)})\rangle+2v_1^{(0)}\langle\phi_l\,|\,u_{1X\tau}^{(1)}\rangle$$
$$+2v_1^{(1)}\langle\phi_l\,|\,u_{1X\tau}^{(0)}\rangle-2v_0^{(1)}v_1^{(0)}\langle\phi_l\,|\,u_{1XX}^{(0)}\rangle-2(v_0^{(1)}v_2^{(0)}+v_1^{(0)}v_1^{(1)})\langle\phi_l\,|\,u_{0XX}^{(0)}\rangle$$
$$-(v_1^{(0)})^2\langle\phi_l\,|\,u_{0XX}^{(1)}\rangle+(\dot v_1^{(0)}+\Gamma v_1^{(0)})\langle\phi_l\,|\,u_{1X}^{(1)}\rangle+(\dot v_1^{(1)}+\Gamma v_1^{(1)})\langle\phi_l\,|\,u_{1X}^{(0)}\rangle$$
$$+(\dot v_2^{(0)}+\Gamma v_2^{(0)})\langle\phi_l\,|\,u_{0X}^{(1)}\rangle-\tfrac{1}{2}\langle\phi_l\,|\,(u_1^{(0)})^2u_0^{(1)}U_{uuu}(u_0^{(0)})\rangle-\langle\phi_l\,|\,u_1^{(0)}u_1^{(1)}U_{uu}(u_0^{(0)})\rangle\;. \qquad \text{(A23)}$$

APPENDIX B

In this appendix we shall detail some of the calculations necessary to evaluate the second-order thermal averages discussed in the text. Details of all of these calculations may be found in Ref. 21.

First we shall evaluate the thermal average of $u_2^{(0)}$. From (A17), (A21), and (A22) we find

$$\overline{u_2^{(0)}}=\int_{-\infty}^{\infty}dl\frac{\psi_l(X)}{(\Gamma^2-4\lambda_l)^{1/2}}\int_{-\infty}^{\tau}d\tau'[2\langle\phi_l\,|\,\partial_X\overline{v_1^{(0)}u_{1\tau'}^{(0)}}\rangle-\overline{(v_1^{(0)})^2}\langle\phi_l\,|\,u_{0XX}^{(0)}\rangle+\langle\phi_l\,|\,\partial_X\overline{\dot v_1^{(0)}u_1^{(0)}}\rangle$$
$$+\Gamma\langle\phi_l\,|\,\partial_X\overline{v_1^{(0)}u_1^{(0)}}\rangle-\tfrac{1}{2}\langle\phi_l\,|\,\overline{(u_1^{(0)})^2}U_{uu}(u_0^{(0)})\rangle]$$
$$\times(\exp\{-\tfrac{1}{2}[\Gamma-(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\}-\exp\{-\tfrac{1}{2}[\Gamma+(\Gamma^2-4\lambda_l)^{1/2}](\tau-\tau')\})\;. \qquad \text{(B1)}$$

Using formulas (A7)–(A9) and (34) to determine some of the averages of the quantities in (B1), one finds

$$\overline{v_1^{(0)}u_{1\tau}^{(0)}}=0\;, \qquad \text{(B2)}$$
$$\overline{\dot v_1^{(0)}u_1^{(0)}}=0\;, \qquad \text{(B3)}$$
$$\overline{v_1^{(0)}u_1^{(0)}}=0\;, \qquad \text{(B4)}$$
$$\overline{(v_1^{(0)})^2}=\frac{1}{\beta(N^{(0)})^2}\;. \qquad \text{(B5)}$$

For evaluating $\overline{(u_1^{(0)})^2}$, and also later calculations, we have found it to be convenient to define the function

$$g(z,z')=\int_{-\infty}^{\infty}dl\frac{\psi_l(X)\phi_l(X')}{\lambda_l}\;. \qquad \text{(B6)}$$

For the sine-Gordon equation, the eigenfunctions[5,18] are

$$\psi_b=(\eta/2)^{1/2}\text{sech}z\;, \qquad \text{(B7)}$$
$$\psi_l=\frac{e^{ilX}(l+i\eta\tanh z)}{[2\pi(l^2+\eta^2)]^{1/2}}\;, \qquad \text{(B8)}$$
$$\phi_l=-\psi_{-l}=\frac{e^{-ilX}(l-i\eta\tanh z)}{[2\pi(l^2+\eta^2)]^{1/2}}\;, \qquad \text{(B9)}$$

which gives[8]

$$g(z,z')=\frac{\frac{1}{2}e^{-2z_>}+\frac{1}{2}e^{2z_<}-|z-z'|}{4\kappa\eta\cosh z\cosh z'}\;, \qquad \text{(B10)}$$

where $z_>$ $(z_<)$ is greater (lesser) of $z=\eta X$ and $z'=\eta X'$. Thus the quantity $\overline{(u_1^{(0)})^2}$, reduces to[21]

$$\overline{(u_1^{(0)})^2}=\frac{1}{\beta}g(z,z)=\frac{2-\text{sech}^2z}{4\beta\kappa\eta}\;. \qquad \text{(B11)}$$

Evaluating $N^{(0)}$ by Eq. (A2) gives

$$N^{(0)}=2s(2\eta)^{1/2}\;. \qquad \text{(B12)}$$

Thus $\overline{(v_1^{(0)})^2}$ in (B5) becomes

$$\overline{(v_1^{(0)})^2}=\frac{1}{8\beta\eta}\;. \qquad \text{(B13)}$$

Finally, one obtains

$$\overline{u_2^{(0)}}=-\frac{s\,\text{sech}z}{8\beta\kappa\eta}[z-\tfrac{1}{2}\tanh z(1+\tfrac{1}{3}\text{sech}^2z)]\;. \qquad \text{(B14)}$$

Next we evaluate the thermal averages of the velocity coefficients. First we note that the zeroth-order result follows from (A1), (B7), and (B1):

$$v_0^{(1)}=-\frac{\pi s}{4\Gamma\eta}\;. \qquad \text{(B15)}$$

Next we must evaluate the second-order terms. Due to symmetry, one can show[21] that

$$\overline{v_2^{(0)}}=0\;. \qquad \text{(B16)}$$

The next two higher terms now become much more complex. From (A18), (A19), and (A20) we have

$$\overline{v_2^{(0)}}=-\frac{1}{\Gamma N^{(0)}}[2\langle\psi_b\,|\,\partial_X\overline{v_1^{(0)}u_{1\tau}^{(0)}}\rangle+\langle\psi_b\,|\,\partial_X\overline{\dot v_1^{(0)}u_1^{(0)}}\rangle+\Gamma\langle\psi_b\,|\,\partial_X\overline{v_1^{(0)}u_1^{(0)}}\rangle-\tfrac{1}{2}\langle\psi_b\,|\,\overline{(u_1^{(0)})^2}U_{uu}\rangle]\;, \qquad \text{(B17)}$$

$$\overline{v_2^{(1)}} = -\frac{1}{\Gamma N^{(0)}}\Big[2v_0^{(1)}\langle\psi_b\,|\,\partial_\chi \overline{u_{2\tau}^{(0)}}\rangle + \Gamma v_0^{(1)}\langle\psi_b\,|\,\partial_\chi \overline{u_2^{(0)}}\rangle - \langle\psi_b\,|\,u_0^{(1)}\overline{u_2^{(0)}}U_{uu}(u_0^{(0)})\rangle$$
$$+2\langle\psi_b\,|\,\partial_\chi\overline{v_1^{(0)}u_{1\tau}^{(1)}}\rangle + 2\langle\psi_b\,|\,\partial_\chi\overline{v_1^{(1)}u_{1\tau}^{(0)}}\rangle - 2v_0^{(1)}\langle\psi_b\,|\,\partial_\chi^2\overline{v_1^{(0)}u_1^{(0)}}\rangle$$
$$-\overline{(v_1^{(0)})^2}\langle\psi_b\,|\,u_{0\chi\chi}^{(1)}\rangle + \langle\psi_b\,|\,\partial_\chi\overline{g_1^{(0)}u_1^{(1)}}\rangle + \langle\psi_b\,|\,\partial_\chi\overline{g_1^{(1)}u_1^{(0)}}\rangle + \overline{g_2^{(0)}}\langle\psi_b\,|\,u_{0\chi}^{(1)}\rangle$$
$$-\tfrac{1}{2}\langle\psi_b\,|\,\overline{(u_1^{(0)})^2}u_0^{(1)}U_{uuu}(u_0^{(0)})\rangle - \langle\psi_b\,|\,\overline{u_1^{(0)}u_1^{(1)}}U_{uu}(u_0^{(0)})\rangle\Big]\,. \tag{B18}$$

From (A8), (A10), (A13), (A14), and (34), one finds

$$\overline{g_1^{(0)}u_1^{(1)}}=0\,. \tag{B19}$$

Also, from (A19) one finds

$$\overline{g_2^{(0)}}=0\,. \tag{B20}$$

Then Eq. (B18) for $\overline{v_2^{(1)}}$ reduces only to nine terms,

$$\overline{v_2^{(1)}} = -\frac{1}{\Gamma N^{(0)}}\Big[2v_0^{(1)}\langle\psi_b\,|\,\partial_\chi \overline{u_{2\tau}^{(0)}}\rangle + \Gamma v_0^{(1)}\langle\psi_b\,|\,\partial_\chi \overline{u_2^{(0)}}\rangle - \langle\psi_b\,|\,u_0^{(1)}\overline{u_2^{(0)}}U_{uu}(u_0^{(0)})\rangle$$
$$+2\langle\psi_b\,|\,\partial_\chi\overline{v_1^{(0)}u_{1\tau}^{(1)}}\rangle + 2\langle\psi_b\,|\,\partial_\chi\overline{v_1^{(1)}u_{1\tau}^{(0)}}\rangle - \overline{(v_1^{(0)})^2}\langle\psi_b\,|\,u_{0\chi\chi}^{(1)}\rangle + \langle\psi_b\,|\,\partial_\chi\overline{g_1^{(1)}u_1^{(0)}}\rangle$$
$$-\tfrac{1}{2}\langle\psi_b\,|\,\overline{(u_1^{(0)})^2}u_0^{(1)}U_{uuu}(u_0^{(0)})\rangle - \langle\psi_b\,|\,\overline{u_1^{(0)}u_1^{(1)}}U_{uu}(u_0^{(0)})\rangle\Big]\,. \tag{B21}$$

The fourth, fifth, and ninth terms in Eq. (B21) cannot be evaluated in a closed form, as best as we can determine, although their values can be given as definite integrals. If one defines the six integrals,

$$I_1=\int_0^\infty \frac{\pi A^2 dx}{(x^2+A^2+1)\cosh^2[(\pi/2)x]}\,, \tag{B22}$$

$$I_2=\int_0^\infty \frac{A^2 x\, dx}{(x^2+1)(x^2+A^2+1)\sinh(\pi x)}\,, \tag{B23}$$

$$I_3=\int_0^\infty dy\int_0^\infty dx \frac{A^2(x^2-y^2)}{(y^2+A^2+1)(x^2+1)^2\sinh[(\pi/2)(x-y)]\sinh[(\pi/2)(x+y)]}\,, \tag{B24}$$

$$I_4=\int_0^\infty dy\int_0^\infty dx \frac{(x^2-y^2)^2}{(y^2+A^2+1)(x^2+1)^2\sinh[(\pi/2)(x-y)]\sinh[(\pi/2)(x+y)]}\,, \tag{B25}$$

$$I_5=\int_0^\infty dx \frac{\pi\{4A^2x^2+A^4-A^3[(8x^2+A^2)/(x^2+1)]^{1/2}\}}{[16x^2(x^2+1)+A^2(8x^2+A^2)]\sinh^2(\pi x)}\,, \tag{B26}$$

$$I_6=\int_0^\infty dy\int_0^\infty dx \frac{x^2(x^4+y^4-2x^2y^2-\frac{3}{2}y^2+\frac{1}{2}x^2-\frac{7}{16})}{(1+4y^2)^2(x^2+1)\cosh[\pi(x-y)]\cosh[\pi(x+y)]}\,, \tag{B27}$$

where

$$A^2=\frac{2\Gamma^2}{\kappa\eta^2}\,, \tag{B28}$$

then one finds[21]

$$\overline{v_2^{(1)}}=-\frac{\pi s c}{32\kappa\beta\Gamma\eta^2}\,, \tag{B29}$$

where

$$c=\frac{(11)(169)}{(125)(100)}+\tfrac{1}{4}I_1+2I_2+\tfrac{1}{2}I_3-\tfrac{1}{2}I_4-I_5+8I_6\,. \tag{B30}$$

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