RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 32, NUMBER 12 15 DECEMBER 1985

Many-particle exchanges and the fractional quantized Hall effect

S. T. Chui

Bartol Research Foundation of the Franklin Institute, University of Delaware, Newark, Delaware 19716

(Received 30 August 1985)

Evidence is presented which suggests that many-particle exchanges stabilize the ground state at particular filling factors. Common features are seen in the Laughlin trial wave function, in charge-density-wave functions as discussed by Tosatti and Parrinello and more recently by Chui, Ma, and Hakim, as well as in the eigenstates of small clusters of particles.

One of the puzzles of the fractional quantized Hall effect (FQHE) is the apparent stability of the ground state when the Landau levels are filled to a specific ratio ν, where ν is a rational number with an odd integer as denominator. This is called the odd denominator rule. Different explanations for this have been proposed. (1) Tosatti and Parrinello[1] (TP) pointed out that a multiparticle exchange around a polygon on the uncorrelated charge-density-wave (CDW) lattice leads to a phase change which is proportional to the magnetic flux enclosed in the polygon. In particular, for a filling factor of $1/\nu$, this phase factor is $\exp(in\pi/\nu)$, where n is the number of particles interchanged. For $1/\nu =$ odd integer, this phase factor is $(-1)^n$ and would lead to a commensuration energy. For a class of correlated lattice studied by us[2] the above phase factor is approached only when the number of particles exchanged becomes large. (2) Laughlin[3] suggested a trial wave function of the ground state by using the angular-momentum gauge. He then pointed out that this wave function is only well defined when the filling factor ν is an odd number. (3) The above two explanations make specific assumptions about the long-range order of the ground state, which may not be crucial to the argument. It is thus important to isolate the essential features which are necessary for the effect. An explanation in the Landau gauge was recently suggested by us.[4] As a function of the separation of the particles, the Hamiltonian has a short-range attractive part because of the exchange (see Fig. 1). As a consequence, the nonhopping diagonal terms possess two local minima, corresponding to a cluster distribution of the particles favored by the exchange and a uniform distribution favored by the direct energy. Because of the hopping term, the energy will be lowered if the wave function forms a linear combination from these two minima. However, a linear combination can be formed for odd denominators because only then are the y momenta of the two minima the same. In the numerical calculations of diagonalizing the Hamiltonian of up to 16 particles in the Landau gauge, the clustering effect, as is illustrated in Fig. 2(a), is indeed observed. The tendency towards clustering has been observed in earlier work by Lai, Yu, Su, and Yu[5] in the calculation of the eigenvalues of 4–7 particles in the circular gauge.

In this Rapid Communication we propose that these explanations represent different ways of looking at the same effect. Our physical picture is as follows: Because of the multiparticle exchange effect, the electrons have a clustering tendency. In the language of TP, one can always enhance the multiparticle exchange if the phase factor and hence the polygons have commensurate areas. For a filling factor without an odd denominator the exchange can be enhanced at the expense of a nonuniform density. On the other

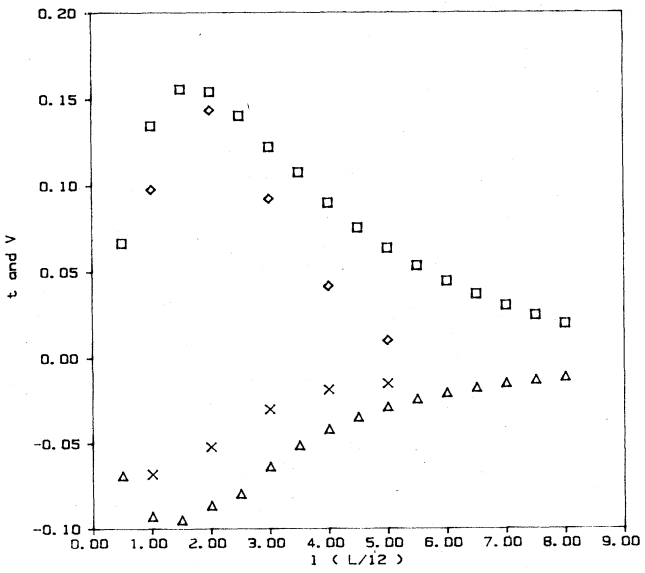

FIG. 1. The Hamiltonian parameters $V(l), t(l)$ as a function of l for 16 (□ and △), and for 4 (× and ◇) particles.

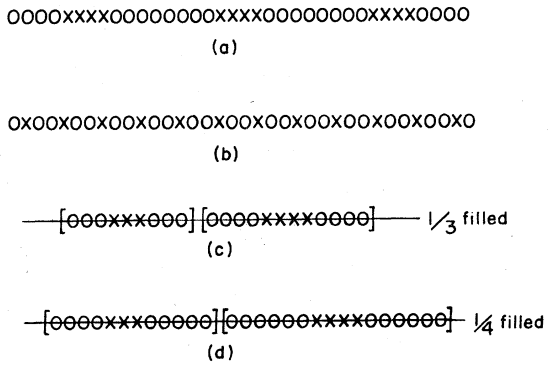

FIG. 2. (a) A lattice of clusters for the $\frac{1}{3}$-filled case with 48 sites; (b) an example of a lattice with a uniform angular momentum M_u; (c) and (d) a schematic way of breaking up a lattice of clusters into subunits enclosed by the square brackets for (c) $\frac{1}{3}$ filled and (d) $\frac{1}{4}$ filled. The optimum cluster size is assumed to be between three and four for illustrative purposes. Whenever it is not possible to have equal numbers of empty sites on both the left and the right in a subunit, we have assumed that the right has more empty sites.

hand, to maximize the direct energy, a uniform density is appropriate. Since densities uniquely determine the mean-square radii and hence the total angular momenta, the angular momentum enhancing the multiparticle exchange and that enhancing the direct energy are equal only at odd filling factors. This tendency is seen no matter what gauge one works in and is what we found in explanation (3) above, starting from a more general perspective. The clustering in m space is built into Laughlin's trial wave function, as we shall show in this Rapid Communication. If Fermi statistics are demanded, then only certain angular momenta are allowed which, again, equal that for maintaining a uniform density only at odd denominators.

(a) We first provide the details of explanation (3). In both the Landau and the angular-momentum gauge the single-particle basis function is specified by a one-dimensional label, the angular momentum m on the one hand and the y momentum j on the other. The total angular momentum M is a constant of motion in one case, while the total y momentum J is also a constant of motion in the other. Let us first discuss the results for the Landau gauge.

The basis set can be written as product wave functions of Landau orbitals given by

$$\phi_j(\mathbf{r}) = \exp[ix_j y - (x-x_j)^2/2]/(\pi^{1/2}L_y)^{1/2} \ , \tag{1}$$

$$x_j = (L_y/N_s)j \ . \tag{1a}$$

Here distances have been expressed in units of the Larmor radii, and will continue to be so in the rest of this Rapid Communication. L_y is the width in the y direction. N_s is the total number of possible states. The Hamiltonian in second-quantized form can be written, except for trivial constants, as

$$H = \sum_{[j]} A(j_1,j_2,j_3,j_4) C_{j_1}^{\dagger} C_{j_2}^{\dagger} C_{j_3} C_{j_4} \ . \tag{2}$$

The A's are integrals of the Coulomb potential and the Landau orbitals (ϕ_j's).

There is a similarity between (2) and the Hubbard model. To bring out this similarity and to gain more insight, we decompose the Hamiltonian as a sum of a diagonal H_d and an off-diagonal H_0 term as

$$H = H_d + H_0 \ , \tag{3}$$

$$H_d = \sum_{k,j} V(k) n_j n_{j+k} \ ,$$

$$H_0 = \sum_{l=1} \sum_{k=1} t_l(k) \sum_i C_{i-l}^{\pm} C_i C_{i+k} C_{i+k+l}^{\pm} + \text{c.c.} \tag{4}$$

The t's are the hopping integrals. To illustrate the behavior of these parameters, we show $V(k)$ and the $t_1(k)$ as functions of the distance k for the 12 and 48 sites cases in Fig. 1. Note that $V(k)$ attains a maximum and then comes down as one approaches the origin. This comes from the exchange. More precisely,

$$V(k) = 2[A(j_{23}=0, j_{13}=k) - A(j_{23}=k, j_{13}=0)] \ , \tag{5}$$

where $j_{ab} = j_a - j_b$; the first (second) is the direct (exchange) contribution. At small distances k, these two terms are comparable in magnitude; the net value of V is reduced. As k increases, the exchange contribution dies off exponentially fast and only the first term remains. The distance l_t at which V turns from repulsive to attractive is independent of the sample size N_s.

The diagonal term H_d exhibits two local minima. Because of the attractive part of V, the particles have a tendency to form clusters of size l_0 less than l_t with local y momentum J_c. For example, at $\frac{1}{3}$ filled the particles tend to form clusters of 2 for the 12-site case and clusters of 4 in the 48-site case. This is indicated schematically in Fig. 2(a). In general, because the number of states covered by the attractive part of the Hamiltonian is of the order of $\sqrt{N}$, where N is the total number of particles. We expect that the number of particles in a cluster is of the order of $\sqrt{N}$. Our numerical results[6] indicated that indeed a substantial component of the ground-state wave function comes from this configuration. On the other hand, a uniform distribution with total y momentum J_u is favored by the direct energy acting together with the hopping term. This distribution is illustrated schematically in Fig. 2(b). J_c is equal to J_u only for odd denominators.

In the angular-momentum gauge the basis function in the lowest Landau level is given by

$$\psi_m(r) = z^m \exp(-r^4/4)/(2^n n! 2\pi)^{0.5} \ , \tag{6}$$

where $z = x + iy$. One can express the electron-electron interaction in this basis set as

$$H = \sum_{[m]} A(m_1,m_2,m_3,m_4) C_{m_1}^{+} C_{m_2}^{+} C_{m_3} C_{m_4} \ , \tag{7}$$

where

$$A(m_a,m_b',m_b,m_a') = \frac{\int d\mathbf{r}\, d\mathbf{r}' \psi_{m_a}(\mathbf{r})^* \psi_{m_b'}(\mathbf{r}')^* \psi_{m_b}(\mathbf{r}) \psi_{m_a'}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{7a}$$

A is given by[5]

$$A = \left(\prod_i m_i\right)^{1/2} \sum_{r=0}^{m_<} \sum_{s=0}^{m_<'} \frac{(-)^{r+s}\Gamma(q+r+s+\frac{1}{2})}{(q+r)!(q+s)!r!s!2^{q+r+s+1}} \ , \tag{7b}$$

where $q = m_> - m_<$. Furthermore, the angular momentum is conserved; i.e., $m_1 + m_2 = m_3 + m_4$. In the limit of large m, the exchange matrix elements have a range $m_> - m_<$ of the order of $\sqrt{m}$. These can best be understood from the fact that the mean radius R of ψ_m is of the order of $\sqrt{m}$ and the fluctuation of the radius is of the order of unity. Hence the overlap of two wave functions is substantial if the difference in their radii is of the order of unity; this happens if the difference of their angular momenta is of the order of $\sqrt{m}$.

Analogous to the case for the Landau gauge, we can separate the Hamiltonian into a part that is diagonal in m and an off-diagonal part. Because of the exchange term that we just mentioned, these terms will have an attractive part of range $\sqrt{m}$. These attractive interactions will then favor the formation of clusters with a range of $\sqrt{m}$ at angular momentum m. Again, the diagonal part of the Hamiltonian possesses two local minima. The first corresponds to fluctuations around a uniform density distribution mandated by the direct Coulomb interaction; an example of this is illustrated by states occupying m values equal to n/ν for positive integers n. This state has total angular momentum M_u. On the other hand, because of the exchange, a second state corresponding to a cluster distribution of the m's that we just described is also possible. This state is of angular momentum M_c. We now argue that these two values are equal only for odd filling factors.

For a general sample with N_s states, we find some clusters with an even number of particles and some with an odd number of particles, so that the total number of states in a cluster is of the order $\sqrt{m}$ around state $|m\rangle$. We now divide the lattice of clusters into subunits consisting of the clusters of j electrons with their share of $[(1/\nu)-1]j$ empty sites around them, shown schematically in Figs. 2(c) and 2(d), where, for illustrative purposes, we have assumed that the optimal cluster size is between 4 and 3. We shall look at the angular momentum of each cluster measured relative to its left end. The total angular momentum is just the sum of the angular momentum of each subunit and that of its left end. For odd denominators, $(1/\nu)-1$ is an even number, and there are $[(1/\nu)-1]j/2$ empty sites on both sides of the cluster in a subunit. We obtain

$$M_c = j(1+j/\nu)/2 \tag{8}$$

for that cluster. For the uniformly spaced case, we assume that the electrons are located at $n+m/\nu$, where n is a nonnegative integer less than $1/\nu$; $m=0,1\ldots(j-1)$. One finds that

$$M_u = j(j+1)/(2\nu) - nj = j(j/\nu) + (1/\nu) - 2n)/2 \ . \tag{9}$$

For $n=[(1/\nu)-1]/2$ we obtain

$$M_c = M_u \ .$$

For even denominators and j even, Eqs. (8) and (9) are still correct. However, $(1/\nu)-1$ is no longer an even number, and we have to pick $n=[(1/\nu)-1\pm1]/2$ in Eq. (9) and then $M_c \neq M_u$. For j odd, there can only be $\{[(1/\nu)-1]j\pm1\}/2$ empty sites on the left of the cluster in the subunit. To be definite, we shall pick the negative sign. We obtain

$$M_c = j(j/\nu)/2 \ . \tag{10}$$

On the other hand, the formula for M_u remains unchanged. If we pick $n=1/2\nu$, then the two M's are the same.

This even-odd behavior is also observed in the calculation of Lai *et al.*[5] They found that a downward cusp is observed at $\nu=\frac{1}{2}$ only for an odd number of electrons. Presumably in their calculation the number of electrons is small enough so that only one (rather than many) cluster is formed.

The total angular momentum is made up of the angular momentum of each subunit. The factor n that we pick has to be the same for each subunit; otherwise the subunits cannot be joined together uniformly. Hence if M_u is larger than M_c for one subunit, it is also larger for the other subunits. When the angular momenta of the subunits are added up, the possibility of cancellation among subunits is thus ruled out. Hence, except for the very special case for which all cluster sizes are odd, M_u is not equal to M_c for the even filling factor cases.

(b) In the circular gauge, the connection between the above argument and that of TP is easier to visualize. Before discussing correlated solids, we first paraphrase the argument of TP for the uncorrelated solid wave function ϕ_s in the angular-momentum gauge.

ϕ_s is defined as an antisymmetrized product wave function of uncorrelated Landau orbitals $g_{\mathbf{R}}(\mathbf{r})$ (Ref. 7) located on a triangular lattice at $\mathbf{R}_i$, viz.,

$$\phi_s = \sum_P (-1)^P M(P) \ , \tag{11}$$

$$M(P) = \prod_i g_{\mathbf{R}_i}(r_{Pi}) \ , \tag{12}$$

$$g_{\mathbf{R}}(\mathbf{r}) = C\exp(-r^2/4 + zZ^*/2) \ . \tag{13}$$

C is a normalization constant, $Z=X+iY$. The exponential in Eq. (13) can be expanded in an infinite series as

$$g_{\mathbf{R}}(\mathbf{r}) = C\exp(-r^2/4)\sum_m (zZ^*)^m/m! \ . \tag{14}$$

This infinite series is dominated by terms with an m close to m_0 such that $R^2=2m_0$. Because the spread in r is of the order of unity, the fluctuation in m about m_0 is of the order of $\sqrt{m}$. The probability density $|g|^2$ is given approximately by

$$|g|^2 = \frac{(R|z|)^{m_0}}{m_0!}\sum_{m,m'} \exp[i(m-m')(\theta-\theta_Z)] \ . \tag{15}$$

Here θ is the phase of z. It is the sum over m,m' that localizes the particle around θ_Z.

In ϕ_s, the electrons are forced apart to form a regular lattice and the density is fixed. Even though it is not an eigenstate of the total angular momentum, from the relationship between the mean-square angular momentum and the density for a system with a uniform density, we expect the most probable angular momentum to be M_u.

We now argue that the clustering tendencies are also built into ϕ_s for odd denominators. In ϕ_s, electrons located at the same distance $|\mathbf{R}|$ will have similar angular momentum and tend to form clusters if the "phase factors" Z are right. The probability density $|\phi_s|^2$ is proportional to $\sum_P (-1)^P\langle M(P)|M(1)\rangle$. The clustering tendencies will be enhanced if those contributions P for circular exchanges involving particles with the same $|\mathbf{R}|$ interfere constructively. This will enhance the overlap and the exchange energy of the wave function. We have, from Eqs. (14) and (12),

$$\langle M(P)|M(1)\rangle = \left\langle \sum_{m_i}\prod_i (z_{Pi}Z_i^*)^{m_i}\exp(-r_i^2/4)/m_i!\,\middle|\,\sum_{l_i}\prod_i (z_iZ_i^*)^{l_i}\exp(-r_i^2/4)/l_i!\right\rangle \tag{16}$$

$$= \sum_{m_i} \text{real const}\prod_i (Z_{Pi}Z_i^*)^{m_i} \ . \tag{17}$$

The phase on the right-hand side is equal to the area enclosed by the exchange. For an n-particle exchange on a regular polygon, it is just $\exp(in\pi/\nu)$. For odd-denominator filling factors $(-1)^P\langle M(P)|M(1)\rangle$ is positive for this type of exchange and the probability for these configurations is enhanced.

TP have applied their argument to a case where there are no correlations among the Landau orbitals at different sites. We have recently studied a class of correlations and found that their argument can be extended to these cases only if the area enclosed by the particles exchanged becomes large.

The wave function that we are interested in can be written as

$$f = N\exp\left(-\sum_i (r_i^2/4 + \sum_{i,j}(z_iZ_j^* - \delta z_i\delta z_j a_{i,j}/2)\right) \ .$$

TABLE I. The coefficients $A([m])$ for $N=3$.

$[m]$	$A([m])$
[036]	1
[126]	3
[045]	3
[135]	6
[234]	15

TABLE II. The coefficients $A([m])$ for $N=4$.

$[m]$	A	$[m]$	A	$[m]$	A
[9630]	1	[8721]	9	[7821]	9
[9621]	3	[8640]	6	[7650]	15
[9540]	3	[8631]	12	[7641]	27
[9531]	6	[8541]	9	[7632]	6
[9432]	15	[8532]	27	[7542]	45
[8730]	3	[7830]	3	[6543]	105

Here N is a normalization constant and a is a real function. We now expand the exponential function into its dominant angular-momentum components and its fluctuations m_0+m. The overlap integral can be written as

$$\langle M(P)|M(1)\rangle=\sum_{[m_i]}\text{const}\prod_{m,m'}\int(z_{Pi}^{*}Z_{Pi})^{m_i+m_0}(z_iZ_i^{*})^{m_i'+m_0}(-\delta z_i\delta z_j a_{i,j}/2-\delta z_{Pi}^{*}\delta z_{Pj}^{*}a_{i,j}/2)^{m_1}/m_1!$$

The fluctuations m,m' are of the order of $m_0^{1/2}$. Since the fluctuation $\langle\delta z_i\delta z_j\rangle$ is of the order of $\ln R$, we expect m_1 to be of the order of $\sqrt{\ln m_0}$. This is much smaller than m_0 in the limit of large m_0, and hence provides an insignificant modification to the phase factor, as we have claimed.

To summarize, corresponding to any density there is an angular momentum M_u such that the direct Coulomb energy is minimized. On the other hand, many-particle exchanges on a polygon involve a phase factor proportional to the area enclosed by the polygon. For a filling factor without an odd denominator, the exchange can be enhanced at the expense of a nonuniform density. Hence there are particular average densities and hence angular-momenta M_c's such that the multiparticle exchange and the clustering effect that we just discussed is enhanced. $M_u=M_c$ only when $1/v$ is odd.

(c) Laughlin proposed that the electrons form a fluid with wave function ϕ_0 given by

$$\phi_0=G([z_i])\exp\left(-\sum_i r_i^2/4\right)\ , \tag{18}$$

where

$$G([z_i])=\prod_{i>j}^{N}(z_i-z_j)^m\ .$$

To appreciate the distribution of the m's of this wave function, we have expanded the polynomial G with the symbol manipulation program REDUCE for small values of N (3,4). In general, one can write G as

$$G=\sum_P(-1)^P\prod_{i=1}^{N}A([m_i])z_{Pi}^{m_i}\ .$$

The coefficients A of different distributions $[m_i]$ for $N=3,4$ are given in Tables I and II. On the average, the cluster distributions have a larger coefficient A. For example, for $N=3$, the largest A is 15 and these correspond to a distribution $[m]=[2,3,4]$. For $N=4$, the largest A (105) corresponds to $[m]=[3,4,5,6]$. Mathematically, these come about because the A's are similar to the multinomial coefficients, and generally the multinomial coefficients are largest when the exponents of the multinomials are close to each other. The large values of the coefficients A for small clusters also show up in the larger clusters because we can think of larger clusters as being made up from smaller clusters. Specifically, let us split up N as $N=N_1+N_2$, where N_i is an arbitrary positive number. Then one of the terms that contribute to G will come from terms of the form

$$\prod_{i>j=1}^{N_1}(z_i-z_j)^m\prod_{i'>j'=N_{1+1}}^{N_2}(z_i-z_j)^m\prod_{i_1>N_{1+1},j_1<N_1}(z_{i1}-z_{j1})^m\ ,$$

and the coefficients A of this term are made up of the product of the coefficients $A(N_1)$ and $A(N_2)$ and other terms. An example of this effect can be clearly seen in the A of the $N=4$ case for $[m]=[2,3,4,9]$. This just comes from a product of terms for $N_1=3$ and $N_2=1$ and $[m_1]=[2,3,4]$ and $[m_2]=[9]$. Hence a wave function of the form

$$G(\alpha,[z_i])=\prod_{i>j}^{N}(z_i-z_j)^{\alpha}$$

has a clustering tendency built in. On the other hand, the total angular momentum $M=0.5N(N-1)\alpha$ is clearly determined by the density. In fact, following Laughlin's argument, one finds that $M=0.5N(N-1)/v$, where v is the filling factor. As is well known for this case, the above is only compatible with Fermi statistics for α equal to an odd integer.

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