Sine-Gordon solitons under weak stochastic perturbations

Pedro J. Pascual and Luis Vázquez
Departamento de Física Teórica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
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We study the behavior of a Sine-Gordon soliton in the presence of certain weak stochastic perturbations. A perturbation analysis based on the modulation of the speed and position of the soliton is considered, and fairly good agreement with the numerical computations is obtained.

I. INTRODUCTION

The effect of external fields on the motion of solitary waves is important in the dynamics analysis of systems characterized by non-linear-wave equations. On the other hand, the interactions of solitons with spatial inhomogeneities in long Josephson junctions, as well as in other systems, are of considerable interest for applications.[1] In this framework we study the behavior of a sine-Gordon soliton under weak stochastic perturbations which can simulate a thermal noise and a random medium, as well as a fluctuating external field.[2,3] In this case, the system is governed by the perturbed sine-Gordon (SG) equation

$$\phi_{tt}-\phi_{xx}+\sin\phi+\alpha\phi_t+V(x,t)\phi+F(x,t)=0\ , \tag{1}$$

where V and F are functions localized in space and varying randomly in time, while $\alpha\phi_t$ represents a loss term.

In this work we study numerically the behavior of a soliton under the effect of either an additive noise F or a multiplicative noise V. And in both cases the effect of the dissipation is analyzed. In addition, a perturbation approach is considered by assuming modulations of the speed and position for the soliton under the stochastic perturbations, and we get a qualitative and quantitative agreement between the perturbation analysis and the numerical results.

The organization of the paper is as follows. In Sec. II we consider the features of the discretization which is used in our computations with the perturbed sine-Gordon equation. While in Sec. III the perturbation analysis for a single soliton is established. In Sec. IV we present the computational results. Finally, a summary of the results and their significance is presented in Sec. V.

II. COMPUTER SIMULATION DETAILS

The numerical integration of Eq. (1) is carried out through the finite-difference scheme

$$\frac{\phi_l^{n+1}-2\phi_l^n+\phi_l^{n-1}}{(\Delta t)^2}-\frac{\phi_{l+1}^n-2\phi_l^n+\phi_{l-1}^n}{(\Delta x)^2}-\frac{\cos\phi_l^{n+1}-\cos\phi_l^{n-1}}{\phi_l^{n+1}-\phi_l^{n-1}}+\alpha\frac{\phi_l^{n+1}-\phi_l^{n-1}}{2\Delta t}+\tfrac{1}{4}(V_l^{n+1}+V_l^{n-1})(\phi_l^{n+1}+\phi_l^{n-1})+\tfrac{1}{2}(F_l^{n+1}+F_l^{n-1})=0\ . \tag{2}$$

The properties of this discretization are the following:

(1) For the SG equation ($\alpha=0$, $V=0$, $F=0$) there is a discrete energy which is constant:

$$E^n=\tfrac{1}{2}\sum_l\Delta x\left[\frac{\phi_l^{n+1}-\phi_l^n}{\Delta t}\right]^2+\tfrac{1}{2}\sum_l\Delta x\left[\frac{\phi_{l+1}^{n+1}-\phi_l^{n+1}}{\Delta x}\right]\left[\frac{\phi_{l+1}^n-\phi_l^n}{\Delta x}\right]+\sum_l\Delta x\left[1-\frac{\cos\phi_l^{n+1}+\cos\phi_l^n}{2}\right]\ . \tag{3}$$

The stability and convergence of the scheme, in this case, is studied by Guo Ben-Yu *et al.*[4] The scheme is a generalization of the Strauss-Vázquez technique.[5]

(2) When $V=0$ and $F=0$, we get

$$\frac{E^n-E^{n-1}}{\Delta t}=-\alpha\sum_l\Delta x\left[\frac{\phi_l^{n+1}-\phi_l^{n-1}}{\Delta t}\right]^2\ . \tag{4}$$

Thus, in our discretization the sign of the energy variation is the same as in the nondiscretized SG equation with dissipation.

(3) In our computations we use the discrete momentum

$$P^n=-\sum_l\left[\frac{\phi_{l+1}^{n+1}-\phi_{l-1}^{n+1}}{2\Delta x}\right]\left[\frac{\phi_l^{n+1}-\phi_l^n}{\Delta t}\right]\Delta x\ . \tag{5}$$

This is constant when we particularize our scheme for the linear wave equation $\phi_{tt}-\phi_{xx}=0$, while for the nonperturbed SG equation, the variation is

$$\frac{P^n-P^{n-1}}{\Delta t}=-\tfrac{1}{2}\sum_l\frac{\cos\phi_l^{n+1}-\cos\phi_l^{n-1}}{\phi_l^{n+1}-\phi_l^{n-1}}(\phi_{l+1}^n-\phi_{l-1}^n)\ . \tag{6}$$

However, this discretization effect is very small, being around 10^{-6} times the momentum value. The great accuracy of (5) also was verified in the case of the soliton with dissipation, for which an exponential decay of the momentum occurs.

(4) The velocity and the center of the soliton are computed as follows:

$$V^n=\frac{P^n}{E^n}, \tag{7}$$

$$X_c^n=\frac{1}{E^n}\sum_l\left\{\tfrac{1}{2}x_l\left[\left[\frac{\phi_l^{n+1}-\phi_l^n}{\Delta t}\right]^2+(2-\cos\phi_l^{n+1}-\cos\phi_l^n)\right]+\tfrac{1}{4}(x_l+x_{l+1})\left[\frac{\phi_{l+1}^{n+1}-\phi_l^{n+1}}{\Delta x}\right]\times\left[\frac{\phi_{l+1}^n-\phi_l^n}{\Delta x}\right]\right\}\Delta x\ . \tag{8}$$

These expressions were checked in the case of a free soliton. When this is centered at the origin, $x_c\approx 10^{-14}$, while the variation of the velocity value is of order 10^{-6} times its value.

(5) At each time step the scheme (2) requires solving a simple functional equation for the unknown ϕ_l^{n+1}. To accomplish this we use Newton's method. The mesh sizes $\Delta x = 2\,\Delta t = 0.05$ were chosen.

In all our computations we study the effects of a noise acting on a soliton initially near the origin. The space interval of our simulations is limited to $[-30,30]$, while V and F are localized in the interval $[-10,10]$; in this way the extent of the soliton is small compared to the width of the impurity potential. The time of the simulation is until $t=12.5$, thus we chose fixed boundary conditions because in this case the computations are reliable until $t=20$.

For the stochastic perturbations we chose

$$V(x,t),F(x,t)=\begin{cases}S(t), & x\in[-10,10],\\ 0, & x\notin[-10,10],\end{cases} \tag{9}$$

where $S(t)$ is a Gaussian white noise with vanishing average

$$\langle S(t)\rangle=0,\quad \langle S(0)S(t)\rangle=2D\,\delta(t)\ . \tag{10}$$

If we interpret Eq. (1) in the Stratonovitch sense,[6] the driving white noise, with correlation given by Eq. (9), can be inserted into our algorithm at the first order in Δt (Ref. 7) through a generator of random numbers S^n with Gaussian distribution, zero mean value and variance $\sigma^2=2D/\Delta t$ being

$$S^n=\frac{1}{\Delta t}\int_{n\,\Delta t}^{(n+1)\Delta t}S(t)dt\ .$$

As a consequence of the time discretization, S^n can simulate a white noise $S(t)$ only in the region of the frequency spectrum below $(\Delta t)^{-1}$ which is, however, large with respect to the inverse of the time scales introduced in Eq. (1).

What we did in the computations was to simulate 30 particular time evolutions ("trajectories") consistent with the stochastic equation, which correspond to particular realizations of the stochastic term (a certain sequence of random numbers). And the average of the relevant quantities was considered.

III. PERTURBATION ANALYSIS FOR A SINGLE SOLITON

Let us consider the perturbed SG equation

$$\phi_{tt}-\phi_{xx}+\sin\phi=f(\phi,\phi_t;x,t)\ . \tag{11}$$

When $f=0$ we have the following soliton-antisoliton solutions:

$$\phi_\pm=4\tan^{-1}\{\exp[\pm\gamma(x-X)]\}\ , \tag{12}$$

where $\gamma=(1-U^2)^{-1/2}$. U is the velocity and X locates the center of the soliton, being $X=X_0+Ut$.

Since we are considering weak perturbations we assume that the predominant effect of these on a single soliton is to modulate its center X and velocity U. Thus we reduce the problem to solve a set of ordinary differential equations. The general analysis of this perturbation method was made by McLaughlin and Scott.[8] However, a simple deduction of the general dynamical equations which govern the response of a single soliton to a generic perturbation, partially reported in Ref. 8, is presented here. From Eq. (11) we get

$$\frac{dE}{dt}=\int_{-\infty}^{\infty}f\phi_t\,dx\ , \tag{13}$$

where $E=\int_{-\infty}^{\infty}\epsilon\,dx$ and

$$\epsilon=\tfrac{1}{2}\phi_t^2+\tfrac{1}{2}\phi_x^2+(1-\cos\phi)\ . \tag{14}$$

For the momentum we have

$$\frac{dP}{dt}=-\int_{-\infty}^{\infty}f\phi_x\,dx\ , \tag{15}$$

with

$$P=-\int_{-\infty}^{\infty}\phi_x\phi_t\,dx\ . \tag{16}$$

On the other hand, the center of the soliton can be defined as follows:

$$X=\frac{\int_{-\infty}^{\infty}x\epsilon\,dx}{E}\ , \tag{17}$$

its equation of motion being

$$\frac{dX}{dt}=U+\frac{1}{E}\int_{-\infty}^{\infty}(x-X)f\phi_t dx\ . \tag{18}$$

For other definitions of X see Bergmann *et al.*[9]

By inserting the ansatz

$$\begin{bmatrix}\phi\\ \phi_t\end{bmatrix}=\begin{bmatrix}4\tan^{-1}\left[\exp\dfrac{x-X(t)}{[1-U^2(t)]^{1/2}}\right]\\ \dfrac{-2U(t)}{[1-U^2(t)]^{1/2}}\operatorname{sech}\left[\dfrac{x-X(t)}{[1-U^2(t)]^{1/2}}\right]\end{bmatrix} \tag{19}$$

in Eqs. (13) and (18), we get the following system for the two parameters of the soliton:

$$\frac{dU}{dt}=-\tfrac{1}{4}(1-U^2)\int_{-\infty}^{\infty} f(\phi,\phi_t;x,t)\operatorname{sech}\theta\, dx \ , \tag{20}$$

$$\frac{dX}{dt}=U-\tfrac{1}{4}U(1-U^2)^{1/2}\int_{-\infty}^{\infty} f(\phi,\phi_t;x,t)\theta \operatorname{sech}\theta\, dx \ , \tag{21}$$

where

$$\theta=x-X(t)/[1-U^2(t)]^{1/2} \ .$$

Also with the ansatz (19) we get for the momentum,

$$\frac{dP}{dt}=-2[1-U^2(t)]^{-1/2}\int_{-\infty}^{\infty} f(\phi,\phi_t;x,t)\operatorname{sech}\theta\, dx \ , \tag{22}$$

being in this equation equivalent to (20). With this approach we reduce our problem in solving a set of stochastic nonlinear ordinary differential equations.

In the cases in which we are interested the perturbations f are localized in space and vary randomly in time, presenting also for studies the effect of a dissipation term. Thus the dynamical equations (20) and (22) are stochastic and the corresponding Fokker-Planck equation must be considered in order to compare the average values with those computed numerically, as was indicated in Sec. II.

For the studied perturbations the spatial extent of the soliton is small compared to the length of the region in which the stochastic perturbation is localized. In this way, to evaluate the integral of Eqs. (20) and (22), we can assume that the perturbative term f is defined in the whole space $(-\infty,\infty)$. The following cases were studied.

A. Additive noise without dissipation

In this case,

$$f=-F(t) \ , \tag{23}$$

such that $\langle F(t)\rangle=0$ and $\langle F(0)F(t)\rangle=2D\,\delta(t)$. The stochastic equations for the momentum, velocity, and center of the soliton are

$$\dot{P}=2\pi F(t) \ , \tag{24}$$

$$\dot{U}=\frac{\pi}{4}(1-U^2)^{3/2}F(t) \ , \tag{25}$$

$$\dot{X}=U \ . \tag{26}$$

Equation (24) corresponds to the elementary random-walk problem, being the associated Fokker-Planck equation[10]

$$\frac{\partial W}{\partial t}=4\pi^2 D\frac{\partial^2 W}{\partial P^2} \ , \tag{27}$$

from which we get the mean values

$$\langle P\rangle=P_0, \ \ \langle (P-\langle P\rangle)^2\rangle=8\pi^2 Dt \ , \tag{28}$$

supporting the fact that no steady motion in momentum space exists.

For the equations of the velocity and center (25) and (26) the Fokker-Planck equation is

$$\frac{\partial W}{\partial t}=-U\frac{\partial W}{\partial X}+\frac{3\pi^2}{16}D\frac{\partial}{\partial U}[U(1-U^2)^2 W] +\frac{\pi^2}{16}D\frac{\partial^2}{\partial U^2}[(1-U^2)^3 W] \ . \tag{29}$$

In spite of the difficulty of this equation, if we have a soliton initially at rest it is possible to consider the nonrelativistic limit $|U|\ll 1$, getting the following mean values:

$$\begin{aligned}&\langle U\rangle=\langle U\rangle_0, \ \ \langle U^2\rangle=\langle U^2\rangle_0+\frac{\pi^2}{8}Dt \ ,\\ &\langle X\rangle=\langle X\rangle_0+\langle U\rangle_0 t \ ,\\ &\langle X^2\rangle=\langle X^2\rangle_0+2\langle XU\rangle_0+\langle U^2\rangle_0 t^2+\frac{\pi^2}{24}Dt^3 \ .\end{aligned} \tag{30}$$

Assuming the initial conditions

$$\begin{aligned}&\langle X\rangle_0=0, \ \ \langle U\rangle_0=0 \ ,\\ &\langle X^2\rangle_0=0, \ \ \langle U^2\rangle_0=0, \ \ \langle XU\rangle_0=0 \ ,\end{aligned} \tag{31}$$

we get for small velocities

$$\begin{aligned}&\langle U\rangle=0, \ \ \langle X\rangle=0 \ ,\\ &\langle U^2\rangle=\frac{\pi^2}{8}Dt, \ \ \langle X^2\rangle=\frac{\pi^2}{24}Dt^3 \ .\end{aligned} \tag{32}$$

B. Multiplicative noise without dissipation

The perturbation is

$$f=-V(t)\phi \tag{33}$$

with $\langle V(t)\rangle=0$ and $\langle V(0)V(t)\rangle=2D\,\delta(t)$.

The stochastic equations are

$$\dot{P}=2\pi^2 V(t) \ , \tag{34}$$

$$\dot{U}=\frac{\pi^2}{4}(1-U^2)^{3/2}V(t) \ , \tag{35}$$

$$\dot{X}=U+HU(1-U^2)V(t) \ , \tag{36}$$

where

$$H=\int_{-\infty}^{\infty} z\operatorname{sech}z\tan^{-1}e^z dz\simeq 2.1 \ .$$

As before we get for the momentum

$$\langle P\rangle=P_0, \ \ \langle (P-\langle P\rangle)^2\rangle=8\pi^4 Dt \ . \tag{37}$$

And for the nonrelativistic limit $|U|\ll 1$ of Eqs. (35) and (36),

$$\begin{aligned}&\langle U\rangle=0, \ \ \langle X\rangle=\frac{\pi^2}{4}HDt \ ,\\ &\langle U^2\rangle=\frac{\pi^4}{8}Dt, \ \ \langle X^2\rangle=\frac{3}{16}\pi^4 H^2 D^2 t^2+\frac{\pi^4}{24}Dt^3 \ ,\end{aligned} \tag{38}$$

where we assume the initial conditions (31).

By comparing the multiplicative and additive noise of the same variance and without dissipation, some remarks must be made.

(1) For the multiplicative case a dependence on t^2 appears in the mean value $\langle X^2 \rangle$; however, this is not relevant for small noises due to the term D^2.

(2) The effect of the multiplicative noise is stronger, as we can see by comparing the mean square of the center and the velocity in both cases.

(3) There is a linear dependence in t for $\langle X \rangle$ in the multiplicative noise that is due to the asymmetry of the noise in the equations. Its effect is stronger in the region where $\phi \approx 2\pi$ than in the other one where $\phi \approx 0$. For the antisolitons, the effect is the opposite because for them $H<0$.

C. Additive noise with dissipation

The perturbation becomes

$$f=-\alpha\phi_t - F(t)\ , \tag{39}$$

and the dynamical equations are

$$\dot{P}=-\alpha P+2\pi F(t)\ , \tag{40}$$

which is the Langevin equation for the Brownian motion, and

$$\dot{U}=\frac{\pi}{4}(1-U^2)^{3/2}F(t)-\alpha U(1-U^2)\ , \tag{41}$$

$$\dot{X}=U\ . \tag{42}$$

In the same way as before we get

$$\begin{aligned} &\langle P \rangle = \langle P \rangle_0 e^{-\alpha t}\ ,\\ &\langle (P-\langle P \rangle)^2 \rangle = \frac{4\pi^2}{\alpha}D(1-e^{-2\alpha t})\ . \end{aligned} \tag{43}$$

And for the limit of small velocities, we obtain with the initial conditions (31)

$$\begin{aligned} &\langle U \rangle = 0,\quad \langle X \rangle = 0\ ,\\ &\langle U^2 \rangle = \frac{\pi^2}{16\alpha}D(1-e^{-2\alpha t})\ ,\\ &\langle X^2 \rangle = \frac{\pi^2}{16\alpha^2}D\left[2t-\frac{3}{\alpha}+\frac{4}{\alpha}e^{-\alpha t}-\frac{1}{\alpha}e^{-2\alpha t}\right]\ . \end{aligned} \tag{44}$$

D. Multiplicative noise with dissipation

The perturbation is

$$f=-\alpha\phi_t - V(t)\phi\ , \tag{45}$$

and we get the stochastic equations

$$\dot{P}=-\alpha P+2\pi^2 V(t)\ , \tag{46}$$

$$\dot{U}=-\alpha U(1-U^2)+\frac{\pi^2}{4}(1-U^2)^{3/2}V(t)\ , \tag{47}$$

$$\dot{X}=U+HU(1-U^2)V(t)\ . \tag{48}$$

For the momentum we get the following mean values:

$$\begin{aligned} &\langle P \rangle = \langle P \rangle_0 e^{-\alpha t}\ ,\\ &\langle (P-\langle P \rangle)^2 \rangle = \frac{4\pi^4}{\alpha}D(1-e^{-2\alpha t})\ . \end{aligned} \tag{49}$$

While in the limit of $|U| \ll 1$, we have with the initial conditions (31)

$$\begin{aligned} &\langle U \rangle = 0,\quad \langle X \rangle = \frac{\pi^2}{4}HDt\ ,\\ &\langle U^2 \rangle = \frac{\pi^4}{16\alpha}D(1-e^{-2\alpha t})\ ,\\ &\langle X^2 \rangle = \frac{\pi^4}{16\alpha^2}D\left[2t-\frac{3}{\alpha}+\frac{4}{\alpha}e^{-\alpha t}-\frac{1}{\alpha}e^{-\alpha t}\right]\\ &\qquad +\frac{\pi^4 H^2 D^2}{16\alpha^2}(e^{-2\alpha t}+2\alpha t-1)+\frac{\pi^4}{16}H^2D^2t^2\ . \end{aligned} \tag{50}$$

In comparing these equations with those of the additive noise with dissipation, we have the same remarks regarding those considered in the case without dissipation.

IV. NUMERICAL RESULTS

In a previous work[11] it was observed that if the variance of the noise is $\sigma \le 0.1$, the solitonic structure is preserved. In this way our perturbation analysis has been considered below this critical value for the variance. In particular, the results we report here correspond to the value $\sigma=0.01$ ($D=1.25\times10^{-6}$). On the other hand, we have considered for the dissipation coefficient the value $\alpha=0.1$. We can summarize our results by distinguishing two kinds of initial conditions.

A. Soliton initially at rest at the origin

This simulates the initial conditions (31) used in the nonrelativistic approximation, which provides a simple analytic tractable case of the stochastic differential equations for U and X. The mean-square values of the velocity $\langle U^2 \rangle$ and the center $\langle X^2 \rangle$ of the soliton are represented in Figs. 1, 2, 3, and 4 for the additive and multiplicative noise, either with dissipation or not. As we can see, the agreement between the analytic and numerical computations is fairly good. The dashed curves are those obtained by considering the Fokker-Planck equation in the nonrelativistic limit, while the continuous lines represent the average over 30 particular time evolutions consistent with the stochastic equations.

B. Soliton with initial velocity

In this case it is possible to obtain mean values directly from the associated Fokker-Planck equation for the momentum without any approximation relative to the velocity. In Figs. 5, 6, 7, and 8, the average of the momentum $\langle P \rangle$ and the mean square of the fluctuation are represented for the initial conditions $X_0=-0.5$, $U_0=0.1$. As before, we have a qualitative and quantitative agreement between the values obtained from the perturbation analysis (dashed curves) and those obtained numerically (solid curves).

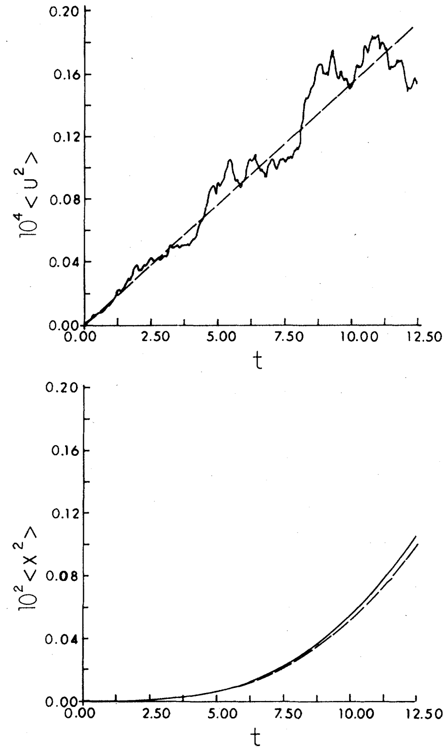

FIG. 1. Mean-square values of the center $\langle X^2 \rangle$ and velocity $\langle U^2 \rangle$ of the soliton for the *additive noise without dissipation:* the dashed lines are obtained from the perturbation approach, while the solid ones corresponds to the numerical computation.

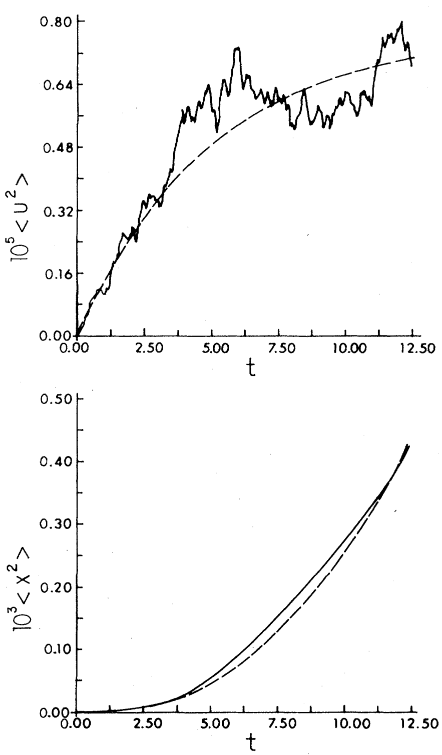

FIG. 2. Same as in Fig. 1, but for the *multiplicative noise without dissipation.*

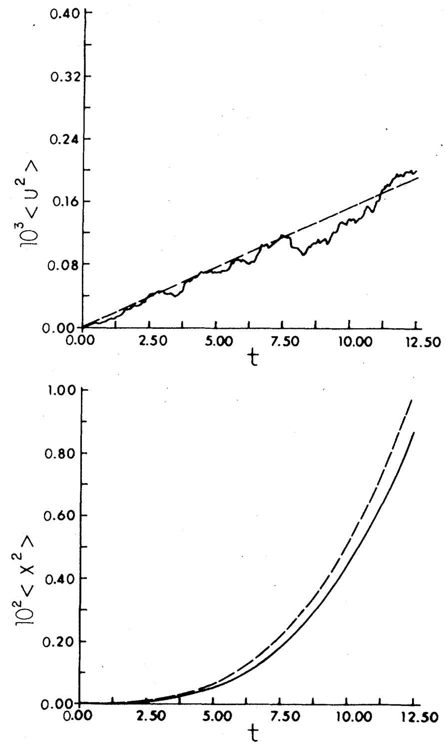

FIG. 3. Same as in Fig. 1, but for the *additive noise with dissipation.*

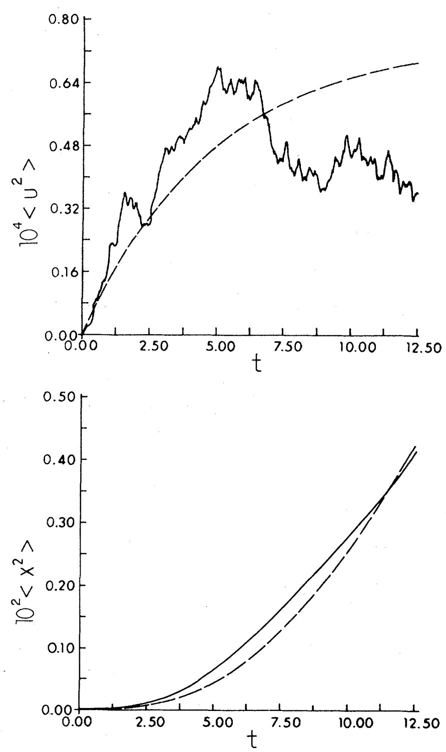

FIG. 4. Same as in Fig. 1, but for the *multiplicative noise with dissipation.*

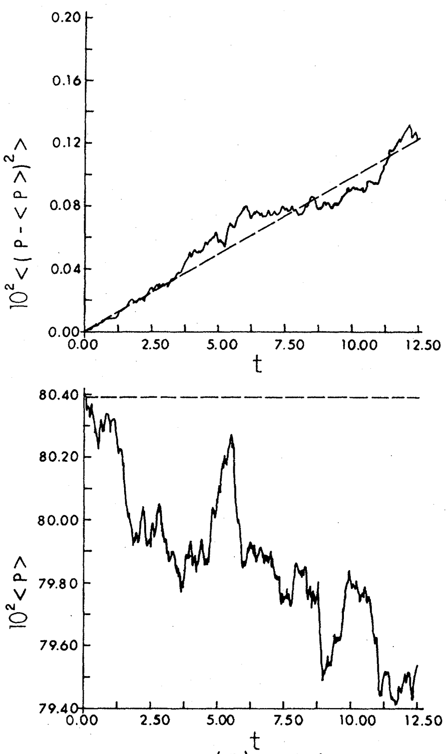

FIG. 5. Mean momentum $\langle P \rangle$ and the mean square of the fluctuations $\langle (P-\langle P \rangle)^2 \rangle$ for the *additive noise without dissipation:* the dashed lines are obtained from the perturbation approach, while the solid ones correspond to the numerical computation.

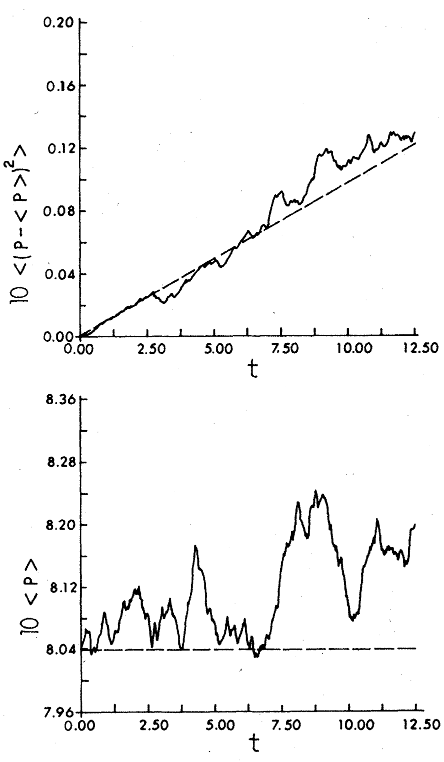

FIG. 6. Same as in Fig. 5, but for the *multiplicative noise without dissipation.*

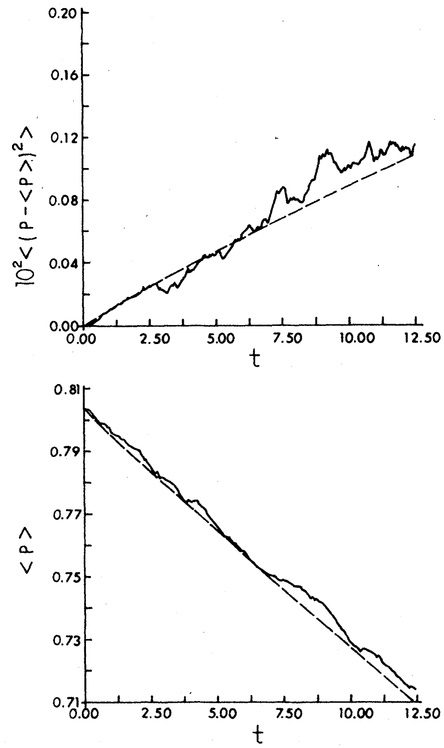

FIG. 7. Same as in Fig. 5, but for the *additive noise with dissipation.*

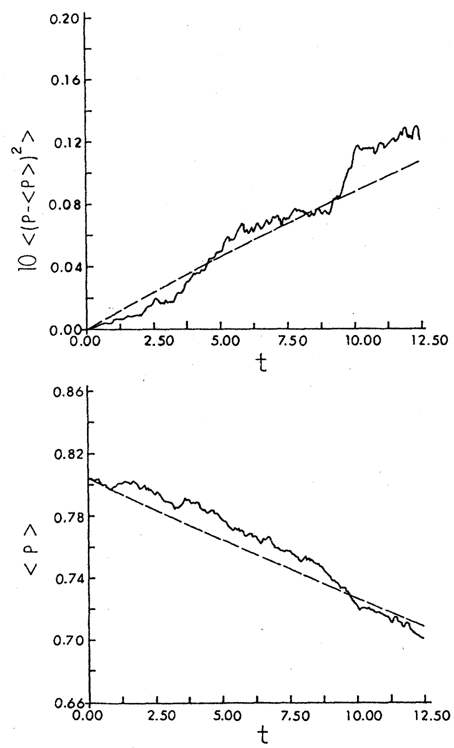

FIG. 8. Same as in Fig. 5, but for the *multiplicative noise with dissipation.*

V. SUMMARY

We have studied the effect on a SG soliton of a weak localized noise, either additive or multiplicative; also the effect of a dissipation is analyzed. It is very surprising that the numerical results fit very well in the analytical approach. That must be justified later in a very general mathematical framework.

On the other hand, we plan to extend our numerical computations in order to fit them into the general problem of the Josephson transmission lines, with impurities and under thermal effects,[12] by considering more general noises.

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