Temperature dependence of the conductivity for the two-dimensional electron gas: Analytical results for low temperatures

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We present analytical results for the static conductivity of the two-dimensional electron gas at low temperature. The conductivity is expressed as $\sigma(T)=\sigma(0)[1-C(\alpha,n)T/\varepsilon_F - D(\alpha,n)(T/\varepsilon_F)^{3/2}+O(T^2)]$. Analytical expressions for $\sigma(0)$ in the case of impurity and surface roughness scattering are given. $C(\alpha,n)$ and $D(\alpha,n)$ are universal functions, depending on the $q^{2\alpha}$ dependence of the scattering potential via $C(\alpha)$ and the density n of electrons via $C(n)$. It is shown that the energy-dependent conductivity at temperature zero has a singularity at the Fermi energy. Our results are compared with other theoretical results and with experiments in Si-MOS (metal-oxide-semiconductor) systems.

I. INTRODUCTION

Transport properties of two-dimensional electron systems are of great practical interest because of the industrial applications of metal-oxide-semiconductor (MOS) systems; for a review see Ref. 1. From the theoretical point of view, great attention has been induced by the discovery of absence of diffusion in a noninteracting two-dimensional electron gas at zero temperature.[2] The corresponding logarithmic temperature decrease of the conductivity for decreasing temperature has been established for Si accumulation and inversion layers in the relatively small mobility specimens, see Ref. 1. But in the samples with relatively high mobility, a linear increase has been found when temperature was decreased.[3] This dependence cannot be explained in terms of the electron-phonon scattering processes[4] and has been explained by the temperature dependence of the screening function for elastic scattering.[5]

The "approximately linear increase" of the mobility by decreasing the temperature was found numerically within lowest-order coupling of the scattering potential with the electrons.[5] It is anomalous in the sense that the Kubo-Greenwood formula[6] predicts only even powers for the temperature dependence within a Sommerfeld expansion, when a noninteracting electron gas is considered. In the following we give an analytical expression for the conductivity at low temperatures, including linear temperature corrections to the conductivity. It will be shown that the linear term is proportional to the mobility if the electron concentration is held constant. Analytical results for the temperature zero conductivity in the case of charged impurity scattering and surface roughness scattering are given and compared with numerical results. The analytical coefficient for the linear temperature dependence is derived and compared with experiments. The finite extension of the wave functions of the electron gas perpendicular to the Si-SiO_2 interface and local field corrections are taken into account.

The paper is organized as follows. In Sec. II we discuss our theory and the approximations. The analytical results are presented in Sec. III. Our expressions are compared with other theoretical results and with experiments in Sec. IV. A conclusion of the paper is given in Sec. V.

II. THEORY AND APPROXIMATIONS

A. Compressibility of the free-electron gas

The compressibility of the noninteracting electron gas $g^0(q,T,\mu)$ for wave number q, temperature T, and chemical potential μ is given by[7]

$$g^0(q,T,\mu)=\frac{1}{4T}\int_0^\infty d\varepsilon \frac{g^0(q,T,\varepsilon)}{\cosh^2[(\mu-\epsilon)/2T]} . \tag{1a}$$

$\hbar$ and k_B are set equal to unity in this paper. $g^0(q,T=0,\epsilon)$ is the compressibility at zero temperature.[8] Explicitly one gets

$$g^0(q,T,\mu)=\tfrac{1}{2}\rho_F\left[1+\tanh\left[\frac{\mu}{2T}\right]\right] - 2\rho_F\left[\frac{Tm}{q^2}\right]^{1/2}\int_0^{q^2/16mT} dx \frac{\sqrt{x}}{\cosh^2(x+a)} , \tag{1b}$$

with $a=(\mu-q^2/8m)/2T$. $\rho_F=g_v m/\pi$ is the density of states, m is the electron mass, and g_v the valley degeneracy.

For low temperatures and small q the integral in Eq. (1b) can be extended to infinity and we find

$$g^0(q,T,\mu)=\rho_F\left\{\frac{1}{2}\left[1+\tanh\left[\frac{\mu}{2T}\right]\right] - \left[2\pi\frac{Tm}{q^2}\right]^{1/2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{1/2}}e^{-2|a|n}\right\} . \tag{2a}$$

In the following we make no difference between μ and the Fermi energy ϵ_F, because we restrict our calculations to

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low temperatures and $\epsilon_F=\mu+T\ln(1+e^{-\mu/T})$. For $q=2k_F$, we receive from Eq. (2a) for $T\ll\epsilon_F$,

$$g^0(q=2k_F,T,\epsilon_F)=\rho_F\left[1-0.268\left[\frac{T}{\epsilon_F}\right]^{1/2}\right]. \quad (2b)$$

Our result for the compressibility is in good agreement with numerical results[5,7] for $T\ll\epsilon_F$.

The dielectric function $\epsilon(q,T)$ for the interacting electron gas at $T=0$ is expressed as

$$\epsilon(q,T=0)=1+V(q)g^0(q,T=0,\epsilon_F)[1-G(q)]. \quad (3)$$

$V(q)$ is the interaction potential of the electron gas in two dimensions,

$$V(q)=\frac{2\pi e^2}{\epsilon_L}\frac{1}{q}F(q). \quad (4)$$

ϵ_L is the dielectric constant of the host material, e is the electron charge, and $F(q)$ is a form factor for the finite extent of the wave function perpendicular to the two-dimensional sheet of the electrons and is specified later. $G(q)$ is the local field correction in Hubbard's approximation:[9] $G(q)=(1/2g_v)q/(q^2+k_F^2)^{1/2}$. k_F is the Fermi momentum. According to Eqs. (1) and (3) the temperature dependence of the compressibility is transferred to a temperature dependence of the dielectric function.[10]

B. Energy-dependent relaxation time

Normally the Kubo-Greenwood formula is written as[6]

$$\sigma(\epsilon_F,T)=\frac{1}{4T}\int_0^\infty d\epsilon\frac{\sigma(\epsilon,T=0)}{\cosh^2[(\epsilon-\epsilon_F)/2T]}, \quad (5)$$

and $\sigma(\epsilon,T=0)=(ne^2/m)\tau(\epsilon,T=0)$. $1/\tau(\epsilon,T=0)$ is the momentum relaxation rate for an electron with energy $\varepsilon=k^2/2m$ and is given by[1]

$$\frac{1}{\tau(\epsilon,T=0)}=\frac{1}{2\pi\varepsilon}\int_0^{2k}dq\frac{q^2}{(4k^2-q^2)^{1/2}}\frac{\langle|U(q)|^2\rangle}{[\epsilon(q)]^2}. \quad (6)$$

$\langle|U(q)|^2\rangle$ is the Fourier transform of the random scattering potential and is defined later. With

$$\epsilon_1(q)=1+V(q)[1-G(q)]F(q)\rho_F, \quad (7a)$$

we take into account the anomalous screening behavior for $q>2k_F$ and write

$$\epsilon(q)=\epsilon_1(q)\left[1-\frac{V(q)[1-G(q)]F(q)\rho_F}{\epsilon_1(q)}\times\left[1-\frac{4k_F^2}{q^2}\right]^{1/2}\Theta(q^2-4k_F^2)\right]. \quad (7b)$$

$\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$. With Eq. (7), we express $1/\tau(\epsilon,T=0)$ in the form

$$\frac{1}{\tau(\epsilon,T=0)}=\frac{1}{\tau_0(\epsilon,T=0)}+\sum_{n=1}^\infty\frac{1}{\tau_n(\epsilon,T=0)} \quad (8a)$$

with

$$\frac{1}{\tau_0(\epsilon,T=0)}=\frac{1}{2\pi\varepsilon}\int_0^{2k}dq\,q^2\frac{1}{(4k^2-q^2)^{1/2}}\frac{\langle|U(q)|^2\rangle}{[\epsilon_1(q)]^2}, \quad (8b)$$

and

$$\frac{1}{\tau_n(\epsilon,T=0)}=\frac{n+1}{2\pi\epsilon}\int_{2k_F}^{2k}dq\,q^{2-n}\frac{(q^2-4k_F^2)^{n/2}}{(4k^2-q^2)^{1/2}}\frac{\langle|U(q)|^2\rangle}{[\epsilon_1(q)]^2}\times\left[\frac{V(q)[1-G(q)]F(q)\rho_F}{\epsilon_1(q)}\right]^n\times\Theta(\epsilon-\epsilon_F). \quad (8c)$$

C. Temperature dependence due to screening

The relaxation time $\tau(\epsilon_F,T=0)$ determines the static conductivity via $\sigma(\epsilon_F,T=0)=(ne^2/m)\tau(\epsilon_F,T=0)$ and the mobility via $\mu(\epsilon_F,T=0)=(e/m)\tau(\epsilon_F,T=0)$. n is the density of electrons. In the Born approximation one gets

$$\frac{1}{\tau(\epsilon_F,T=0)}=\frac{1}{2\pi\varepsilon_F}\int_0^{2k_F}dq\,q^2\frac{1}{(4k_F^2-q^2)^{1/2}}\frac{\langle|U(q)|^2\rangle}{[\epsilon(q,T=0)]^2}. \quad (9)$$

To take into account the temperature dependence of the screening, we replace in Eq. (9) $\epsilon(q,T=0)$ by $\epsilon(q,T)$,

$$\frac{1}{\tau(\epsilon_F,T)}=\frac{1}{2\pi\epsilon_F}\int_0^{2k_F}dq\,q^2\frac{1}{(4k_F^2-q^2)^{1/2}}\frac{\langle|U(q)|^2\rangle}{[\epsilon(q,T)]^2}. \quad (10)$$

With Eq. (2a), written as $g^0(q,T,\epsilon_F)=\rho_F-\Delta g^0(q,T,\epsilon_F)$, we expand Eq. (10) into the form

$$\frac{1}{\tau(\epsilon_F,T)}=\frac{1}{\tau(\epsilon_F,T=0)}+\sum_{n=1}^\infty\frac{1}{\Delta\tau_n(\epsilon_F,T)}, \quad (11a)$$

with

$$\frac{1}{\tau(\epsilon_F,T=0)}=\frac{1}{2\pi\epsilon_F}\int_0^{2k_F}dq\,q^2\frac{1}{(4k_F^2-q^2)^{1/2}}\times\frac{\langle|U(q)|^2\rangle}{[\epsilon(q,T=0)]^2}, \quad (11b)$$

and

$$\frac{1}{\Delta\tau_n(\epsilon_F,T)}=\frac{n+1}{2\pi\epsilon_F}\int_0^{2k_F}dq\frac{q^2}{(4k_F^2-q^2)^{1/2}}\frac{\langle|U(q)|^2\rangle}{[\epsilon(q,T=0)]^2}\times\left[\frac{V(q)[1-G(q)]\Delta g^0(q,T,\epsilon_F)}{\epsilon(q,T=0)}\right]^n. \quad (11c)$$

Equation (11b) is the classical formula for the $T=0$ momentum relaxation time, and we will see later that Eq. (11c) gives a linear T contribution to $1/\tau(T)$.

D. Scattering mechanisms

In the following we discuss scattering by charged impurities of density n_i, located in a two-dimensional plane. Then the random potential is expressed as[11]

$$\langle |U(q)|^2\rangle = n_i \left[\frac{2\pi e^2}{\epsilon_L}\frac{1}{q}\right]^2 [F_i(q)]^2 . \tag{12}$$

$F_i(q)$ is a form factor because of the finite extension of the wave function. The scattering by the surface roughness is expressed as[12]

$$\langle |U(q)|^2\rangle = \pi\Delta^2\Lambda^2 q_s^2 \epsilon_F^2 (1+2N_D/n)^2 e^{-q^2\Lambda^2/4} . \tag{13}$$

Δ and Λ are the length parameters of the surface-roughness scattering, q_s is the Fermi screening wave number at $T=0$, and N_D is the depletion density. When the scattering potential is characterized by the q dependence,

$$\langle |U(q)|^2\rangle = \frac{\pi}{\alpha!}(q\Lambda/2)^{2\alpha}\Lambda^2 U^2 e^{-q^2\Lambda^2/4} , \tag{14}$$

with a power 2α, a strength U and a range Λ, then the impurity scattering corresponds to $\alpha=-1$ and the surface-roughness scattering corresponds to $\alpha=0$.

In the following we consider a Si(100) MOS system with $m^*=0.19m_0$, m_0 is the electron mass in vacuum, $g_v=2$, and $\epsilon_L=7.7$. The form factor $F(q)$ is expressed as[11]

$$F(q)=\frac{1}{2}\left[1+\frac{\epsilon_1}{\epsilon_2}\right]\left[1+\frac{9}{8}\frac{q}{b}+\frac{3}{8}\frac{q^2}{b^2}\right]\Big/\left[1+\frac{q}{b}\right]^3+\frac{1}{2}\left[1-\frac{\epsilon_1}{\epsilon_2}\right]\Big/\left[1+\frac{q}{b}\right]^6 .$$

$\epsilon_1=3.9$ is the dielectric constant of the SiO_2 and $\epsilon_2=11.5$ is the dielectric constant of the Si. $1/b$ is the length parameter for the thickness of the electron gas and given by[13] $b=(48\pi m_\perp e^2 N^*/\epsilon_1)^{1/3}$. $m_\perp=0.916$ is the electron mass for the motion perpendicular to the Si-SiO_2 interface, $N^*=N_D+\frac{11}{32}n$. $F_i(q)$ is the form factor for the impurity electron interaction, and when the impurities are located in the Si-SiO_2 interface, it is given by[11] $F_i(q)=1/(1+q/b)^3$. When finite extension effects of the electron gas are neglected, we have $F(q)=F_i(q)=1$. Now our model is completely specified and we discuss the results of our theory.

III. ANALYTICAL RESULTS

A. Conductivity at zero temperature

Because of the square-root singularity in Eq. (11b) the main contribution from the q integral comes from the $q\approx 2k_F$ region. For $\langle |U(q)|^2\rangle \propto q^{2\alpha}$ we evaluate then Eq. (11b) in the form $\int_0^1 dx x^{2\alpha+4}/(1-x^2)^{1/2}$ and use the involved form factors at $q=2k_F$. For impurity scattering, we obtain

$$\frac{1}{\tau(\epsilon_F,T=0)}=\frac{\pi}{g_v}\epsilon_F\frac{n_i}{n}\left[\frac{F_i(2k_F)}{F(2k_F)[1-G(2k_F)]+2k_F/q_s}\right]^2 \tag{15}$$

and for surface-roughness scattering we receive, for $k_F\Lambda \ll 1$,

$$\frac{1}{\tau(\epsilon_F,T=0)}=\frac{3\pi}{2}\epsilon_F(\Delta\Lambda k_F)^2\left[\frac{1+2N_D/n}{F(2k_F)[1-G(2k_F)]+2k_F/q_s}\right]^2 . \tag{16}$$

For the scattering potential, defined in Eq. (14), we get for $k_F\Lambda \ll 1$:

$$\frac{1}{\tau(\epsilon_F,T=0)}=\frac{\pi}{\alpha!}\frac{(3+2\alpha)!!}{(4+2\alpha)!!}\epsilon_F(\Lambda k_F)^{2\alpha+2}\left[\frac{2k_F}{q_s}\right]^2\frac{U^2}{\epsilon_F^2}\left[\frac{1}{F(2k_F)[1-G(2k_F)]+2k_F/q_s}\right]^2 . \tag{17}$$

In Fig. 1 we compare our analytical formulas with the exact numerical results. For $n<5\times10^{12}$ cm^{-2} the differences between the exact expressions and our analytical formulas are smaller than 50% in the case of impurity scattering and smaller than 30% in the case of surface-roughness scattering. The higher the power in q of the random potential and the smaller the density, the better is our approximative evaluation of the q integral. At high electron density surface-roughness scattering is the relevant scattering mechanism in Si-MOS systems.[1] For $n>8\times10^{12}$ cm^{-2}, one expects that the second subband becomes occupied, and so we believe that our analytical formulas can be used to analyze very easily conductivity measurements. Especially we propose to use our analytical results to get some insight into local field corrections. In three dimensions, the behavior of the local field corrections at $q=2k_F$ is very controversial, for a review see Ref. 14, and has some relevance in connection with phase transitions. The influence of $G(q)$ on the relaxation time has been discussed for impurity scattering[15] and surface-

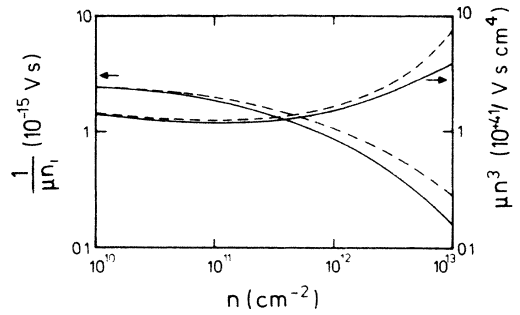

FIG. 1. $1/\mu n_i$ and μn^3 versus density for impurity scattering and surface-roughness scattering, respectively. The full lines are our analytical results, the dashed lines the numerical results. The parameters are $\Delta=4$ Å, $\Lambda=15$ Å, and $N_D=0$.

roughness scattering.[16]

B. Linear correction for finite temperature

With Eq. (11c) for $1/\tau_n(\epsilon_F,T)$, we proceed in the following way. From our Eq. (1b) it is easily seen that the temperature gives the strongest effect at $q\approx 2k_F$. Then we transfer the q integral in Eq. (11c) into a $\epsilon=2k_F-q$ integral. The integration interval $0\le q\le 2k_F$ is extended to $0\le\epsilon<\infty$ and only the most singular term $1/\epsilon^{1/2}$ is used for the integration. This gives again a factor $T^{1/2}$ in addition to $T^{1/2}$ from Δg^0, see Eq. (2a), and $1/\Delta\tau_1(\epsilon_F,T)$ is proportional to T. The calculation is exact in the limit $T\to 0$ and we receive

$$\frac{1}{\Delta\tau_1(\epsilon_F,T)}=(T\ln 2)k_F^2\frac{\langle\,|U(2k_F)|^2\rangle}{\epsilon_F^2}\times\frac{V(2k_F)\rho_F[1-G(2k_F)]}{\{[1+V(2k_F)\rho_F[1-G(2k_F)]\}^3}\,. \tag{18}$$

Explicitly we find for impurity scattering,

$$\frac{1}{\Delta\tau_1(\epsilon_F,T)}=T\frac{2\pi}{g_v}\ln 2\frac{n_i}{n}\times\frac{F_i(2k_F)^2F(2k_F)[1-G(2k_F)]}{\{F(2k_F)[1-G(2k_F)]+2k_F/q_s\}^3}\,, \tag{19a}$$

and for surface-roughness scattering for $k_F\Lambda\ll 1$,

$$\frac{1}{\Delta\tau_1(\epsilon_F,T)}=T4\pi\,\ln 2(\Delta\Lambda k_F)^2\left[1+\frac{2N_D}{n}\right]^2\times\frac{F(2k_F)[1-G(2k_F)]}{\{F(2k_F)[1-G(2k_F)]+2k_F/q_s\}^3}\,. \tag{19b}$$

Equation (18) gives the linear temperature dependence due to the temperature dependence of the screening function. The next leading term in the expansion of Eq. (11) gives a $T^{3/2}$ behavior:

$$\frac{1}{\Delta\tau_2(\epsilon_F,T)}=\left[\frac{T}{\epsilon_F}\right]^{3/2}\frac{3}{\pi 2^{1/2}}k_1\frac{\langle\,|U(2k_F)|^2\rangle}{\epsilon_F}\times\frac{\{V(2k_F)\rho_F[1-G(2k_F)]\}^2}{\{1+V(2k_F)\rho_F[1-G(2k_F)]\}^4} \tag{20}$$

and

$$k_1=\left[\frac{\pi}{2}\right]^{3/2}\sum_{n,m=1}^{\infty}\frac{(-1)^{n+m}}{(n^2m+m^2n)^{1/2}}\approx 0.60\,. \tag{21}$$

C. Energy-dependent conductivity

Because of the $2k$ singularity in Eq. (8b) we use the same trick for the evaluation of the integral as in Sec. II A. For $|(\epsilon-\epsilon_F)/\epsilon_F|\ll 1$, we can write

$$\frac{1}{\tau_0(\epsilon,T=0)}=\frac{1}{\tau(\epsilon_F,T=0)\left[1+A(\epsilon_F)\dfrac{\epsilon-\epsilon_F}{\epsilon_F}\right]}\,, \tag{22}$$

with $A=-\epsilon_F\partial[1/\tau(\epsilon_F,T=0)]/\partial\epsilon_F$. For Eq. (8c) we get

$$\frac{1}{\tau_n(\epsilon,T=0)}=\frac{n+1}{2\pi\epsilon}(2k_F)^{2-n}\frac{\langle\,|U(2k_F)|^2\rangle}{\epsilon_1(2k_F)^2}\times\left[\frac{V(2k_F)[1-G(2k_F)]\rho_F}{\epsilon_1(2k_F)}\right]^n\times N_n\Theta(\epsilon-\epsilon_F)\,, \tag{23a}$$

with

$$N_n=\int_{2k_F}^{2k}dq\frac{(q^2-4k_F^2)^{n/2}}{(4k_F^2-q^2)^{1/2}}\,. \tag{23b}$$

Explicitly we find for $\epsilon-\epsilon_F/\epsilon_F=t\ll 1$,

$$N_1=\frac{\pi}{2}k_F\frac{\epsilon-\epsilon_F}{\epsilon_F}[1+O(t^2)]\,, \tag{24a}$$

$$N_2=\frac{8}{3}k_F^2\left[\frac{\epsilon-\epsilon_F}{\epsilon_F}\right]^{3/2}[1+O(t^2)]\,. \tag{24b}$$

With the Eqs. (15)–(17) for $1/\tau(\epsilon_F,T=0)$, the energy-dependent conductivity at temperature zero is given by

$$\sigma(\epsilon,T=0)=\sigma(\epsilon_F,T=0)\left\{1-\frac{C(\alpha)C(n)}{\ln 2}\Theta(\epsilon-\epsilon_F)\frac{\epsilon-\epsilon_F}{\epsilon_F}\times\left[1+\frac{8}{2\pi}C(n)\left[\frac{\epsilon-\epsilon_F}{\epsilon_F}\right]^{1/2}+O(t)\right]-A(\epsilon_F)\frac{\epsilon-\epsilon_F}{\epsilon_F}+O(t^2)\right\} \tag{25}$$

with

$$C(\alpha)=\ln 2\frac{(4+2\alpha)!!}{(3+2\alpha)!!}\ , \tag{26a}$$

$$C(n)=\frac{F(2k_F)[1-G(2k_F)]}{F(2k_F)[1-G(2k_F)]+2k_F/q_s}\ . \tag{26b}$$

Most important in Eq. (25) are the $\Theta(\epsilon-\epsilon_F)$ contributions. They arise from the anomalous q dependence of the screening function (or of the compressibility). The Kubo-Greenwood average of these terms gives anomalous T contributions for the conductivity. The coefficient of ϵ dependence can be written as a product of two factors. One, $C(\alpha)$, is a number depending only on the power in q of the random potential. For impurity scattering we obtain $C(-1)=2\ln 2$ and for surface-roughness scattering we obtain $C(0)=\frac{8}{3}\ln 2$. The second factor, $C(n)$, depends on the density because of the form factor $F(2k_F)$ and the local field correction. For $F(2k_F)=1$ and $G(2k_F)=0$ this factor is given by $C(n)=1/(1+2k_F/q_s)$. In Fig. 2 the influence of finite N_D on $C(n)$ and in Fig. 3 the influence of $G(2k_F)$ on $C(n)$ is demonstrated. N_D does not influence $C(n)$ in a strong way, but $G(2k_F)$ does.

D. Temperature dependence of the conductivity

Because of the temperature dependence of the screening, we have found a T and a $T^{3/2}$ behavior of $1/\tau(\epsilon_F,T)$. To take this into account, we write the Kubo-Greenwood formula, Eq. (5), in the form

$$\sigma(\epsilon_F,T)=\frac{1}{4T}\int_0^\infty d\epsilon\frac{\sigma(\epsilon,T\approx 0)}{\cosh^2[(\epsilon-\epsilon_F)/2T]}\ , \tag{27a}$$

with

$$\sigma(\epsilon,T\approx 0)=\frac{ne^2}{m}\tau(\epsilon,T)\ , \tag{27b}$$

and

$$\frac{1}{\tau(\epsilon,T)}=\frac{1}{\tau(\epsilon,T=0)}+\sum_{n=1}^{\infty}\frac{1}{\Delta\tau_n(\epsilon_F,T)}\ . \tag{27c}$$

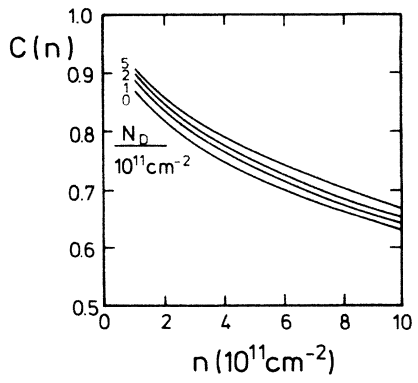

FIG. 2. $C(n)$ versus density according to Eq. (26b) for various values of N_D.

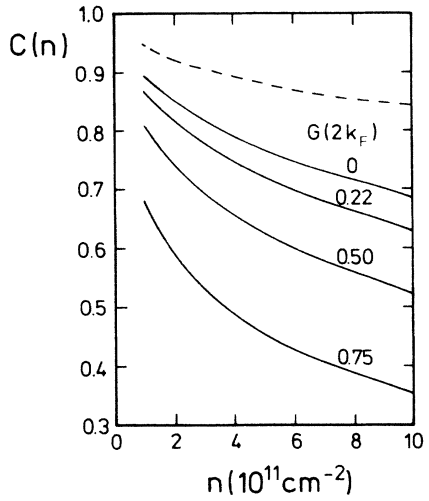

FIG. 3. $C(n)$ versus density according to Eq. (26b) for various values of $G(2k_F)$. The dashed line is for $F=1$, $G=0$, and $N_D=0$.

The Kubo-Greenwood average of $1/\tau(\epsilon,T=0)$ gives T and $T^{3/2}$ contributions due to the $\Theta(\epsilon-\epsilon_F)$ function. These anomalous T corrections come from the q dependence of the compressibility for $q>2k_F$. Anomalous T corrections $1/\Delta\tau_1(\epsilon_F,T)$ and $1/\Delta\tau_2(\epsilon_F,T)$ come from the q dependence of the compressibility for $q<2k_F$. Then the low temperature expansion of the conductivity is expressed as

$$\sigma(\epsilon_F,T)=\sigma(\epsilon_F,T=0)\left[1-C(\alpha,n)\frac{T}{\epsilon_F}-D(\alpha,n)\left[\frac{T}{\epsilon_F}\right]^{3/2}+O(T^2)\right], \tag{28}$$

with

$$C(\alpha,n)=2C(\alpha)C(n)\ , \tag{29a}$$

and

$$D(\alpha,n)=2.45C(\alpha)[C(n)]^2\ . \tag{29b}$$

Clearly Eq. (28) is only valid for $C(\alpha,n)T/\epsilon_F\ll 1$ and $D(\alpha,n)(T/\epsilon_F)^{3/2}\ll 1$. Equations (28) and (29) are the main result of our paper. The analytical results specify the "approximately linear increase" found in a numerical calculation by Stern[5] into "linear increase." The linear temperature dependence of the conductivity, found in experiment[3,4] and explained numerically in terms of the temperature-dependent screening function, is achieved analytically.

The temperature-dependent conductivity correction is given by the mobility of the sample:

$$\Delta\sigma(T)=\rho_F C(\alpha,n)\mu(T=0)T\ . \tag{30}$$

Equation (30) explains the fact that this anomalous T correction was only found recently[3,4] in samples with high mobility. The result that $\Delta\mu(T)$ is nearly proportional to

$\mu(0)$, has been found independently in Ref. 17 for impurity scattering.

Some experiments show departure from the linear T dependence at low temperature.[3] This effect cannot be explained by T^2 terms in a Sommerfeld expansion of the conductivity. Weak localization[2] also gives a decreasing conductivity with decreasing temperature, but the effect is too small to explain the experimental results of Ref. 3. Samples with the above-mentioned anomalies at low-temperature exhibit in a weak magnetic field a temperature-dependent Hall voltage.[18] We suggest that this is due to a temperature-dependent electron density.

When two scattering mechanisms are present, we receive Eq. (28) with $[\tau_i(0)=\tau_i(\epsilon_F,T=0)]$,

$$\sigma(\epsilon_F,T=0)=\frac{ne^2}{m}\frac{\tau_1(0)\tau_2(0)}{\tau_1(0)+\tau_2(0)}\ , \tag{31a}$$

$$C(\alpha,n)=C(\alpha_1,n)\left[\frac{1+[\tau_1(0)/\tau_2(0)]C(\alpha_2)/C(\alpha_1)}{1+\tau_1(0)/\tau_2(0)}\right], \tag{31b}$$

and we conclude that for $C(\alpha_1)<C(\alpha_2)$,

$$C(\alpha_1,n)<C(\alpha,n)<C(\alpha_2,n)\ . \tag{32}$$

This behavior for $C(\alpha,n)$ is shown in Fig. 4 for two values of the surface-roughness parameter $\Delta\Lambda$ and $\alpha_1=-1$ and $\alpha_2=0$.

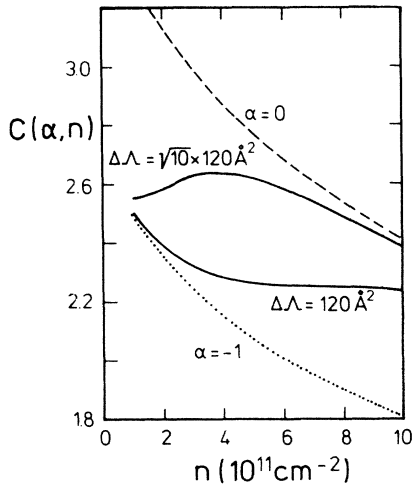

FIG. 4. $C(\alpha,n)$ versus density according to Eq. (31b) for surface-roughness scattering ($\alpha=0$) and impurity scattering ($\alpha=-1$). The dashed and dotted lines are for one scattering mechanism alone. The parameters are $n_i=2\times10^{10}$ cm^{-2} and $N_D=2\times10^{11}$ cm^{-2}.

E. Long-ranged surface-roughness scattering

Our analytical formula for the surface-roughness scattering, Eq. (16), holds for $k_F\Lambda\ll1$. For $k_F\Lambda\gg1$, the scattering potential is long-ranged and the q integral in Eq. (9) with Eq. (13) gives

$$\frac{1}{\tau(\epsilon_F,T=0)}=3\pi^{1/2}\epsilon_F\frac{1}{k_F\Lambda}\frac{\Delta^2}{\Lambda^2}\frac{1}{\{F(2/\Lambda)[1-G(2/\Lambda)]+2/q_s\Lambda\}^2}\left[1+2\frac{N_D}{n}\right]^2. \tag{33}$$

Hence, for high density, $n\gg N_D$, the mobility is given by $\mu\propto1/n^{1/2}$ as found before.[16] In Fig. 5 we discuss μ as a function of the density for various N_D. In the density range used, a finite depletion density has a strong effect on the mobility. The peak structure comes from the last factor in Eq. (33) and is not a consequence of our approximative evaluation of the integral. So we find the new result that the surface-roughness scattering alone is able to exhibit a peak structure in the mobility, if $k_F\Lambda\gg1$. Normally one argues that the interplay between impurity scattering and surface-roughness scattering is responsible for the peak structure in the mobility.[1]

For $1/\Delta\tau_1(\epsilon_F,T)$ we get the same expression as in Eq. (19b), with an additional factor $e^{-k_F^2\Lambda^2}$. The temperature dependence is determined by

$$C(O,n)=\frac{8\pi^{1/2}}{3}\ln2C(n)\frac{\{F(2/\Lambda)[1-G(2/\Lambda)]+2/q_s\Lambda\}^2}{\{F(2k_F)[1-G(2k_F)]+2k_F/q_s\}^2}(k_F\Lambda)^5e^{-k_F^2\Lambda^2} \tag{34}$$

Because of the exponential factor, $C(O,n)$ is strongly reduced for high density; the behavior of $C(O,n)$ depends strongly on the value of Λ, as shown in Fig. 6, and this effect should be useful to determine Λ in the region $k_F\Lambda\gg1$. But the experimental verification of that region needs more experimental work with high-mobility samples.

IV. DISCUSSION

In the following we compare our results with other theoretical results obtained numerically and with experiments in Si-MOS systems and heterostructures.

A. Comparison with theoretical results

In Ref. 5, numerical results for the "linear T dependence" of the inverse mobility have been given for $n=2\times10^{12}$ cm^{-2}, $n_i=1\times10^{11}$ cm^{-2}, $N_D=3\times10^{11}$ cm^{-2}, $\Delta=6$ Å, and $\Lambda=13$ Å. Analogous results have been reported later.[19] Local field corrections have not been taken into account and numerically calculated form factors have been used in (Ref. 5). From the figures of Refs. 5 and 19 we receive the corresponding $C(\alpha,n=2\times10^{12}$ cm$^{-2})$ values of our theory, and these values are shown in Table I together with our results.

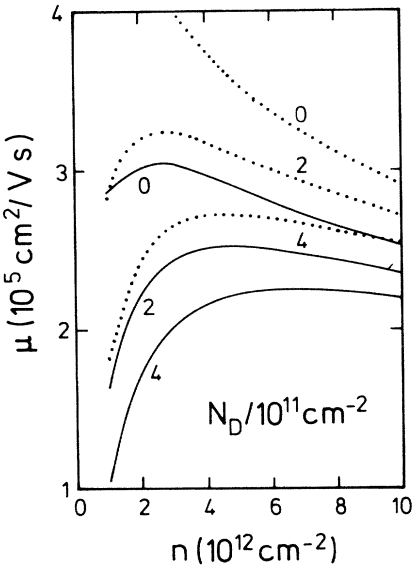

FIG. 5. Mobility versus density for surface-roughness scattering and various values of N_D ($\Delta=2$ Å, $\Lambda=120$ Å). The solid lines are according to our analytical formula, Eq. (9) and Eq. (13), the dotted lines are according to Eq. (33).

From our Eq. (28) follows that a strict linear temperature behavior can only be found if $D(\alpha,n)(T/\epsilon_F)^{1/2} \ll C(\alpha,n)$. With Eq. (29) we then conclude that

$$T \ll T^* \tag{35a}$$

and

$$T^* = \epsilon_F \left[\frac{0.82}{C(n)} \right]^2 \tag{35b}$$

must be fulfilled to see this linear T dependence. For $n=2\times10^{12}$ cm^{-2}, see Table I, we get then from Eq. (35) that $T^*=264$ K in the case of surface-roughness scattering. Our values for $C(\alpha,n)$ are greater than those given by Stern. The numerically calculated form factors of Ref. 5 could be the origin of this difference. But perhaps the

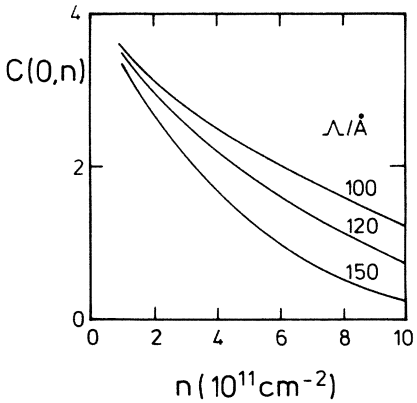

FIG. 6. $C(O,n)$ versus density for surface-roughness scattering and various values of Λ according to Eq. (34), but with $1/\tau(\epsilon_F, T=0)$ according to Eq. (9) ($\Lambda=2$ Å, $N_D=0$).

TABLE I. $C(\alpha,n=2\times10^{12}$ cm$^{-2})$ values extracted from the theory of Refs. 5 and 19 in comparison with our results for $G(q)=0$.

$C(\alpha,2\times10^{12}$ cm$^{-2})$	Stern (Ref. 5)	Lai and Ting (Ref. 19)	Present work
Impurity	0.8	~0.3 (~0.6)	1.7
Surface roughness	1.4	2.7 (4.1)	2.2
Both	1.6	1.3 (1.8)	2.1

temperature range of the numerical results is overly large to determine the coefficients $C(\alpha,n)$ in a correct way from Refs. 5 and 19. But, in any case, it is not clear why Stern gets for both scattering mechanisms a value greater than for one scattering mechanism alone, in contradiction to our Eq. (32). In Ref. 19 the Kubo-Greenwood formula was used for σ and $1/\sigma$. The authors argue that the difference in the T-dependent conductivity between these two procedures should be a T^2 effect. But from Fig. 1 of Ref. 19 we get two different coefficients for $C(\alpha,n)$. The values in brackets in Table I from Ref. 19 are due to the Kubo-Greenwood formula for $1/\sigma$. We cannot reproduce the great value for surface-roughness scattering found there.

For impurity scattering with $n_i=1\times10^{12}$ cm^{-2}, $N_D=1\times10^{11}$ cm^{-2}, and $T=4.5$ K, the equation

$$\frac{1}{\Delta\tau(4.5\text{ K})} = (9.23\times10^{-2}\text{ MeV}) \left[\frac{n}{2\times10^{12}\text{ cm}^{-2}} \right]^{-1.9} \tag{36}$$

was received numerically.[5] From our Eq. (30) we get for $G(2k_F)=0$ for $n=1\times10^{12}$ cm^{-2}, 0.48 MeV instead of 0.34 MeV from Eq. (36) and for $n=5\times10^{12}$ cm^{-2}, 0.022 MeV instead of 0.016 MeV. The origin of this 30% difference between our results and the results of Ref. 5 could be due to the numerically calculated form factors of Ref. 5 in contrast to our analytical ones.

Our expression for

$$\frac{\mu(T_1)-\mu(T_2)}{\mu(T_2)} = \frac{T_2-T_1}{\epsilon_F} C(\alpha,n) \frac{1}{1-C(\alpha,n)T_2/\epsilon_F} \tag{37}$$

is in good agreement with numerical results found recently for $\mu(1\text{ K})-\mu(4.5\text{ K})/\mu(4.5\text{ K}) \approx N_s^{-1.2}$,[20] and the prefactor of this equation is specified by our Eq. (37). Again we mention that Eq. (37) is only valid for $C(\alpha,n)T_2/\epsilon_F \ll 1$. The prefactor is in good agreement with theoretical results.[17]

B. Comparison with experiments in Si-MOS systems (T)

In Fig. 1 of Ref. 3, the temperature dependence of the resistance has been found for two samples. The corresponding values for $C(\alpha,n)$ are 1.7 and 1.0 for $n=1.2\times10^{12}$ cm^{-2} and 1.3×10^{12} cm^{-2}, respectively. Detailed comparison with our theory is not possible because the density dependence of the conductivity is not given and so the relevant scattering mechanism is unknown.

From Ref. 4 we extract from Fig. 3 the values $C(\alpha, 2.7\times10^{11}\ \mathrm{cm}^{-2})=2.9$, $C(\alpha, 3.9\times10^{11}\ \mathrm{cm}^{-2})=2.7$, and $C(\alpha, 5.1\times10^{11}\ \mathrm{cm}^{-2})=2.6$. Our values for short-ranged surface-roughness scattering, Eq. (29), are 2.9, 2.8, and 2.7 for the according densities. So our theory is in excellent agreement with the experiment of Ref. 4.

Our Eq. (30) explains the experimental results for $\Delta\sigma(T)$ versus n of Ref. 3 in their Fig. 3, curve 2 for $n_i=6\times10^{11}\ \mathrm{cm}^{-2}$ and $G=0$, and $n_i=4.8\times10^{11}\ \mathrm{cm}^{-2}$ for $G=0.224$. Experimentally, an oxide fixed charge density of $8\times10^{11}\ \mathrm{cm}^{-2}$ was found.[3] Essential for the agreement is the density dependence of the form factors $F(2k_F)$ and $F_i(2k_F)$ in Eq. (26b). Analogous experimental results of Ref. 21 can also be explained.

C. Comparison with experiments in Si-MOS systems ($T^{3/2}$)

Anomalous temperature corrections to the conductivity have been first reported by Cham and Wheeler.[22] A $T^{3/2}$ behavior has been found, but prefactors have not been given there. A $T^{1.8}$ behavior in low-mobility samples and a $T^{2.0}$ behavior in high-mobility samples have been found by the authors of Ref. 21. Our theoretical result for a $T^{3/2}$ behavior is new and needs experimental verification. Preliminary results of our theory have already been published.[23] But there the anomalous T dependences from the energy dependence of the relaxation rate have been neglected.

D. Experimental results from heterostructures

A negative coefficient for the temperature-dependent conductivity was also found in GaAs-$Al_xGa_{1-x}As$ heterostructures.[18,24] There $m^*=0.076m_0$, $g_v=1$, and $\epsilon_L=12.9$. The corresponding values for $C(\alpha,n)$ are given in Table II. Unfortunately the temperature-dependent part of the energy relaxation time in this structure is in the same order of magnitude as the temperature-dependent part of the momentum relaxation time,[18] and the experimental results are not conclusive for a determination of $C(\alpha,n)$. For more experimental data see Ref. 25.

V. CONCLUSION

We discussed the low-temperature-dependent conductivity of the two-dimensional electron gas in the presence of charged impurity scattering and surface-roughness scattering. New analytical results are obtained for the zero-temperature conductivity and for the conductivity correction up to $T^{3/2}$. When one scattering mechanism is dominant, then the anomalous linear temperature dependence shows a universal behavior with characteristic density dependence. The possibility of long-ranged surface-roughness scattering is pointed out.

TABLE II. $C(\alpha,n)$ values from experimental results in Refs. 18 and 24. The theoretical value $C(n)$, Eq. (26b), for $G(q)=0$ and $F(q)=1$ is also given.

n	$C(\alpha,n)$	$C(n)$	Ref.
$1.27\times10^{11}\ \mathrm{cm}^{-2}$	0.44	0.56	24
$1.89\times10^{11}\ \mathrm{cm}^{-2}$	1.31	0.51	24
$2.66\times10^{11}\ \mathrm{cm}^{-2}$	0.12	0.46	18

Including corrections due to weak localization[26] and interaction anomalies,[27] the low-temperature-dependent conductivity is given as

$$\sigma(T)=\sigma(0)\left[1-C(\alpha,n)\frac{T}{\epsilon_F}-D(\alpha,n)\left[\frac{T}{\epsilon_F}\right]^{3/2}\right]+\frac{e^2}{\hbar}\beta\ln\left[\frac{T}{T_0}\right]. \tag{38}$$

β is a sum of two coefficients due to weak localization and interaction, depending on the nature of the inelastic scattering process and on a screening function. Multiple scattering effects also modify the compressibility.[28] Accordingly, $\sigma(0)$ and $C(\alpha,n)$ are modified by multiple scattering effects.[15,29]

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