Comments

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Comment on "Annihilation and creation of a Korteweg–de Vries soliton"

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The expression reported recently as a new solution of the Korteweg–de Vries equation is discussed and reinterpreted. It is shown that depending on the choice of sign one obtains two different solutions corresponding to an interaction of rational quasistatic singularity with regular or singular solitons, respectively. In both cases the interaction can be described in terms of a two-soliton collision. The two interacting objects can preserve or exchange their identities; they are, however, neither annihilated nor created during

In a recent paper[1] Au and Fung (hereafter referred to as AF) discussed a new solution of the Korteweg–de Vries (KdV) equation:

$$u_t + 12uu_x + u_{xxx} = 0 \ . \tag{1}$$

Starting with the static rational solution $u = -1/x^2$ and applying the Bäcklund transformation, AF obtained the following, rather complicated expression:[1]

$$u = \frac{1}{x^2} + \lambda - \left\{ \frac{1}{x^2(\lambda x^2-1)^2} + \frac{2\lambda^{3/2}x}{(\lambda x^2-1)^2}\left[\frac{\exp(\sqrt{\lambda}\xi)-\exp(-\sqrt{\lambda}\xi)}{\exp(\sqrt{\lambda}\xi)+\exp(-\sqrt{\lambda}\xi)}\right] + \frac{\lambda^3 x^4}{(\lambda x^2-1)^2}\left[\frac{\exp(\sqrt{\lambda}\xi)-\exp(-\sqrt{\lambda}\xi)}{\exp(\sqrt{\lambda}\xi)+\exp(-\sqrt{\lambda}\xi)}\right]^2 \right\} , \tag{2}$$

where

$$\xi = x + \frac{1}{2\sqrt{\lambda}} \ln\frac{|\sqrt{\lambda}x-1|}{|\sqrt{\lambda}x+1|} - 4\lambda t, \quad \lambda > 0 \ .$$

Unfortunately, both the mathematical form of Eq. (2) and its physical interpretation give rise to serious doubts. From the mathematical point of view, expression (2) satisfies the KdV equation in three separate regions ($x < -1/\sqrt{\lambda}$, $-1/\sqrt{\lambda} < x < 1/\sqrt{\lambda}$, $x > 1/\sqrt{\lambda}$); it is, however, discontinuous at the boundaries $x = \pm 1/\sqrt{\lambda}$. Thus, strictly speaking, Eq. (2) cannot be regarded as a solution of the KdV equation valid over the whole x axis.

On the other hand, the Bäcklund transformation may be described roughly as a procedure of adding a new soliton to the already existing "old" solution.[2] In this context, the very complicated analysis reported by AF (annihilation and creation of a soliton, additional disturbance interpreted as a "message") is unclear and physically ambiguous.

In this clarifying Comment we show that the above doubts follow simply from the misinterpretation of Eq. (2). In fact, we deal with *two* distinct solutions, and the interaction between a soliton and a rational quasistatic singularity can be easily described in well-known terms as a limiting case of the two-soliton solution.

The detailed derivation of Eq. (2) is not given by AF, but it seems that the absolute value of the argument in $\ln(|\sqrt{\lambda}x-1|/|\sqrt{\lambda}x+1|)$ has been introduced in order to ensure a real value of the ln function. Note, however, that in the particular case of Eq. (2) the logarithm enters the formalism through the exponential function. As a result, both positive and negative values of the argument $\pm(\sqrt{\lambda}x-1)/(\sqrt{\lambda}x+1)$ lead to a real solution u (although the logarithm itself may be complex). The above observation is crucial for the further analysis and enables us to explain the main misunderstanding in Ref. 1.

Indeed, let us assume in Eq. (2) $\ln[(\sqrt{\lambda}x-1)/(\sqrt{\lambda}x+1)]$ over the whole x axis. It can be easily checked that the resulting solution is real and continuous for all x (except for singularity). After some algebraic transformations we obtain the following simple form:

$$u = -k^2 \frac{\exp(2\theta) - 2[2(kx)^2+1]\exp(\theta)+1}{[(kx-1)\exp(\theta)+(kx+1)]^2} \ , \tag{3}$$

where $\theta = 2kx - 8k^3 t$; $k = \sqrt{\lambda}$.

Similarly, assuming in (2) $\ln[(1-\sqrt{\lambda}x)/(1+\sqrt{\lambda}x)]$, we have

$$u = -k^2 \frac{\exp(2\theta) + 2[2(kx)^2+1]\exp(\theta)+1}{[(kx-1)\exp(\theta)-(kx+1)]^2} \ . \tag{4}$$

The solution (4) is again real and continuous for all x (except for singularities). It is, however, essentially different from the solution (3).

It is interesting that Eqs. (3) and (4) can be derived in a much simpler way from the well-known multiple-soliton formula.[3,4] Indeed, for the KdV equation given by (1), the two-soliton solution can be written in the Hirota form as

$$u=(\ln f)_{xx} \ , \tag{5}$$

where

$$f=1+\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_1+\eta_2+A_{12}) \ ,$$
$$\exp A_{12}=(k_1-k_2)^2/(k_1+k_2)^2 \ ,$$
$$\eta_i=k_ix-k_i^3t+\eta_i^{(0)}, \quad i=1,2 \ ;$$

k_i, $\eta_i^{(0)}$ are arbitrary constants.

For k_i and $\eta_i^{(0)}$ real, Eq. (5) describes a nonlinear interaction of two regular solitons having an asymptotic form

$$\tfrac{1}{4}k_i^2\,\mathrm{sech}^2[\tfrac{1}{2}(k_ix-k_i^3t)] \ .$$

However, for k_i real and $\mathrm{Im}\eta_i^{(0)}=\pi$, instead of regular soliton we obtain a singular soliton of the type $-\frac{1}{4}k_i^2 \times \mathrm{csch}^2[\frac{1}{2}(k_ix-k_i^3t)]$. The singular-soliton solution can be further transformed into a rational static form[4] by taking the limit $k_i \to 0$:

$$-\tfrac{1}{4}k_i^2\,\mathrm{csch}^2[\tfrac{1}{2}(k_ix-k_i^3t)] \to -1/x^2 \ .$$

Let us assume $\eta_1^{(0)}=0$, $\eta_2^{(0)}=i\pi$, $k_2 \to 0$. In other words, we transform the second soliton into rational form, while keeping the first one unchanged. According to Eq. (5) we obtain

$$f \propto (4-k_1x)\exp(\eta)-k_1x \ , \tag{6}$$

where $\eta=k_1x-k_1^3t$. Changing the variables $x \to x+2/k_1$, $t \to t+2/k_1^3$, and substituting $k_1=2k$, we have

$$f \propto (1-kx)\exp(\theta)-(1+kx) \ , \tag{7}$$

where $\theta=2kx-8k^3t$ as in (3). It can be easily verified that $(\ln f)_{xx}$ for f given by (7) yields the solution which is identical with Eq. (3). (Note that the proportionality factor is unimportant in the logarithmic derivative of f.)

Similarly, one can show that the substitution $\eta_1^{(0)}=\eta_2^{(0)}=i\pi$, $k_2 \to 0$ leads to a nonlinear interaction of a singular soliton with rational component,

$$f \propto (kx-1)\exp(\theta)-(kx+1) \ . \tag{8}$$

In this case, the resulting solution $(\ln f)_{xx}$ is identical to Eq. (4).

It should be mentioned here that Eq. (3) is equivalent to the "quasisoliton"[5] and "pole-soliton"[6] solutions reported some years ago. On the other hand, the solution (4) has not been quoted so far in the literature, but its existence can be deduced from general considerations on the Hirota multiple-soliton formula.[7]

The time evolution of Eqs. (3) and (4) for $k=1$ has been shown in Figs. 1 and 2, respectively. It is clear that the interaction between two "objects" is much simpler than that reported by AF. In accordance with the derivation of Eqs. (7) and (8), Fig. 1 represents a nonlinear superposition of a regular soliton and quasistatic rational singularity, while in Fig. 2 we can see similar (asymptotically the same) quasi-

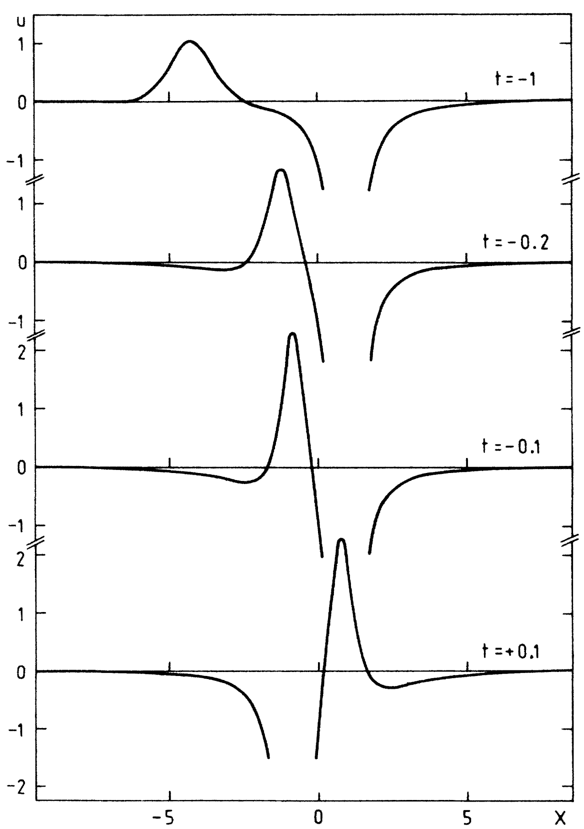

FIG. 1. Solution (3) as a function of x for $k=1$ and $t=-1$, -0.2, -0.1, $+0.1$. For $t=0$ the moving soliton becomes infinitely tall, while its width tends to zero.

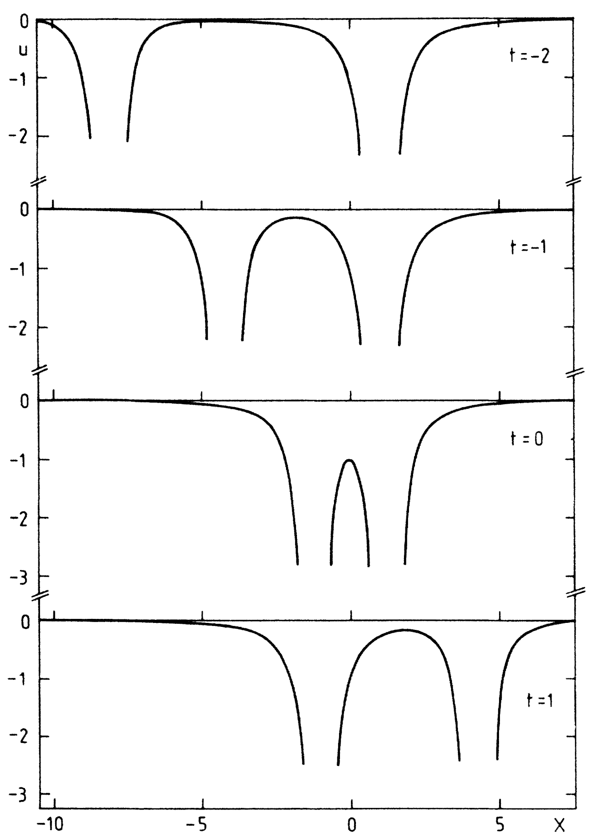

FIG. 2. Solution (4) as a function of x for $k=1$ and $t=-2$, $-1, 0, +1$.

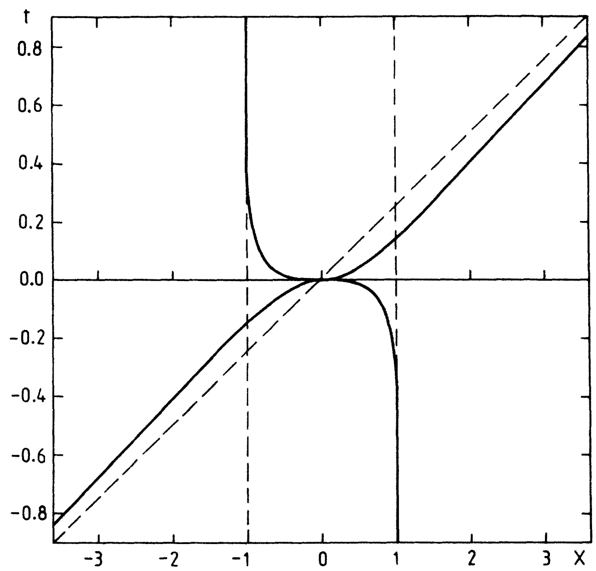

FIG. 3. x-t graph corresponding to Eq. (3). Solid line, trajectories of two interacting objects; dashed line, asymptotes for $|t|\rightarrow\infty$.

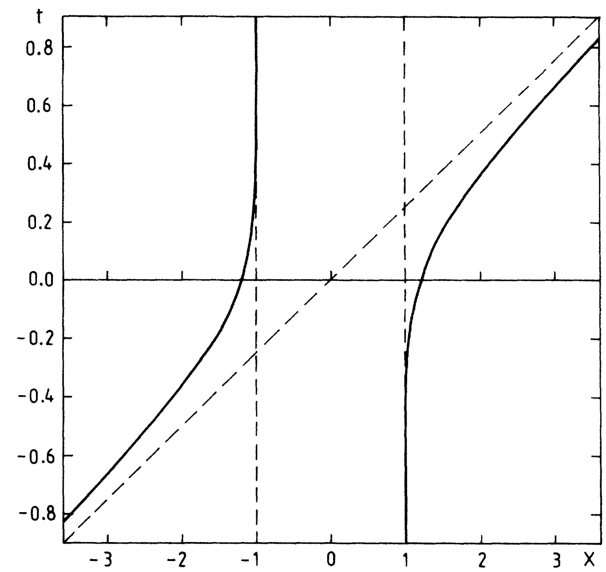

FIG. 4. The same as in Fig. 3, but for Eq. (4).

static singularity interacting with a singular rather than regular soliton.

In order to discuss the interaction process in more detail, the trajectories corresponding to Eqs. (3) and (4) have been shown in Figs. 3 and 4, respectively. For singular objects we have identified their positions in the x-t plane with the corresponding poles of Eqs. (3) and (4). The position of the regular soliton, however, has been determined uniquely by the (x,t) coordinates of its peak. Such an approach seems more reasonable than the criterion $\xi=0$ suggested by AF, although other definitions of the soliton position are also possible.[8]

It should be noted that for both (3) and (4) the net phase shift of the moving soliton is zero; i.e., the trajectory tends asymptotically to the line $x-4t=0$ for $|t|\rightarrow\infty$. On the other hand, the quasistatic rational singularity experiences a shift by $\Delta x=-2$. Such behavior is in complete agreement with general features of multiple-soliton collisions.[9] Indeed, for two interacting solitons the phase shift measured along the x axis can be expressed as follows:

$$\Delta x_1=-A_{12}/k_1,\ \ \Delta x_2=+A_{12}/k_2,\ \ k_1>k_2\ , \tag{9}$$

where A_{12} has been defined in (5). As shown earlier, Eqs. (3) and (4) can be derived from the two-soliton solution by taking the limit $k_2\rightarrow 0$. Thus, substituting into (9) $k_1=2k$, $k_2\rightarrow 0$, we obtain the following for $k=1$: $\Delta x_1=0$, $\Delta x_2=-2$.

The question of whether solitons maintain or exchange their identities is still open. Recently, Bowtell and Stuart[8] have shown, on the basis of pole dynamics in the complex domain, that two regular KdV solitons exchange rather than maintain their identities during an interaction. A similar situation is illustrated in Figs. 2 and 4, where the incident singular soliton decelerates and eventually stops at $x=-1$, while another singular object (being a stationary rational solution for $t\rightarrow-\infty$) starts from $x=1$ and accelerates, attaining asymptotically the shape and velocity of the incident soliton. The problem becomes more complicated for the interaction described by Eq. (3) (see Figs. 1 and 3), since the trajectories cross each other. However, careful analysis in the complex domain shows that the regular soliton simply passes through the singularity and the two objects maintain their identities for all t. Thus, two different mechanisms of the interaction are possible, depending on the singular or regular nature of interacting objects.

In summary, in this Comment we have shown that Eq. (2) reported by AF represents, in fact, two distinct solutions, (3) and (4). Both solutions can be easily obtained from the well-known multiple-soliton expression by performing an appropriate limiting procedure. In this formalism the quasistatic rational singularity may be viewed as a limiting case of a singular-soliton solution, and there is no eason to interpret $u=-1/x^2$ as an "interaction field." 'or both (3) and (4) we deal with two well-defined objects vhich (a) exist for each t, (b) interact according to the vell-known principles of multiple-soliton collisions, and (c) either preserve or exchange their identities during the interaction.

Thus, it seems that the far-reaching conclusions drawn by AF on the annihilation and creation of KdV solitons, unified description of particles and fields, etc., are misleading or at least premature. On the other hand, the soliton velocity and shape varying as a function of x are nothing new, since similar phenomena are also observed in multiple-soliton collisions. The difference is merely quantitative, since the effective interaction region is much larger for the rational solution $u=-1/x^2$ than for the rapidly decreasing soliton profiles.

Finally, it should be mentioned that the above discussion shows clearly the usefulness of the Hirota method for obtaining and interpreting various solutions of the KdV equation. As pointed out in Ref. 7, the Hirota multisoliton expression can describe in a unified way regular and singular solitons, rational solutions as well as their various combinations. The two solutions discussed in this Comment belong to the above class and can be named "rational-regular soliton" and "rational-singular soliton," respectively.

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