Finite-temperature aspects of the quantum Hall effect: A Boltzmann-equation approach

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A quantum Boltzmann equation developed previously [M. Charbonneau *et al.*, J. Math. Phys. **23**, 318 (1982)] is employed to evaluate the dc electrical conductivity in two dimensions in the presence of very strong magnetic fields. The resulting formulas for the conductivity components σ_{yx} and σ_{xx} are very simple and valid for all temperatures. σ_{yx} is independent of any scattering potentials in the first Born approximation; besides, it shows analytically the inadequacy of the independent-electron theory to account for the fractional Hall effect. The quantization becomes obvious whenever the Fermi level lies in an energy gap. For an integer Landau level filling factor ν, good agreement is obtained with the experimental results for the plateau values, for the deviation from the integer plateau values as a function of temperature, magnetic field, effective mass, and position of the Fermi level, and for the temperature dependence of the plateau widths. Besides, we indicate that the formalism can incorporate electron-electron interaction necessary for the fractional Hall effect ($\nu<1$).

I. INTRODUCTION

Since its discovery[1] the quantum Hall effect of a two-dimensional electron gas in a strong magnetic field has been the subject of numerous theoretical investigations.[2–12] At present the theoretical interest has been shifted to the fractional quantum Hall effect,[13] and the normal (or integer) effect is supposed to be well understood. There are, however, important questions left in the normal case, some of which have been recently discussed by Joynt and Prange.[14] In this paper following a new approach, which is closer to the spirit of traditional transport theory, we discuss some finite-temperature aspects of the normal case, such as accuracy of the plateau values in σ_{yx}, deviations from the values ie^2/h, i integer, and the temperature dependence of the plateau widths. At the same time we will indicate how the formalism can deal with electron-electron interaction necessary for the fractional Hall effect.

The new approach consists in using a quantum Boltzmann equation derived previously[15] for the evaluation of the dc electrical conductivities. This equation has been derived as follows. Applying Zwanzig's projection operator technique to the von Neumann equation for the density operator, Van Vliet[16] derived in the context of linear response theory, and within the Van Hove limit (equivalent to the first Born approximation), two inhomogeneous master equations for the diagonal and nondiagonal parts of the density operator. Then with the method of moments the diagonal,[16] nondiagonal,[15] and total Boltzmann[15] equations have been derived. The Hamiltonian used in the von Neumann equation was

$$H=H^0+\lambda V-AF(t)\ . \tag{1.1}$$

In (1.1), H^0 is the largest part of H which can be diagonalized, λV is the interaction, assumed nondiagonal (the diagonal part is to be incorporated in H^0), and $-AF(t)$ is the external field Hamiltonian, with $F(t)$ being a generalized force and A the conjugate operator.

The total (diagonal plus nondiagonal) Boltzmann equation derived in this way for fermion-boson interaction (e.g., electron-phonon or electron-impurity interaction) reads (in second-quantization language)

$$\begin{aligned}&\frac{\partial\langle c^\dagger_{\xi_1}c_{\xi_2}\rangle_t}{\partial t}-F(t)\frac{1-e^{-\beta(\varepsilon_{\xi_1}-\varepsilon_{\xi_2})}}{\varepsilon_{\xi_1}-\varepsilon_{\xi_2}}\langle n_{\xi_2}\rangle_{\rm eq}(1-\langle n_{\xi_1}\rangle_{\rm eq})(\xi_2\,|\,\dot\alpha\,|\,\xi_1)\\&-\beta F(t)\langle n_{\xi_1}\rangle_{\rm eq}(1-\langle n_{\xi_1}\rangle_{\rm eq})\sum_{\xi'}\{[(\xi'\,|\,\alpha\,|\,\xi')-(\xi_1\,|\,\alpha\,|\,\xi_1)][w_{\xi_1\xi'}(1-\langle n_{\xi'}\rangle_{\rm eq})+w_{\xi'\xi_1}\langle n_{\xi'}\rangle_{\rm eq}]\}\delta_{\xi_1\xi_2}\\&=\sum_{\xi'}[w_{\xi'\xi_1}\langle n_{\xi'}\rangle_t(1-\langle n_{\xi_1}\rangle_t)-w_{\xi_1\xi'}\langle n_{\xi_1}\rangle_t(1-\langle n_{\xi'}\rangle_t)]\delta_{\xi_1\xi_2}-(i/\hbar)(\varepsilon_{\xi_2}-\varepsilon_{\xi_1})\langle c^\dagger_{\xi_1}c_{\xi_2}\rangle_t\ .\end{aligned} \tag{1.2}$$

Further,

$$H^0=\sum_\xi \mathbf{n}_\xi\varepsilon_\xi+\sum_\eta \mathbf{N}_\eta E_\eta\ , \tag{1.3}$$

$$\lambda V=\sum_{\xi',\xi'',\eta',\eta''}(\xi'',\eta''\,|\,\lambda v\,|\,\xi',\eta')c^\dagger_{\xi''}c_{\xi'}b^\dagger_{\eta''}b_{\eta'}\ , \tag{1.4}$$

$$|\gamma\rangle=|\{n_\xi\},\{N_\eta\}\rangle=|\{n\}\rangle\otimes|\{N\}\rangle\ . \tag{1.5}$$

In these expressions $|\gamma\rangle$ are the many-body eigenstates of H^0, $\{|\xi)\},\{|\eta)\}$ denote the sets of one-particle states for fermions and bosons with eigenvalues ε_ξ and E_η, respectively. $\mathbf{n}_\xi=c^\dagger_\xi c_\xi$ and $\mathbf{N}_\eta=b^\dagger_\eta b_\eta$ are the occupation

number operators with eigenvalues n_ξ, N_η, while the c's and b's are the creation and annihilation operators for fermions and bosons, respectively. The fermion transition rates $w_{\xi\xi'}$ are given by

$$w_{\xi\xi'}=\sum_{\eta',\eta''} Q(\xi'',\eta'';\xi',\eta')\langle N_{\eta''}(1+N_{\eta'})\rangle_{\rm eq}$$
$$\approx \sum_{\eta',\eta''} Q(\xi'',\eta'';\xi',\eta')\langle N_{\eta''}\rangle_{\rm eq}(1+\langle N_{\eta'}\rangle_{\rm eq}) . \quad (1.6)$$

As usual, it has been assumed that the bosons remain at equilibrium (eq). The latter equality is based on a truncation rule and the Q's are the binary transition rates (see Ref. 15). Further, $\beta=1/kT$, T is the temperature, k is Boltzmann's constant, and λ is the interaction constant.

In the case of an externally applied electric field $\mathbf{E}(t)$, $\mathbf{F}(t)=q\mathbf{E}(t)$, and $A=\sum_i(\mathbf{r}_i-\langle\mathbf{r}_i\rangle_{\rm eq})=\sum_i\alpha_i$, where q is the charge of the carriers (fermions), $\mathbf{r}_i$ their positions, and $\langle\mathbf{r}_i\rangle_{\rm eq}$ their positions before switching on the electric field. The electrical current Schrödinger operator is $\mathbf{J}=q\sum_i\mathbf{v}_i/V_0$, where V_0 is the volume of the system ($\mathbf{v}_i=\dot{\alpha}_i$).

Before proceeding to the calculations we write a few results from Refs. 15 and 17. In the Van Hove limit ($\lambda\to 0$, $t\to\infty$, $\lambda^2 t=$finite) the average current coming from the diagonal part of (1.2) is given by (d stands for diagonal, $\xi_1=\xi_2=\xi$)

$$\langle(J_\mu)_d\rangle_t=\frac{q}{V_0}\sum_\xi(-\mathscr{B}_\xi\langle n_\xi\rangle_t\alpha_{\mu\xi}+\langle n_\xi\rangle_t\dot{\alpha}_{\mu\xi}) , \quad \mu=x,y,z \quad (1.7)$$

where the quantity $\mathscr{B}_\xi\langle n_\xi\rangle_t$ is given by the first term on the right-hand side of the diagonal part of (1.2). The first term of (1.7) has been termed "collisional" current, since it represents the many-body contribution of collisions to the current ($\mathscr{B}_\xi\langle n_\xi\rangle_t$ is the collision integral); the second term is the usual ponderomotive current. The full ponderomotive current is, of course, given by

$$\langle J_\mu\rangle_{\rm pond}=\frac{q}{V_0}\sum_{\substack{\xi,\xi',\\ \rm spin}}\langle c_\xi^\dagger c_{\xi'}\rangle_t(\xi\,|\,\dot{\alpha}_\mu\,|\,\xi') . \quad (1.8)$$

If there is only collisional current the dc conductivity resulting from (1.7) and the diagonal part of (1.2) is given by [see (2.83) and Sec. 7 of Ref. 15]

$$\sigma^d_{\mu\nu}(0)=\frac{\beta q^2}{V_0}\sum_{\xi,\xi',\rm spin}\langle n_{\xi'}\rangle_{\rm eq}(1-\langle n_\xi\rangle_{\rm eq})\times w_{\xi'\xi}(R_{\nu\xi'}-R_{\nu\xi})R_{\mu\xi} ,$$
$$R_\xi=(\xi\,|\,(\mathbf{r}-\mathbf{r}^{\rm eq})\,|\,\xi) . \quad (1.9)$$

We further note that the total dc conductivity tensor $\sigma_{\mu\nu}(0)$ is given by

$$\sigma_{\mu\nu}(0)=\sigma^d_{\mu\nu}(0)+\sigma^{\rm nd}_{\mu\nu}(0) , \quad (1.10)$$

where nd stands for nondiagonal, i.e., the contribution from the nondiagonal part of (1.2).

In the next section we will evaluate the components σ_{yx},σ_{xx} using (1.2). In Sec. III we will discuss the zero temperature limit of the results and the deviations from it for finite temperatures; further, we will discuss the validity of the first Born approximation for the interaction λV. We will finish with some remarks and conclusions.

II. THE QUANTUM HALL EFFECT

We consider a two-dimensional electron gas, such as realized in the inversion layer of a MOSFET (metal-oxide-semiconductor field-effect transistor),[1] in a very strong magnetic field B perpendicular to the surface and pointing in the z direction. In the Landau gauge, the one-particle Hamiltonian, states, and eigenvalues read

$$h^0=(\mathbf{p}+e\mathbf{A})^2/2m^*, \quad \mathbf{A}=(0,Bx,0) , \quad (2.1)$$
$$|\,\xi)=|\,N,k_y)=\phi_N(x+l^2k_y)e^{ik_y y}/L_y^{1/2} , \quad (2.2)$$
$$\varepsilon_\xi=(N+\tfrac{1}{2})\hbar\omega_0, \quad N=0,1,2,\ldots \quad (2.3)$$

where $\omega_0=eB/m^*$ is the cyclotron frequency, m^* is the effective mass, and $l=(\hbar/m^*\omega_0)^{1/2}$ is the radius of the orbit. N denotes the various Landau levels, $\mathbf{A}$ is the vector potential, and $A_0=L_xL_y$ is the area. We set $x_0=l^2k_y$. In the representation (2.2) the matrix elements, necessary for the evaluation of the components σ_{yx},σ_{xx}, are

$$(\xi\,|\,x\,|\,\xi')=x_0\delta_{NN'}\delta_{k_yk_y'}+l'(\sqrt{N+1}\,\delta_{N',N+1}-\sqrt{N}\,\delta_{N',N-1})\delta_{k_yk_y'} , \quad (2.4)$$
$$(\xi\,|\,y\,|\,\xi')=(L_y/2)\delta_{NN'}\delta_{k_yk_y'}=y^{\rm eq}\delta_{NN'}\delta_{k_yk_y'} , \quad (2.5)$$
$$(\xi\,|\,\dot{\alpha}_x\,|\,\xi')=il'(-\sqrt{N+1}\,\delta_{N',N+1}+\sqrt{N}\,\delta_{N',N-1})\delta_{k_yk_y'} , \quad (2.6)$$
$$(\xi\,|\,\dot{\alpha}_y\,|\,\xi')=l'(\sqrt{N+1}\,\delta_{N',N+1}+\sqrt{N}\,\delta_{N',N-1})\delta_{k_yk_y'} , \quad (2.7)$$

where $l'=l/\sqrt{2}$.

The last two equations show that there is no diagonal ponderomotive current neither in the x nor in the y direction [cf. (1.7), (1.8)]. We are thus left as far as diagonal contributions are concerned with the first term of (1.7), i.e., we have only collisional current. Returning now to Eq. (1.9) we see immediately that $\sigma^d_{yx}(0)=0$ since, by (2.5), $(\xi\,|\,(y-y^{\rm eq})\,|\,\xi)=0$. Now, Eq. (2.4) indicates that in the absence of an external electric field there are stable orbits, with center x_0. Since, however, by (2.4) $(\xi\,|\,(x-x^{\rm eq})\,|\,\xi')$ is nonzero in its diagonal and nondiagonal forms it is seen from (1.9) that $\sigma^d_{xx}(0)$ does not vanish.

In this way we are led to evaluate the components

$$\sigma_{yx}^{\rm nd}\equiv\sigma_{yx}^{\rm nd}(0),\sigma_{xx}^{\rm nd}\equiv\sigma_{xx}^{\rm nd}(0)\ , \tag{2.8}$$

and

$$\sigma_{xx}^{d}\equiv\sigma_{xx}^{d}(0)\ .$$

Assuming, however (see below), that the Fermi energy lies in an energy gap or region of localized states it is obvious that no energy can be absorbed from the small electric field (to be taken in the x direction) and so the longitudinal current σ_{xx} is zero[18] (it will be shown later that $\sigma_{xx}^{\rm nd}$ vanishes identically). Thus, we will not evaluate σ_{xx}^{d} but only $\sigma_{yx}^{\rm nd},\sigma_{xx}^{\rm nd}$ with the help of the nondiagonal Boltzmann equation, i.e., the nondiagonal part of (1.2). The equation for that part reads

$$\frac{\partial\langle c_{\zeta_1}^{\dagger}c_{\zeta_2}\rangle_t}{\partial t}+eE_\nu(t)\frac{1-e^{\beta(\varepsilon_{\zeta_1}-\varepsilon_{\zeta_2})}}{\varepsilon_{\zeta_1}-\varepsilon_{\zeta_2}}\langle n_{\zeta_2}\rangle_{\rm eq}(1-\langle n_{\zeta_1}\rangle_{\rm eq})(\zeta_2\,|\,\dot\alpha\,|\,\zeta_1)=-\frac{i}{\hbar}(\varepsilon_{\zeta_2}-\varepsilon_{\zeta_1})\langle c_{\zeta_1}^{\dagger}c_{\zeta_2}\rangle_t,\quad \zeta_1\neq\zeta_2\ . \tag{2.9}$$

Note that the interaction λV does not appear in (2.9).

Now the solution of (2.9) upon making the initial random-phase assumption $\langle c_{\zeta_1}^{\dagger}c_{\zeta_2}\rangle_{t=0}=0$ is

$$\langle c_{\zeta_1}^{\dagger}c_{\zeta_2}\rangle_t=e\int_0^t dt'\,E_\nu(t')e^{-i\Delta(t-t')/\hbar}\frac{1-e^{\beta\Delta}}{\Delta}\langle n_{\zeta_2}\rangle_{\rm eq}(1-\langle n_{\zeta_1}\rangle_{\rm eq})(\zeta_2\,|\,\dot\alpha\,|\,\zeta_1),\quad \Delta=\varepsilon_{\zeta_2}-\varepsilon_{\zeta_1}\ . \tag{2.10}$$

Combining (2.10) with (1.8) gives an expression for the current density which, upon comparing with the linear-response formula

$$\int_0^t dt'\,\phi(t-t')E_\nu(t')\ ,$$

gives an expression for the response function $\phi(t)$. The Laplace transform of $\phi(t)$ gives the conductivity $\sigma_{\mu\nu}^{\rm nd}(\omega)$, whose dc limit is easily obtained:

$$\sigma_{\mu\nu}^{\rm nd}=-\frac{e^2}{iA_0}\sum_{\substack{\zeta_1,\zeta_2,\\ \text{spin}}}\frac{1-e^{\beta\Delta}}{\Delta^2}\langle n_{\zeta_2}\rangle_{\rm eq}(1-\langle n_{\zeta_1}\rangle_{\rm eq})\times(\zeta_2\,|\,\dot\alpha_\nu\,|\,\zeta_1)(\zeta_1\,|\,\dot\alpha_\mu\,|\,\zeta_2)\ . \tag{2.11}$$

This result satisfies $\sigma_{yx}^{\rm nd}=-\sigma_{xy}^{\rm nd}$, as can be easily seen from (2.6) and (2.7). It has also been obtained for the three-dimensional case ($A_0\to V_0$, $\varepsilon_\zeta\to\varepsilon_\zeta+\hbar^2k_z^2/2m^*$) from the solution of the nondiagonal master equation.[15,16] At high temperatures it led to the ordinary Hall effect[15] and at very low temperatures to the oscillatory Hall effect.[19] The most important features of (2.11) are the following:

(i) It is valid for all temperatures.

(ii) It is independent of the interaction λV.

(iii) ζ_1,ζ_2 label different Landau levels.[20]. ($\zeta_1\neq\zeta_2$ [cf. (2.9)]; actually for $\zeta_1=\zeta_2$ the matrix element $(\zeta\,|\,\dot\alpha\,|\,\zeta)$ vanishes [cf. (2.6) and (2.7)]; so there is no diagonal conductivity associated with (2.11)). The second point needs some clarification since in (2.11) the nearly-free-electron energies appear [m^* is the effective mass; cf. (2.3)]. In fact, Eqs. (1.2) and (2.9) are the result of taking the trace of the corresponding master equations, and that was taken in the subdynamics of H^0 which contains the periodic part of the interaction expressed by the effective mass. The fact that the interaction λV does not appear is due to the Van Hove limit ($\lambda\to0$, $t\to\infty$, $\lambda^2t=$finite) in which Eq. (2.9) contains terms proportional to λ. Thus, provided that the Van Hove limit (or the first Born approximation) applies, the interaction λV does not have to be specified further (see the Introduction); the only additional assumption is that the Fermi energy lies in a gap as a result of this interaction.

Proceeding now to the explicit evaluation of (2.11), we remark that due to (2.3), (2.6), and (2.7), the quantity Δ in (2.11) is given by

$$\Delta=\varepsilon_{\zeta_2}-\varepsilon_{\zeta_1}=(N_2-N_1)\hbar\omega_0=\pm\hbar\omega_0\ . \tag{2.12}$$

Since $|\zeta)=|N,k_y)$ there will be one summation over k_y which, with periodic boundary conditions for k_y, will give

$$\sum_{k_y}\to\frac{L_y}{2\pi}\int_{-L_x/2l^2}^{L_x/2l^2}dk_y=\frac{A_0}{2\pi l^2}\ , \tag{2.13}$$

since the functions $\phi_N(x+l^2k_y)$ oscillate around the point $x_0=l^2k_y$. In (2.13) it has been assumed that only one spin state is occupied; if both spin states are occupied then $\sum_{k_y}\to A_0/\pi l^2$. In the following we set $\langle n_\zeta\rangle_{\rm eq}=f_N$ and $E=\beta\hbar\omega_0$.

A. Evaluation of $\sigma_{yx}^{\rm nd}$

Using (2.11), (2.6), (2.7), and the relations just stated for Δ, $\sum_{k_y}$, $\langle n_\zeta\rangle_{\rm eq}$, and E, we easily find for $\mu=y$, $\nu=x$ the expression

$$\sigma_{yx}^{\rm nd}=\frac{e^2}{2h}\sum_N[(N+1)f_N(1-f_{N+1})(1-e^{-E})-Nf_N(1-f_{N-1})(1-e^{E})]\ . \tag{2.14}$$

In the second term we change $N\to N+1$; the result is

$$\sigma_{yx}^{\rm nd}=\frac{e^2}{2h}\sum_N(N+1)[f_N(1-f_{N+1})(1-e^{-E})-f_{N+1}(1-f_N)(1-e^{E})]\ . \tag{2.15}$$

If we now use the equilibrium Fermi-Dirac expression for f_N and (2.3), we easily find that

$$f_N(1-f_{N+1})(1-e^{-E})=-f_{N+1}(1-f_N)(1-e^{E})\ . \tag{2.16}$$

Thus (2.15) becomes

$$\sigma_{yx}^{\rm nd}=\frac{e^2}{h}\sum_N (N+1)f_N(1-f_{N+1})(1-e^{-E})\ . \qquad (2.17)$$

This is the final result, valid for all temperatures. Since the diagonal contribution σ_{yx}^{d} vanishes (see above), (2.17) is the result for the total σ_{yx} but we stress the fact that this is not a nearly-free-electron result [in (2.17) only the effective mass appears]; simply the part due to the interaction other than the periodic one vanishes identically. For ellipsoidal energy surfaces (2.17) remains unaltered except that m^* is replaced by $m^*=(m_1m_2)^{1/2}$, where m_1 and m_2 are the effective masses in the x and y directions, respectively.

B. Evaluation of $\sigma_{xx}^{\rm nd}$

Similar to (2.15) we find

$$\sigma_{xx}^{\rm nd}=\frac{ie^2}{2h}\sum_N (N+1)[f_N(1-f_{N+1})(1-e^{-E}) +f_{N+1}(1-f_N)(1-e^{E})]=0\ , \qquad (2.18)$$

due to (2.16). Thus, as stated previously, this contribution vanishes identically.

III. ZERO TEMPERATURE LIMIT, FINITE TEMPERATURE DEVIATIONS

In the presence of the interaction λV the density of states $N(\varepsilon)$, which in its absence was a series of δ functions, becomes[21] a series of broadened δ functions; this is shown schematically in Fig. 1, where the center of the broadened level is assumed to consist of extended states and the tails of localized states (for a justification of this band structure see Refs. 2 and 7).

Our main result (2.17) makes the quantization apparent whenever the Fermi level lies in the region of localized states. In fact, it is obvious that for $\varepsilon_N<\varepsilon_F<\varepsilon_{N+1}$ (see Fig. 1)

$$\lim_{T\to 0}\sigma_{yx}^{\rm nd}=(N+1)\frac{e^2}{h},\quad N=0,1,2,\ldots \qquad (3.1)$$

since only the Nth term contributes (the terms $N'<N$ do not contribute due to the factor $1-f_{N+1}$). Note that we have arrived at this result without further assumptions about the density of states and the scattering, in contrast with Streda,[6] for example. Note also that if both spin states are occupied, the result (3.1) is multiplied by 2.

For finite temperatures the deviations from the result (3.1) can easily be obtained. Assuming $e^{-E}=\delta\ll 1$ and setting $0<b=(\varepsilon_F-\varepsilon_N)/\hbar\omega_0<1$ we find, upon expanding the exponential of the factors f_N, $1-f_{N+1}$, the following result:

$$\sigma_{yx}=\frac{e^2}{h}(N+1)(1-\delta^b-\delta^{1-b}+\delta^{2b}+\delta^{2(1-b)}+\cdots),\qquad \delta^b,\delta^{1-b}\ll 1\ ,\varepsilon_N<\varepsilon_F<\varepsilon_{N+1}\ . \qquad (3.2)$$

Thus the deviation $\delta\sigma=-\delta^b-\delta^{1-b}+\delta^{2b}+\delta^{2(1-b)}$ from the integer value (at zero temperature) $(e^2/h)(N+1)$ depends on b exponentially. The quantities appearing in b are the effective mass m^*, the magnetic field B, the temperature T, and the position of the Fermi level with respect to ε_N. The corresponding behavior of the Hall coefficient $R_H=1/\sigma_{yx}$ has been observed experimentally and deduced theoretically but in a less explicit fashion by means of percolation arguments.[22] It is of course possible that other finite temperature factors neglected here (e.g., hopping between localized states) modify $\delta\sigma$ quantitatively. The plateau values in σ_{yx} obtained, for finite temperatures, from direct numerical evaluation of (2.17) compared well with the experimental[13,23,24] values; they were correct to the nth decimal place with $5\le n\le n'$, $n'\ge 8$. The accuracy was found to vary from sample to sample and to be limited, for $T\to 0$ from the machine's efficiency (too-large αb numbers). The most important factor in determining the deviation $\delta\sigma$, as (2.17) stands, was the temperature, whereas the position of the Fermi level, within the shaded region[25] in Fig. 1, almost did not matter at all. (If, however, the Fermi level for each value of B is determined from the free-particle density, i.e., without level broadening, in the manner

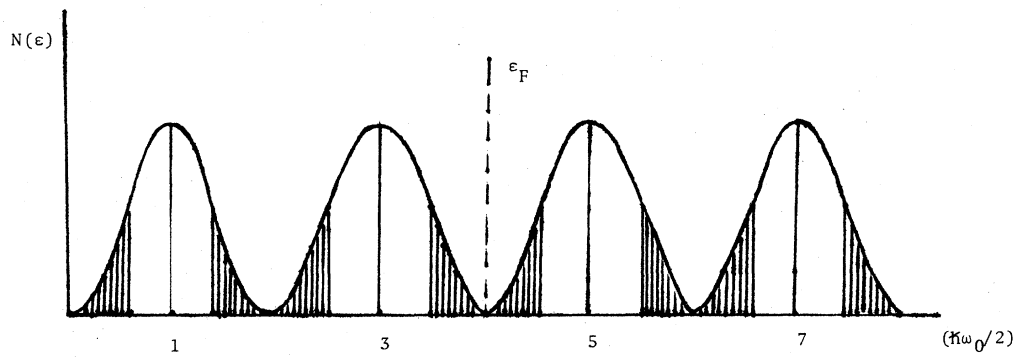

FIG. 1. Density of states of broadened Landau levels.

$$n_0=\frac{N_0}{A_0}=\frac{1}{2\pi l^2}\sum_N \frac{1}{e^{E(\varepsilon_N-\varepsilon_F)}+1},\quad N=0,1,2,\ldots \tag{3.3}$$

then this results for the plateau corresponding to lower B fields to roughly up to two decimal places loss in the accuracy [since the interaction λV is absent from (3.3) the Fermi level is shifted downwards in energy and the assignment of an integer N for values of the magnetic field B at the edges becomes dubious].) In agreement also with (3.2) it was found that the accuracy of the plateau values became poorer as B decreased. This is of course related to the validity of the Born approximation which we now discuss.

If we assume that the electrons interact with uniformly distributed ionized impurities, the approximation implies that the magnetic length l is much smaller than the average impurity separation d. This condition, however, is met only for high magnetic fields roughly higher than $10T$ whereas for most of the experiments B is smaller than $10T$ and l is of the order of d (the best agreement between this theory and the experiment was obtained with the higher B values of Ref. 24). This casts doubt on the still reasonable agreement of our formulas with the experiment for the B values, for which the condition $l \ll d$ is not met.

This restriction, however, can be relaxed by the following reasoning. We can include the diagonal part V_d of the interaction λV in H^0 thus weakening the strength of λV. Using projection operators[16] V_d can be written as $V_d=\sum_\gamma |\gamma\rangle\langle\gamma|\,\langle\gamma|V_d|\gamma\rangle$ and the matrix element can be expressed, e.g., in the symmetric gauge, in terms of definite integrals involving Laguerre polynomials.[26] This will modify slightly the eigenvalues ε_ζ ($\varepsilon_\zeta \to \varepsilon_\zeta+\Delta\varepsilon_\zeta$), but the eigenfunctions $|\zeta\rangle$ will remain the same. Since $\Delta'=\varepsilon_{\zeta_2}+\Delta\varepsilon_{\zeta_2}-\varepsilon_{\zeta_1}-\Delta\varepsilon_{\zeta_1}\approx\varepsilon_{\zeta_2}-\varepsilon_{\zeta_1}=\Delta$, the result (2.11) will be affected only through the factors $\langle n_{\zeta_1}\rangle_{\rm eq}$, $\langle n_{\zeta_1}\rangle_{\rm eq}$. This, in turn, may affect the accuracy[26] of the result (2.17) for finite temperatures but will not change (2.18) [(2.11) evaluated in the symmetric gauge with $\Delta\varepsilon_\zeta=0$ leads to the same results[27]].

A similar reasoning can be applied to electron-electron interaction if one works in the relative frame of Ref. 28. For an exact treatment at zero temperature, see Ref. 7.

Before closing this section we note that the result (2.17) can be cast into a collision broadened version as follows: We write

$$f_N(1-f_{N+1})=f_N(1-f_N)\frac{1-f_{N+1}}{1-f_N}$$
$$\approx\frac{\delta(\varepsilon_N-\varepsilon_F)}{\beta}\frac{1-f_{N+1}}{1-f_N} \tag{3.4}$$

for the very low temperatures of interest here. Using (3.4) and a Lorentzian for the δ function (for simplicity of zero shift), we find

$$\sigma_{yx}^{\rm nd}\approx\frac{e^2}{h}\frac{2\tanh(E/2)}{\pi\beta}\sum_N (N+1)\frac{\Gamma_N}{(\varepsilon_N-\varepsilon_F)^2+\Gamma_N^2}, \tag{3.5}$$

where Γ_N is the width of the level ε_N. In contrast to (2.17), however, (3.5) reproduces poorly, as far as accuracy is concerned, the zero temperature limit (3.1) for Γ_N proportional to kT. Whereas in (2.17) the terms $N'<N$ do not contribute at all, in (3.5) they do. For example, setting $\varepsilon_F-\varepsilon_N=b\Gamma_N$ leads, for ε_F in the middle of the gap, to a ratio roughly $10(N+1)$, for $b\geq 2$, between the Nth and the $(N-1)$th terms. This indicates that the Lorentzian approximation of the δ function and the replacement of ε_N by ε_F in the factor $(1-f_{N+1})/(1-f_N)$ are only approximately correct.

The temperature behavior of the plateau widths[29] can be deduced directly from the exact result (2.17). At $T=0$, we have the sharp steps expressed by (3.1); as the temperature increases, provided that $e^{-E}\ll 1$, $f_{N+1}\simeq 0$ and the terms $N'<N$ do not contribute, the plateaus between the various levels increase due to the "washing-out" of the Fermi step function f_N; see also Ref. 22 for percolation arguments.

IV. FURTHER REMARKS AND CONCLUSIONS

In order to arrive at (3.1) and (3.2) it was sufficient to assume that the Fermi level lies in a region of localized states without discussing how, if at all, the assumed weak interaction potentials λV do pin it there; this has been discussed recently by Joynt and Prange,[14] whose estimates show that the potentials λV are close to being weak; *a posteriori* evidence for that is, of course, the fact that the effect has been observed in good (i.e., clean) samples but not in poor ones.

We have also assumed that one or both spin states of a Landau level were fully occupied; this is equivalent to saying that the occupation fraction $\nu=2\pi n_0 l^2$ is an integer. If, however, the Fermi level lies within, say, the lower spin level, then $\nu<1$, and the above analysis does not apply; this is the case of the fractional quantum Hall effect.[13]

The physical reason is that the Landau levels are (highly) degenerate and $\Delta=\varepsilon_{\zeta_2}-\varepsilon_{\zeta_1}$ in (2.11) vanishes, if ζ_1,ζ_2 refer to the same Landau level. This degeneracy, associated with zero conductivity, as given by (2.11), must be lifted before one hopes to account for transport effects when $\nu<1$, or else independent electron theories cannot deal with the case $\nu<1$.

As is well known,[28,30] electron-electron correlations and many-body effects can account quite well for the fractional Hall effect ($\nu<1$).

Now the Boltzmann equation misses the correlations between particles although it contains a many-body effect (collision integral). These correlations can be treated to first order by a transport equation for $\langle n_{\zeta_1}, n_{\zeta_2}\rangle_t$, $\zeta_1\neq\zeta_2$ [cf. Eq. (1.2)] involving electron-electron interaction explicitly; its derivation is a straightforward extension of previous work.[31] We plan to use this equation for the fractional Hall effect; work is in progress.

In summary, using a quantum Boltzmann equation, we have arrived at an exact result for the conductivity σ_{yx} in two dimensions, valid for all temperatures and independent of the interaction potentials. The zero temperature limit and the finite temperature deviations have been easily evaluated and are in reasonable agreement with experiment for an integer Landau level filling factor λ. We expect that the formalism will also work for $\nu < 1$.

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[26]It can be done in the Landau gauge as well; we then end up with integrals involving Hermite polynomials. In both cases the integrals are very cumbersome to handle analytically. From direct numerical evaluation of (3.2) or (2.17) we find that the reported accuracy is not affected for $V_d \sim 0.20\hbar\omega_0$. Larger V_d values can always be allowed for lower temperatures.
[27]Again $\sigma_{yx}^{nd} = -\sigma_{xy}^{nd}$, and this entails $\sigma_{yx}^{d} = -\sigma_{xy}^{d}$ by appealing to Onsager's relation $\sigma_{yx} = -\sigma_{xy}$. Using now (1.9) and detailed balance we can easily show, without any reference to the form of the eigenfunctions, that $\sigma_{yx}^{d} - \sigma_{xy}^{d} = 0$, which implies $\sigma_{yx}^{d} = 0$, as previously [see Eq. (2.5)].
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