Photoinduced macroscopic quantum tunneling

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Recently there has been interest in macroscopic systems which can be described in terms of the *quantum* dynamics of the same collective variables which describe their classical behavior. In particular, there has been considerable interest in the following questions: (1) Is it possible to observe unambiguous evidence of the quantum-mechanical nature of a macroscopic collective coordinate? (2) In what way is the quantum mechanics of a macroscopic variable different from that of a microscopic variable or, in other words, what is the effect of dissipation on the dynamics of a quantum-mechanical system? Spectroscopic evidence of the quantization of energy levels can be found by measuring the system's response to an applied ac perturbation, exploiting the quantum-mechanical relation between energy and frequency. The specific system studied is a superconducting quantum interference device (SQUID). It is shown that a time-varying magnetic field through the superconducting loop in a rf SQUID can cause transitions between different fluxoid states. The role of dissipation is discussed in detail, and it is shown that in the case that dissipation is characterized by linear damping in the classical limit, the photon-absorption process depends critically on the ratio of the resistance of the weak link to the fundamental unit of resistance h/e^2.

I. INTRODUCTION

It is well established that there are many circumstances in which the dynamics of a macroscopic system can be described in terms of the motion of a collective coordinate which satisfies a classical equation of motion. That the system actually has a very large number of microscopic degrees of freedom is reflected solely in the fact that the equations of motion for the collective coordinate are dissipative.[1] The dissipation can, in many cases,[2] be described by a single friction coefficient, η. One example, which will be of prime interest to us here, is the superconducting quantum interference device (SQUID), where the appropriate collective coordinate is the magnetic flux, ϕ, through the superconducting ring.

It is generally expected that under suitable conditions (e.g., at sufficiently low temperature) macroscopic systems can be described in terms of the *quantum* dynamics of the same collective variables which describe their classical behavior. In particular, there has been considerable recent interest[3] in the following questions: (1) Is it possible to observe *unambiguous* evidence of the quantum-mechanical nature of a macroscopic collective coordinate? (2) In what way is the quantum mechanics of a macroscopic variable different from that of a microscopic variable or in other words, what is the effect of dissipation on the dynamics of a quantum-mechanical system? Naively it might seem that this program is impractical due to the long time scales and miniscule quantum of energy generally associated with the motion of macroscopic objects. Thermal effects will therefore tend to swamp quantum effects. However, in current biased Josephson junctions and rf SQUID's (and possibly other systems) where the quantum energy scale is set by the Josephson plasma frequency ω_J, one can readily perform experiments at temperatures low compared to $\hbar\omega_J$.

One possible set of experiments which could address these questions involve observing the decay of a metastable state by quantum tunneling of the macroscopic variable.[4] While it may be the most straightforward way to observe macroscopic quantum tunneling, this particular approach suffers from practical problems. The decay rate is simply a number, and hence the only way we have to recognize that the decay is quantum is that it becomes temperature independent. However, this is not an unambiguous evidence of quantum tunneling; there could, for instance, be a temperature-independent source of noise, external to the system, which is causing the transitions. For this reason it is important in interpreting the experiment to have a quantitative theoretical calculation (prefactors and all) of the decay rate to compare with experiment.[5] While this can be done for simple models of dissipation, the results can only be compared to within the accuracy of the model, and to the extent that the values of *all* the relevant system parameters (capacitance, resistance, critical current, etc.) are known. To some extent these difficulties can be overcome by studying the systematic variation of the decay rates from system to system as the values of the relevant experimental parameters are changed.[6] In practice, however, this is difficult to do in a well-controlled way.

A more direct approach is to look for spectroscopic evidence[7] of the quantization of energy levels by measuring the system's response to an applied ac perturbation and exploiting the quantum-mechanical relation between energy and frequency, $\Delta E = \hbar\omega$, where ΔE is the difference in energy between the initial and final state of the macroscopic system, and ω is the frequency of the applied field. Since $\hbar$ appears explicitly in the relation between two quantities which can be independently determined, such

an experiment could provide incontravertible evidence of the quantum nature of the transition. Moreover, as we will see, the ability of the experimenter to vary the frequency ω makes possible a detailed study of the effect of dissipation on the transition.

In most ways, this type of experiment is similar to a host of analogous optical and infrared experiments on microscopic systems. One difference, however, is that because the system of interest to us is macroscopic, there is one as opposed to $\sim 10^{23}$ systems undergoing a transition. Thus, we cannot observe the transition by measuring the amount of energy absorbed from the ac field. On the other hand, if the transition occurs between macroscopically distinguishable states, we can measure the value of the macroscopic variable directly to determine whether or not a transition has occurred.

In this paper we present in detail the theory of photoinduced macroscopic quantum tunneling in rf SQUID's. A brief version of the present work was reported recently.[7] In this case the macroscopic variable is the flux through the SQUID loop. We show explicitly that an applied microwave field can cause transitions between different fluxoid states of the SQUID. Such a photoinduced transition, if observed, would be striking because it involves a transition between two macroscopically distinguishable states caused by the absorption of a single photon. Furthermore, the energy difference ΔE between the final and initial state can be controlled by adjusting the applied magnetic field through the SQUID. Thus, the explicitly quantum-mechanical relation between ΔE and the threshold frequency ω_T at which the transitions first occur is amenable to experimental verification. Finally, the functional dependence of the transition rate on the frequency of the ac field is highly sensitive to the nature of the dissipation. In the important special case of ohmic dissipation (see Sec. II) the transition rate depends critically on the normal resistance of the SQUID.

The plan of this paper is as follows. In Sec. II we discuss the Caldeira-Leggett model of a SQUID, and in Sec. III we show how in a limited region of the parameter space this model can be mapped onto the simpler problem of a two-state system coupled to a heat bath. This sort of mapping, where it is possible, is very useful as the resulting model is much easier to treat theoretically. In Sec. IV the photoinduced transition rate is calculated in the presence of a general heat bath using linear-response theory, and the range of validity of the results is delineated. In particular, the case of ohmic dissipation is worked out in detail. Finally Sec. V contains a discussion of spontaneous emission, and in Sec. VI we make some general observations and discuss some other possible processes involving photoinduced quantum transitions in macroscopic systems (e.g., quantum resonant activation from a single metastable well).

II. FORMULATION OF THE PROBLEM

It has been argued by Caldeira and Leggett[1] that the quantum dynamics of the collective variable, the total flux ϕ through a SQUID loop, can be well described[8] by the Hamiltonian given by

$$H=\tfrac{1}{2}C\dot{\phi}^2+V(\phi)+\sum_\alpha \tfrac{1}{2}m_\alpha(\dot{x}_\alpha^2+\omega_\alpha^2 x_\alpha^2) +\phi\sum_\alpha f_\alpha x_\alpha+\phi^2\sum_\alpha \frac{f_\alpha^2}{2m_\alpha\omega_\alpha^2} . \tag{2.1}$$

Here C is the capacitance of the weak link in a rf SQUID and the set of variables $\{x_\alpha,\dot{x}_\alpha\}$ represents the degrees of freedom of the environment whose spectral density $J(\omega)$ is

$$J(\omega)=\frac{\pi}{2}\sum_\alpha\left[\frac{f_\alpha^2}{m_\alpha\omega_\alpha}\right]\delta(\omega-\omega_\alpha) . \tag{2.2}$$

All information concerning the effect of the environment on the flux dynamics is contained in $J(\omega)$. It can be shown that if $J(\omega)\to\omega/R$ as $\omega\to 0$ then in the classical limit ϕ obeys the equation of motion given by

$$C\ddot{\phi}+\frac{\phi}{R}=-\frac{dV(\phi)}{d\phi} , \tag{2.3}$$

which is precisely the widely used phenomenological resistively-shunted-junction (RSJ) equation. We shall consider a model in which $J(\omega)=\omega/R$ up to a high-frequency cutoff ω_c. The potential energy $V(\phi)$ is, in the case of a SQUID,

$$V(\phi)=\frac{(\phi-\phi_{\rm ext})^2}{2L}-\left[\frac{I_c\phi_0}{2\pi}\right]\cos\left[\frac{2\pi\phi}{\phi_0}\right] . \tag{2.4}$$

Here $\phi_{\rm ext}$ is the applied external flux (i.e., the flux through the loop due to the applied external magnetic field alone), $\phi_0=h/2e$ is the flux quantum, L is the inductance of the SQUID loop, and I_c is the critical current of the weak link. If $\beta_L=2\pi LI_c/\phi_0$ is greater than 1, $V(\phi)$ consists, in general, of more than one minimum. We consider the situation in which V has two nearly degenerate minima. This case is obtained when $\phi_{\rm ext}$ is biased close to $\frac{1}{2}\phi_0$.

Now, imagine introducing a small-amplitude time-dependent magnetic field through the superconducting loop which produces a time-dependent external flux $\phi_{\rm ext}\to\phi_{\rm ext}+\delta\phi_{\rm ext}\cos\omega t$. Thus

$$H\to H-\frac{\delta\phi_{\rm ext}}{L}(\phi-\phi_{\rm ext})\cos(\omega t)+O((\delta\phi_{\rm ext})^2) . \tag{2.5}$$

We expect that when treated quantum mechanically, the Hamiltonian (2.5) will cause transition between flux states of the double well. It is the purpose of this paper to calculate such transition rates. In order to facilitate this calculation we shall map this Hamiltonian to an effective two-state system coupled to an infinite number of degrees of freedom of the harmonic oscillators. This mapping is discussed extensively in the next section. In the remaining part of the present section we shall formulate the path integral version of the partition function corresponding to the Hamiltonian in Eq. (2.1) in preparation for our discussion in Sec. III.

The partition function Z can be expressed as

$$Z=\int d\phi\, G(\phi,\phi) , \tag{2.6}$$

where

$$G(\phi,\phi)=\int \mathscr{D}\phi \prod_\alpha \mathscr{D}x_\alpha \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\tfrac{1}{2}C\dot{\phi}^2+V(\phi)+\sum_\alpha \tfrac{1}{2}m_\alpha(\dot{x}_\alpha^2+\omega_\alpha^2 x_\alpha^2)+\phi\sum_\alpha f_\alpha x_\alpha+\phi^2\sum_\alpha \frac{f_\alpha^2}{2m_\alpha\omega_\alpha^2}\right]\right]. \tag{2.7}$$

The boundary conditions which follow from the definition of partition function as a trace over the degrees of freedom are $\phi(0)=\phi(\beta\hbar)$ and $x_\alpha(\beta\hbar)=x_\alpha(0)$. The environment degrees of freedom are not of primary interest, but their effect in perturbing the system described by ϕ is. The environment degrees of freedom will therefore be integrated out leaving us with a clear-cut problem involving the system variable only. This operation leads to

$$Z=Z_0 Z[\phi]\,, \tag{2.8}$$

where Z_0 is the partition function of the environment alone and is independent of ϕ. The expectation value of any operator which depends on the system variable alone can equally well be calculated with the help of $Z[\phi]$. More accurately such an expectation value is to be interpreted as the expectation value of the operator acting on the system and simultaneously the unit operator acting on the environment. The integration of the environment degrees of freedom, although completely straightforward in the context of Feynman path integral, is extremely difficult to carry out in the context of Schrödinger equation. This is why a path-integral formulation is used in the present context.[9]

To integrate out the environment degrees of freedom we need to calculate the following path integral:

$$Q=\int \prod_\alpha \mathscr{D}x_\alpha \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\sum_\alpha \tfrac{1}{2}m_\alpha(\dot{x}_\alpha^2+\omega_\alpha^2 x_\alpha^2)+\phi\sum_\alpha f_\alpha x_\alpha\right]\right]. \tag{2.9}$$

The calculation is a standard one and we briefly outline the procedure. Exploiting the boundary conditions on $\phi(\tau)$ and $x_\alpha(\tau)$, one can develop them in terms of Fourier series:

$$\phi(\tau)=\sum_{n=-\infty}^{n=+\infty}\phi_n e^{i\omega_n\tau}\,, \tag{2.10}$$

and

$$x_\alpha(\tau)=\sum_{n=-\infty}^{n=+\infty}x_{\alpha n}e^{i\omega_n\tau}\,, \tag{2.11}$$

where $\omega_n=2\pi n/\beta\hbar$, n being an integer. This allows us to write

$$\int_0^{\beta\hbar} d\tau\left[\sum_\alpha \tfrac{1}{2}m_\alpha(\dot{x}_\alpha^2+\omega_\alpha^2x_\alpha^2)+\phi\sum_\alpha f_\alpha x_\alpha\right]=\frac{\beta\hbar}{2}\sum_{\alpha,n}[m_\alpha(\omega_n^2+\omega_\alpha^2)\,|\,x_{\alpha n}\,|^2+2f_\alpha\phi_n x_{\alpha-n}]$$
$$=\frac{\beta\hbar}{2}\sum_{\alpha,n}m_\alpha(\omega_\alpha^2+\omega_n^2)\,|\,x'_{\alpha n}\,|^2-\beta\hbar\sum_n|\,\phi_n\,|^2\left[\sum_\alpha\frac{f_\alpha^2}{2m_\alpha(\omega_\alpha^2+\omega_n^2)}\right], \tag{2.12}$$

where the new variable $x'_{\alpha n}$ is given by

$$x'_{\alpha n}=x_{\alpha n}+\frac{f_\alpha\phi_n}{m_\alpha(\omega_\alpha^2+\omega_n^2)}\,. \tag{2.13}$$

The Gaussian path integral Q is now easily evaluated. We obtain for $G(\phi,\phi)$

$$G(\phi,\phi)=\int_{\phi(0)=\phi}^{\phi(\beta\hbar)=\phi}\mathscr{D}\phi(\tau)\exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar}d\tau[\tfrac{1}{2}C\dot{\phi}^2+V(\phi)]-\beta\sum_n|\,\phi_n\,|^2K(\omega_n)\right], \tag{2.14}$$

where

$$K(\omega_n)=\omega_n^2\sum_\alpha\frac{f_\alpha^2}{2m_\alpha\omega_\alpha^2}\frac{1}{\omega_\alpha^2+\omega_n^2}$$
$$=\frac{\omega_n^2}{\pi}\int_0^\infty d\omega\frac{1}{\omega}\frac{J(\omega)}{\omega^2+\omega_n^2}\,. \tag{2.15}$$

If we assume that $J(\omega)$ is given by

$$J(\omega)=\frac{\omega}{R}e^{-\omega/\omega_c}\,, \tag{2.16}$$

we obtain

$$K(\omega_n)=\frac{1}{2R}\frac{|\,\omega_n\,|}{1+|\,\omega_n\,|/\omega_c}\,. \tag{2.17}$$

However, we must emphasize that as far as the high-frequency part is concerned there is nothing sacrosanct about the form of $J(\omega)$ given in Eq. (2.16). The low-frequency part is, however, essential in order to obtain the correct RSJ behavior of the classical equation of motion as discussed earlier. As far as the high-frequency part is concerned, it would make no difference to choose other

forms of $J(\omega)$, such as $(\omega/R)\Theta(\omega_c-\omega)$ (where Θ is the usual unit step function). Now

$$K(\tau)=\sum_n K(\omega_n)e^{i\omega_n\tau} , \tag{2.18}$$

can easily be calculated, and in the limit $\omega_c\to\infty$, it is the following:

$$K(\tau)=-\frac{\pi}{2R\beta\hbar}\frac{1}{\sin^2(\pi\tau/\beta\hbar)} . \tag{2.19}$$

Thus for $G(\phi,\phi)$ we obtain

$$G(\phi,\phi)=\int\mathscr{D}\phi\exp\left[-\frac{1}{\hbar}S_{\rm eff}[\phi]\right] , \tag{2.20}$$

where

$$S_{\rm eff}[\phi]=\int_0^{\beta\hbar}d\tau[\tfrac{1}{2}C\dot{\phi}^2+V(\phi)]+\frac{1}{4\pi R}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\frac{[\phi(\tau)-\phi(\tau')]^2}{\{(\beta\hbar/\pi)\sin[\pi(\tau-\tau')/\beta\hbar]\}^2} . \tag{2.21}$$

III. MAPPING OF THE CONTINUOUS DOUBLE-WELL PROBLEM TO AN EFFECTIVE TWO-STATE SYSTEM

In this section we discuss how one can map a continuous double-well problem to an effective two-state system.[10] This mapping facilitates our subsequent calculation of the photoinduced transition rate. It is clear at the very outset that the mapping cannot be an exact one; it is therefore important to stress what we really mean by a mapping, and what are its limits of applicability. We shall show that provided certain conditions are satisfied, a one-to-one correspondence between the partition functions of the continuous double-well problem and an effective two-state system can be set up. The limits of the applicability of such a procedure shall emerge naturally as we continue our discussion.

Consider the path-integral representation of the partition function of the continuous double-well problem discussed in the preceding section, i.e.,

$$Z=\int d\phi\, G(\phi,\phi) , \tag{3.1}$$

where

$$G(\phi,\phi)=\int_{\phi(0)=\phi}^{\phi(\beta\hbar)=\phi}\mathscr{D}\phi(\tau)\exp\{-S_{\rm eff}[\phi(\tau)]/\hbar\} , \tag{3.2}$$

and

$$S_{\rm eff}[\phi(\tau)]=\int_0^{\beta\hbar}d\tau[\tfrac{1}{2}C\dot{\phi}^2+V(\phi)]+\frac{1}{4\pi R}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau' K(\tau-\tau')\times[\phi(\tau)-\phi(\tau')]^2 . \tag{3.3}$$

We shall now evaluate Z by summing over important paths in the low-temperature limit, $\beta\hbar\to\infty$; the relevant ideas are borrowed from the work of Feynman.[11] In the subsequent discussion we shall refer to τ as the time; in reality it is not. Thus the paths do not represent the real motion of the particle; it is simply a formal description of the expression for the partition function. As stressed by Feynman, the true motion of the particle may have some analogy to the motion as a function of τ, but such an analogy need not be drawn. The reason to continue to call τ and $\beta\hbar$ "time" is to help to make the arguments as picturesque as possible so that intuition is most effective. At time zero the initial coordinate of the particle is ϕ; as time proceeds it moves about in such a way that at time $\beta\hbar$ it has the same position ϕ. Each type of motion is weighted by the negative exponential $S_{\rm eff}/\hbar$; the sum is taken over all such motions. Finally an integral is taken over all possible initial positions ϕ. It is therefore clear that paths which make $S_{\rm eff}/\hbar$ large will contribute very little to the sum. For the same reason, the initial positions which are not close to the minima $\pm\phi_m$ of the double well will also contribute very little to the sum. The approximation is to keep carefully those contributions which dominate the total sum. In the process we shall see that there exists a set (infinite) of paths for which $S_{\rm eff}/\hbar$ is close (slightly larger) to the path which gives the lowest value for $S_{\rm eff}/\hbar$; however, as $\beta\hbar\to\infty$ the multiplicity of these near minimum paths make their contribution dominant.

Let us now enumerate the important paths. It is clear that the contribution of a path which is independent of time to $S_{\rm eff}/\hbar$ is simply $-\beta V(\phi)$. In the high-temperature limit, the static paths would dominate, since in the limit $\beta\to 0$ any path which depends on τ will increase $S_{\rm eff}/\hbar$ by a large amount due to both the kinetic energy term and the nonlocal term. For static paths there is no path integral to be done and only the ordinary integral over ϕ remain. This is essentially the classical partition function except for a normalization constant which, however, is dependent on β. As we lower the temperature, the nonstatic paths begin to make significant contribution to the partition function, since at low temperatures their contribution to $S_{\rm eff}/\hbar$ may not be significantly larger (see below) than the static paths which sit at the bottom of the wells; furthermore, the "volume in path space" over which we sum is increased due to their inclusion. Particularly unruly paths are still ruled out because the volume gained in the path space can be offset by their large value of $S_{\rm eff}/\hbar$. So, what kind of nonstatic paths are the most important ones in the limit $\beta\hbar\to\infty$? Consider the paths which stay in one minimum or the other for a long period of time but hop to the other in a short time τ_0. This infrequent hopping can take place back and forth between the two wells. Furthermore, there is no reason for the hops to be equally spaced in time, and so by placing them at all separations (even for a fixed number of hops) we gain enormously large volume in the path space. Since most of the time they sit at the absolute minima, their potential energy is almost the same as the potential energy of the static paths which sit forever at the absolute minima of $V(\phi)$. The hops, however, make a positive contribution to both the kinetic and the nonlocal terms. Of course, to

get quantitative results, we must generalize the class of hopping or tunneling paths by adding quadratic fluctuations about these simple paths. We now turn to a more quantitative formulation of the ideas discussed in this paragraph.

A. Case of zero dissipation: an approximate method

Consider a simple quantum-mechanical problem of a particle in an asymmetric double well [Fig. 1(a)]. The minima are at $\phi=\pm\phi_m$ and the value of the potential at the minima are, respectively, $V(\pm\phi_m)=\pm\epsilon$. We want to examine the low-lying eigenstates using a simple approximate method which we shall find very useful for the discussion of the dissipative case. In the limit $\beta\hbar\to\infty$ it is sufficient to approximate Z [Eq. (3.1)] by $G(+\phi_m,+\phi_m)+G(-\phi_m,-\phi_m)$ for reasons mentioned above. Let us first calculate $G(+\phi_m,+\phi_m)$ which is given by

$$G(+\phi_m,+\phi_m)=\int_{\phi(0)=+\phi_m}^{\phi(\beta\hbar)=+\phi_m}\exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar}[\tfrac{1}{2}C\dot{\phi}^2+V(\phi)]d\tau\right]. \tag{3.4}$$

Let us now assume that G is entirely dominated by the tunneling paths [Fig. 1(b)] in the limit $\beta\hbar\to\infty$. We will approximate these paths by straight line flips of duration τ_0 as shown in the figure. This is not the optimal choice of path; it results in an underestimate of Z. However, the errors produced by adopting this extremely simple ansatz will prove to be rather small and qualitatively unimportant. For a path consisting of $2n$ flips [$n=2$ in Fig. 1(b)], it is easy to show that

$$\tfrac{1}{2}C\int_0^{\beta\hbar}\dot{\phi}^2\,d\tau=C(2\phi_m^2/\tau_0)(2n)\,, \tag{3.5}$$

and that

$$\int_0^{\beta\hbar}d\tau\,V(\phi)=\epsilon\sum_{i=0}^{2n}(-1)^i(t_{i+1}-t_i)+2n\frac{\tau_0}{2\phi_m}\int_{-\phi_m}^{\phi_m}V(\phi)d\phi\,. \tag{3.6}$$

The exponent of the integrand in the functional integral, Eq. (3.4), is therefore minimized by

$$\frac{\phi_m}{\tau_0}=\frac{1}{2(C\phi_m)^{1/2}}\left[\int_{-\phi_m}^{\phi_m}V(\phi)d\phi\right]^{1/2}. \tag{3.7}$$

Given the value of τ_0 from Eq. (3.7), we now have for a path consisting of $2n$ flips

$$-\frac{1}{\hbar}\int_0^{\beta\hbar}[\tfrac{1}{2}C\dot{\phi}^2+V(\phi)]d\tau=-\frac{\epsilon}{2\hbar}\sum_{i=0}^{2n}(-1)^i(t_{i+1}-t_i)-\frac{2n}{\hbar}\frac{\tau_0}{\phi_m}\int_{-\phi_m}^{\phi_m}V(\phi)d\phi\,. \tag{3.8}$$

By defining a quantity $\bar{V}$ given by

$$\bar{V}=\frac{1}{2\phi_m}\int_{-\phi_m}^{\phi_m}V(\phi)d\phi\,, \tag{3.9}$$

we can write

$$-\frac{1}{\hbar}\int_0^{\beta\hbar}[\tfrac{1}{2}C\dot{\phi}^2+V(\phi)]=-\frac{\epsilon}{\hbar}\sum_{i=0}^{2n}(-1)^i(t_{i+1}-t_i)-\frac{2n}{\hbar}(2\phi_m)(2m\bar{V})^{1/2}\,. \tag{3.10}$$

Now summing over all possible $2n$ number of flips, and integrating over their positions over available phase space we obtain

$$G(+\phi_m,+\phi_m)=\sum_{n=0}^{\infty}\left[y^{2n}\int_0^{\beta\hbar}dt_{2n}\frac{1}{\tau_0}\int_0^{t_{2n}}dt_{2n-1}\frac{1}{\tau_0}\cdots\int_0^{t_2}dt_1\frac{1}{\tau_0}\exp\left[-\frac{\epsilon}{\hbar}\sum_{i=0}^{2n}(-1)^i(t_{i+1}-t_i)\right]\right], \tag{3.11}$$

where

$$y=e^{-(2\phi_m/\hbar)\sqrt{2C\bar{V}}} \tag{3.12}$$

is easily identified to be an approximation to the WKB barrier penetration factor. The factors of τ_0 in the integrand of Eq. (3.11) are introduced simply for dimensional reasons, since the partition function Z has to be dimensionless, and since there are no other scales in the problem (given our restriction to the tunneling paths [Fig. 1(b)]) having the dimension of time save τ_0. Of course Gaussian fluctuation around the tunneling paths can change this scale factor by a multiplicative constant of the order unity (see below). The error implicit in ignoring this can be viewed as a small (and quite unimportant) multiplicative error in the computed value of y. It is equally easy to compute $G(-\phi_m,-\phi_m)$, and hence the approximation to the tunneling paths leads to the result ($\beta\hbar\to\infty$)

$$\begin{aligned}Z&\approx G(+\phi_m,+\phi_m)+G(-\phi_m,-\phi_m)\\&\approx 2\sum_{n=0}^{\infty}\left[y^{2n}\int_0^{\beta\hbar}dt_{2n}\frac{1}{\tau_0}\int_0^{t_{2n}}dt_{2n-1}\frac{1}{\tau_0}\cdots\int_0^{t_2}dt_1\frac{1}{\tau_0}\cosh\left[\frac{\epsilon}{\hbar}\sum_{i=0}^{2n}(-1)^i(t_{i+1}-t_i)\right]\right]\\&=2\cosh[\beta(\epsilon^2+\Delta_0^2)^{1/2}]\,,\end{aligned} \tag{3.13}$$

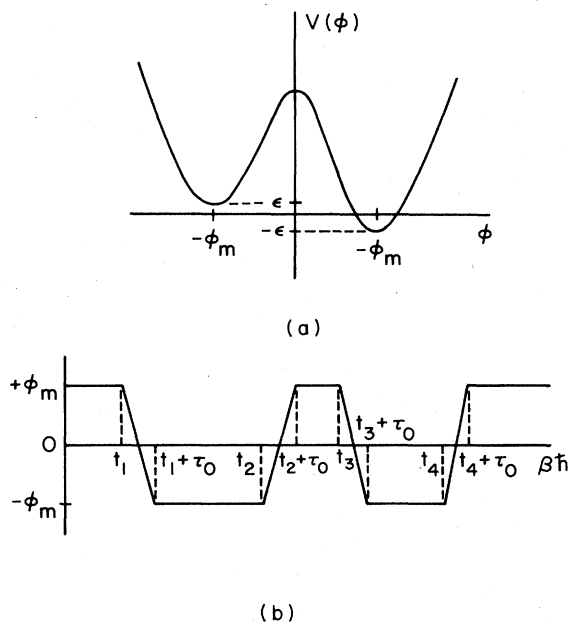

FIG. 1. (a) Asymmetric double-well potential; (b) hopping paths.

where

$$\Delta_0=(\hbar/\tau_0)y \ . \tag{3.14}$$

Consider, on the other hand, a pseudospin Hamiltonian given by

$$H_s=-\epsilon'\sigma_z-\Delta_0'\sigma_x \ , \tag{3.15}$$

where σ's are the usual Pauli matrices. The partition function of this spin Hamiltonian is easily calculated to be

$$Z_s(\beta)=2\cosh\{\beta[(\epsilon')^2+(\Delta_0')^2]^{1/2}\} \ . \tag{3.16}$$

Identifying Eqs. (3.16) and (3.13) we arrive at the conclusion that in the limit $\beta\hbar\to\infty$, the partition function and hence the low-lying levels of the continuous asymmetric double-well problem are identical to those of the pseudospin Hamiltonian [Eq. (3.15)] provided we set $\epsilon'=\epsilon$, and $\Delta_0'=\Delta_0$.

Let us now examine how good an approximation it is to consider the tunneling paths made out of straight line segments [Fig. 1(b)] with only one variational parameter τ_0 as we have considered above. For a quantitative estimate let us examine the symmetric (i.e., $\epsilon=0$) double-well potential:

$$V(\phi)=V_0(\phi^2-\phi_m^2)^2 \ . \tag{3.17}$$

As discussed in the Appendix, this potential is an accurate representation of the physically relevant problem, i.e., a SQUID for which $\phi_{\rm ext}=\phi_0/2$. The tunnel splitting $\delta\Delta$ is given by[12]

$$\delta\Delta=2\Delta_0=4\sqrt{3}\hbar\omega_0(S_0/2\pi\hbar)^{1/2}e^{-S_0/\hbar} \ , \tag{3.18}$$

where

$$S_0/\hbar=\tfrac{16}{3}(\Delta V/\hbar\omega_0)\approx 5.33(\Delta V/\hbar\omega_0) \ . \tag{3.19}$$

The barrier height ΔV is given by $V_0\phi_m^4$ and the small oscillation frequency ω_0 at the bottom of the wells is given by $\omega_0=2\phi_m(2V_0/C)^{1/2}$. From Eqs. (3.7), (3.12), and (3.14) we obtain

$$\begin{aligned}\delta\Delta=2\Delta_0&=\frac{2\hbar}{\tau_0}y\\ &=2\hbar\omega_0(\tfrac{2}{15})^{1/2}\exp[-16(\tfrac{2}{15})^{1/2}(\Delta V/\hbar\omega_0)]\\ &\approx 2(\hbar\omega_0/\pi)\exp[-5.84(\Delta V/\hbar\omega_0)] \ .\end{aligned} \tag{3.20}$$

The error in the exponent is 9.6% (i.e., 5.84 instead of 5.33). The prefactor is close to the naive WBK result ($1/\pi\sim 0.32$, whereas $\sqrt{2/15}\sim 0.36$) and not the correct prefactor given in Eq. (3.18). Note that the approximate result, Eq. (3.20), underestimates the tunnel splitting; both the prefactor and the exponent are smaller.

Within this scheme of dominant paths two improvements can be made.[13] (a) Instead of considering straight-line paths which interpolate between the two minima as in Fig. 1(b), one can, in principle, introduce paths which are smoother, i.e., paths which do not have discontinuity in their derivatives. (b) It is also possible to take into account quadratic fluctuations at the bottom of the minima which would improve the prefactor in Eq. (3.20). It is not our intention here to carry out such improvements, but to simply note that the whole point of such a simple technique of dominant paths made out of a particularly simple class of straight-line paths is threefold: (1) It is a semiquantitative method which can in principle be improved; (2) it leads to the correct structure of the low-lying energy spectrum; and (3) the method is so simple that it can be readily applied to more complicated problems involving dissipation as discussed in the next subsection. It is precisely this simplicity that allows us to obtain the correct structure of the problem of mapping even in a case which is plagued with infrared divergences.

B. Finite dissipation

We proceed along the same lines as discussed in Sec. III A, i.e., we evaluate $G(+\phi_m,+\phi_m)$ of Eq. (3.2) by making use of the tunneling paths shown in Fig. 1(b). Part of $S_{\rm eff}/\hbar$ has already been evaluated in Sec. III A; we simply need to evaluate the last term in Eq. (3.3) (i.e., the nonlocal part of the action). As before we shall be interested only in the limit $\beta\hbar\to\infty$. We shall first concentrate on the problem of ohmic dissipation; subsequently we shall discuss the nonohmic case. The nonlocal term in Eq. (3.3) is therefore given by

$$\frac{\eta}{4\pi}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\left[\frac{\phi(\tau)-\phi(\tau')}{\tau-\tau'}\right]^2 \ . \tag{3.21}$$

In the case of a SQUID, $\eta=1/R$, where R is the resistance of the weak link, and $\phi\sim\phi_0$, where $\phi_0=h/2e$ is the flux quantum. Thus the strength of the nonlocal term, if we denote it by g, is given by

$$g=\frac{1}{\hbar}\frac{1}{4\pi R}\left[\frac{h}{2e}\right]^2=\frac{1}{8}\frac{(h/e^2)}{R}=\frac{3227\ \Omega}{R} \ . \tag{3.22}$$

At first sight, it would seem that a perturbative theory in terms of g would be meaningful if the resistance R is much larger than the fundamental unit of resistance h/e^2. This, however, is not necessarily true. The nature of the perturbation theory depends strongly on the nature of $V(\phi)$. This is not surprising since $V(\phi)$ dictates both the local and the global behavior of the important paths contributing to the partition function, and since the evaluation of the path integral depends both on the local as well as the global behavior of the paths. For example, if $V(\phi)$ has a simple quadratic minimum such as $V(\phi)=\frac{1}{2}C\omega_0^2\phi^2$, the effect of the nonlocal term is somewhat innocuous.[14] However, if $V(\phi)$ consists of a double well (it is irrelevant whether it is symmetric or not) the perturbation theory in g is doomed to fail due to infrared divergences as was first pointed out by one of us and independently by Bray and Moore,[15] and subsequently rederived or rediscussed by a number of authors[16] using a variety of different techniques. Given the simplified form, Eq. (3.21), in the limit $\beta\hbar\to\infty$, it is clear that this term is scale invariant; we have lost all information of the microscopic scale ω_c which served as a cutoff for $J(\omega)$. This should not be a cause for great concern since, as we shall see, $S_{\rm eff}/\hbar$ has a natural frequency scale $1/\tau_0$ which is considerably smaller than ω_c. Even if we explicitly kept track of ω_c, we would find that for the low-energy, low-temperature behavior (i.e., for temperatures lower than $\hbar/k_B$ times the natural frequency) the smaller of the two frequency scales is dominant. On the other hand, if $S_{\rm eff}/\hbar$ were such that it did not have any natural, intrinsic frequency scale, it would make no sense to rid ourselves of the ω_c of the environment.

With the help of two integration by parts we can reexpress Eq. (3.21) as

$$\frac{\eta}{4\pi}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\left[\frac{\phi(\tau)-\phi(\tau')}{\tau-\tau'}\right]^2 = -\frac{\eta}{2\pi}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\,\dot\phi(\tau)\dot\phi(\tau')\ln|\tau-\tau'| = T_1+T_2\ , \tag{3.23}$$

where

$$T_1=-\frac{\eta}{2\pi}\sum_{\substack{i\neq j\\ i,j}}\int_{\tau_i}^{\tau_i+\tau_0}d\tau\int_{\tau_j}^{\tau_j+\tau_0}d\tau'\,\dot\phi(\tau)\dot\phi(\tau')\ln|\tau-\tau'|\ , \tag{3.24}$$

and

$$T_2=-\frac{\eta}{2\pi}\sum_i\int_{\tau_i}^{\tau_i+\tau_0}d\tau\int_{\tau_i}^{\tau_i+\tau_0}d\tau'\,\dot\phi(\tau)\dot\phi(\tau')\ln|\tau-\tau'|\ . \tag{3.25}$$

Assuming that the hops in Fig. 1(b) are rare and consequently widely separated, we can evaluate T_1:

$$T_1\approx-\frac{\eta}{2\pi}\sum_{\substack{i\neq j\\ i,j}}(-1)^{i+j}\ln|\tau_i-\tau_j|\left[\int_{-\phi_m}^{\phi_m}d\phi\right]^2 = -\frac{\eta(2\phi_m)^2}{2\pi}\sum_{\substack{i\neq j\\ i,j}}(-1)^{i+j}\ln|\tau_i-\tau_j|\ . \tag{3.26}$$

From Eq. (3.26) it is clear that T_1 is *very insensitive* to the shape of the hops in Fig. 1(b). The evaluation of T_2, however, depends to some extent on the shape of the paths. We adhere to the straight-line path as shown in Fig. 1(b). It is not difficult to show that

$$T_2=-\frac{\eta(2\phi_m)^2}{2\pi}(2n)(\ln\tau_0-\tfrac{3}{2})\ , \tag{3.27}$$

where $2n$ is the total number of flips. Thus

$$T_1+T_2=-\frac{\eta(2\phi_m)^2}{2\pi}\sum_{\substack{i\neq j\\ i,j}}(-1)^{i+j}\ln\left|\frac{\tau_i-\tau_j}{\tau_0}\right| +3n\frac{\eta}{2\pi}(2\phi_m)^2\ . \tag{3.28}$$

We can now follow the method described in Sec. III A to obtain

$$G(+\phi_m,+\phi_m)=\sum_{n=0}^{\infty}y^{2n}\int_0^{\beta\hbar}d\tau_{2n}\frac{1}{\tau_0}\int_0^{\tau_{2n}-\tau_0}d\tau_{2n-1}\frac{1}{\tau_0}\cdots \times\int_0^{\tau_2-\tau_0}d\tau_1\frac{1}{\tau_0}\exp\left[\alpha\sum_{\substack{i\neq j\\ i,j}}(-1)^{i+j}\ln\left|\frac{\tau_i-\tau_j}{\tau_0}\right| -\frac{\epsilon}{\hbar}\sum_i(-1)^i(\tau_{i+1}-\tau_i)\right], \tag{3.29}$$

where

$$\alpha=\frac{(2\phi_m)^2}{2\pi\hbar R}\ , \tag{3.30}$$

$$y=e^{-\alpha/2}e^{-2\tau_0\bar V/\hbar}\ , \tag{3.31}$$

$$\frac{2\pi}{\tau_0}=\left[\left[\frac{\eta}{C}\right]^2+\frac{2\pi^2\bar V}{C\phi_m^2}\right]^{1/2}-\frac{\eta}{C}\ , \tag{3.32}$$

and, as before,

$$\bar V=\frac{1}{2\phi_m}\int_{-\phi_m}^{\phi_m}V(\phi)d\phi\ . \tag{3.33}$$

It is worth pointing out that the cutoff prescription chosen in Eq. (3.29) (two successive hops are not allowed to be within τ_0) is not unique and is simply a matter of convenience. We strongly believe that any other cutoff prescription would not change the essential structure of $G(+\phi_m,+\phi_m)$. For example, it is possible to use a cutoff prescription such that the factor $\ln|(\tau_i-\tau_j)/\tau_0|$ is replaced by $\ln(1+|\tau_i-\tau_j|/\tau_0)$ and the integral

$$\int_0^{\tau_{2i-1}-\tau_0} d\tau_{2i}\frac{1}{\tau_0}$$

by

$$\int_0^{\tau_{2i-1}} d\tau_{2i}\frac{1}{\tau_0} .$$

As before, let us now consider a two-state system coupled to an environment; the Hamiltonian is given by

$$H=-\epsilon\sigma_z-\Delta\sigma_x+\sum_n \hbar\omega_n(b_n^\dagger b_n+\tfrac{1}{2}) + \phi_m\sigma_z\sum_n f_n\left[\frac{\hbar}{2m_n\omega_n}\right]^{1/2}(b_n^\dagger+b_n) . \tag{3.34}$$

Here b_n's ($b_n^\dagger$'s) are boson annihilation (creation) operators and the spectral density of the environment is given by

$$J(\omega)=\frac{\pi}{2}\sum_n\frac{f_n^2}{m_n\omega_n}\delta(\omega-\omega_n) . \tag{3.35}$$

It is now a textbook exercise[17] to show that for this two-state system, coupled to an environment of harmonic oscillators,

$$G(\uparrow,\uparrow)=\sum_{n=0}^{\infty}(\Delta/\hbar)^{2n}\int_0^{\beta\hbar}d\tau_{2n}\int_0^{\tau_{2n-1}}d\tau_{2n-1}\cdots \times\int_0^{\tau_2}d\tau_1\exp\left[\alpha\sum_{\substack{i\neq j\\ i,j}}(-1)^{i+j}\ln(1+\omega_c|\tau_i-\tau_j|)-\frac{\epsilon}{\hbar}\sum_i(-1)^i(\tau_{i+1}-\tau_i)\right] . \tag{3.36}$$

In deriving Eq. (3.38), we have carried out the trace over the harmonic oscillator degrees of freedom, and have deliberately used a spectral density $J(\omega)$ given by

$$J(\omega)=\eta\omega e^{-\omega/\omega_c} \tag{3.37}$$

to emphasize the ambiguity in the cutoff prescription discussed earlier. Any other $J(\omega)$ with a different high-frequency cutoff prescription such as $J(\omega)=\eta\omega\Theta(\omega_c-\omega)$ should be physically equivalent. The phenomena that we are concerned with depend on the behavior of $J(\omega)$ at low frequencies and not at frequencies comparable to the cutoff frequency. Equations (3.29) and (3.36) are now term by term identical (barring the different but irrelevant choices of the cutoff prescription) provided we identify

$$\Delta=\frac{\hbar}{\tau_0}y=\frac{\hbar}{\tau_0}e^{-\alpha/2}e^{-2\tau_0\bar{V}/\hbar} \tag{3.38}$$

and

$$\omega_c=\frac{1}{\tau_0} .$$

α and ϵ have the same meaning as before. This completes our mapping of the continuous double-well problem to an effective two-state system. All the difficulties due to infrared divergences are still buried in Eq. (3.36). It is worth emphasizing that the specific problem of mapping does not *require* one to solve the problem.

Finally, we note that the same procedure can be carried out with other types of heat baths. The mapping must be done carefully in each case. In each case, there will be a tunneling time, τ_0, which is itself renormalized by the coupling to the heat bath. All heat-bath modes with frequency greater than $1/\tau_0$ serve only to renormalize the tunnel splitting Δ; they do not appear in the two-state Hamiltonian. The result is a two-state system with a renormalized tunneling matrix element Δ, coupled as before to the heat bath which, however, has a new upper cutoff frequency $\omega_c=1/\tau_0$. Finally, it goes without saying that the mapping to the two-state problem is valid only if the conditions $\hbar\omega_0\gg\Delta,\epsilon$ are met.

IV. PHOTOINDUCED TRANSITIONS IN A TWO-STATE SYSTEM

In this section we calculate the photoinduced transition rate of a two-level system which is coupled to a heat bath consisting of noninteracting bosons with spectral weight $J(\omega)$. The Hamiltonian of the two-level system plus the heat bath is given in Eq. (3.34) and, as is clear from Eq. (2.5), the coupling to the electromagnetic field is

$$H_{\rm ext}=\frac{\delta\phi_{\rm ext}}{L}\phi_m\sigma_z\cos(\omega t) . \tag{4.1}$$

Problems of this sort are quite standard. We therefore simply quote familiar results. We will not bother to treat the problem in full generality, although this can in principle be done. We will consider the case in which the external ac field is sufficiently small that we can calculate a transition rate using second-order perturbation theory in $\delta\phi_{\rm ext}$, which is simply Fermi's golden rule. Equivalently, we will calculate the linear response of the system to an ac field. Since Δ is typically small compared to all other energies, we will also specialize to the case in which $\Delta\ll 2\epsilon$. Thus, we can expand the wave functions in powers of $\Delta/2\epsilon$. To zeroth order, the system is entirely localized in

one well or the other; there is no tunneling. The zeroth-order Hamiltonian can be exactly diagonalized. Depending on which well the two-state system occupies, the heat bath bosons are shifted, but their frequencies remain the same. Transitions between wells occur when the first-order corrections to the wave functions are included. We shall see that these make a contribution to the transition rate which is second order in Δ. Shifts in the energy of the system occur first in second order in Δ, and hence can readily be seen to make contributions to the transition rate that are fourth order and higher.

We can consider two photoinduced processes: either we can prepare the system in the lower well and compute the transition rate $\nu_\uparrow$ to the upper well, or the reverse process in which the system starts in the upper well and makes a transition to the lower well $\nu_\downarrow$. In either case we must thermally average over initial states of the heat bath (subject to the constraint that the system is in the specified well) and sum over final states of the heat bath. The resulting expression for the up transition rate as a function of the frequency ω of the external ac field is

$$\nu_\uparrow(\omega)=\frac{2\pi}{\hbar}\left[\frac{\phi_m\,\delta\phi_{\rm ext}}{L}\right]^2\left[\frac{2\Delta}{\hbar\omega}\right]^2 g(\omega-2\epsilon/\hbar) + O((\Delta/\hbar\omega)^4)\ , \tag{4.2}$$

where

$$g(\omega)=\int_{-\infty}^{\infty}\frac{dt}{2\pi\hbar}\exp[it\omega-X(t)]\ , \tag{4.3}$$

$$X(t)=\frac{(2\phi_m)^2}{h}\int_0^\infty d\omega\,\frac{J(\omega)}{\omega^2}\{[1+N(\omega)](1-e^{-i\omega t}) + N(\omega)(1-e^{i\omega t})\}\ , \tag{4.4}$$

and $N(\omega)$ is the boson occupation factor

$$N(\omega)=(e^{\beta\hbar\omega}-1)^{-1}\ . \tag{4.5}$$

Notice that conservation of energy implies that the initial and final states of the entire system must differ in energy by precisely $\hbar\omega$. At zero temperature, when the system cannot absorb any energy from the heat bath, $\nu_\uparrow(\omega)$ must be equal to zero for $\hbar\omega<2\epsilon$. The corresponding expression for the *induced* down transition rate is

$$\nu_\downarrow(\omega)=\frac{2\pi}{\hbar}\left[\frac{\phi_m\,\delta\phi_{\rm ext}}{L}\right]^2\left[\frac{2\Delta}{\hbar\omega}\right]^2 g(2\epsilon/\hbar-\omega) + O((\Delta/\hbar\omega)^4)\ . \tag{4.6}$$

There are two useful properties of $g(\omega)$ that are worth mentioning here. The first is that $g(\omega)$ satisfies a sum rule

$$\hbar\int_{-\infty}^{\infty}d\omega\,g(\omega)=1\ , \tag{4.7}$$

which is a consequence of the conservation of probability. The second is that

$$g(\omega)=e^{\beta\hbar\omega}g(-\omega)\ , \tag{4.8}$$

which is a consequence of microreversibility.

So far the results we have obtained are valid regardless of the form of the spectral function $J(\omega)$. If the heat bath does not have too much spectral density at small ω, $g(\omega)$ will generally consist of two parts,

$$\hbar g(\omega)=\delta(\omega)e^{-X(0)}+\hbar g_{\rm bg}(\omega)\ , \tag{4.9}$$

where the part proportional to $\delta(\omega)$ is the recoilless piece, which is analogous to the Mössbauer line, and a smooth background piece $g_{\rm bg}(\omega)$ whose shape depends on the spectral function $J(\omega)$. Oscillator strength is apportioned between the recoilless and background parts in such a way that the sum rule Eq. (4.7) is satisfied. A necessary and sufficient condition for the existence of a recoilless piece to $g(\omega)$ is that $X(0)$ be finite. For an ohmic heat bath, $X(t)$ diverges as $t\to0$, even at zero temperature. [Note that for $J(\omega)\sim\eta\omega^\nu$, $X(t)$ diverges at finite temperature for $\nu\le2$, and at zero temperature for $\nu\le1$.]

We shall now present the results for an ohmic heat bath in which $J(\omega)$ is given by Eq. (3.37). At $T=0$, $g(\omega)$ is given by the expression

$$g(\omega)=[\hbar\omega_c\Gamma(2\alpha)]^{-1}(\omega_c/\omega)^{1-2\alpha}\exp(-\omega/\omega_c)\Theta(\omega)\ , \tag{4.10}$$

where Θ is the step function, and $\Gamma(\alpha)$ is the gamma function. Thus the transition rate at the threshold diverges as a power law if $\alpha<\frac{1}{2}$ or vanishes as a power law if $\alpha>\frac{1}{2}$. If we believe that the model of the harmonic-oscillator heat bath is reasonable for the purpose of calculating the transition rate at finite temperatures (or at least for low temperatures), it is not difficult to calculate an expression for $g(\omega)$ at finite temperatures as well:

$$g(\omega-2\epsilon/\hbar)=\frac{1}{\pi\hbar}{\rm Re}\left[\int_0^\infty dt\,e^{it(\omega-2\epsilon/\hbar)}\times\left[\frac{\pi t/\beta\hbar}{(1+i\omega_c t)\sinh(\pi t/\beta\hbar)}\right]^{2\alpha}\right].$$

To obtain the threshold behavior at finite temperatures, the integrand can be approximated by its large-t limit and one obtains

$$g(2\epsilon/\hbar)=\frac{\sqrt{\pi}}{2\hbar\omega_c}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})}\left[\frac{\hbar\omega_c}{\pi k_B T}\right]^{1-2\alpha}\ . \tag{4.11}$$

To order $(\beta\hbar\omega_c)^{-1}$ this expression is valid for all α greater than zero. For values of $\omega\sim2\epsilon$, the behavior of $g(\omega)$ is insensitive to any details of the high-frequency behavior of $J(\omega)$; it depends only on the magnitude of the cutoff frequency, ω_c. It is always symptomatic of a problem involving an infrared divergence that all scales up to a high-frequency cutoff contribute.

V. SPONTANEOUS EMISSION

Until now we have treated the flux through the SQUID as a classical external perturbation and have computed the

induced transition rates $\nu_\uparrow$ and $\nu_\downarrow$. However, we know that quantum fluctuations of the electromagnetic field produce an additional process, namely spontaneous emission; if the system starts initially in the upper well, it is possible for it to decay to the lower well by spontaneously emitting a photon. The rate of spontaneous emission could be calculated by semiclassical quantization of the electromagnetic field. Equivalently and more simply it can be computed from the induced transition rate by use of the Einstein relation, which is merely the condition of detailed balance.

We consider an ensemble of SQUID's interacting with a photon field. The intensity of light with frequencies in the range $(\omega, \omega+d\omega)$ and all polarizations we denote by $dI(\omega)$. If the wavelength $\lambda=c/\omega$ is large compared to the dimensions of the SQUID, then $(\delta\phi_{\rm ext})^2$ can be expressed in terms of $dI(\omega)$ (c is the velocity of light)

$$(\delta\phi_{\rm ext})^2=\frac{2\pi S_0^2}{c}(\tfrac{1}{3})dI(\omega)\ , \tag{5.1}$$

where S_0 is the cross-sectional area of the SQUID and the factor of $\frac{1}{3}$ comes from the average over polarizations. In thermal equilibrium, a fraction $n_d=(1+e^{-2\beta\epsilon})^{-1}$ of the SQUID's will be in the lower well, a fraction $n_u=e^{-2\beta\epsilon}n_d$ in the upper well, and there will be a black body distribution of photons such that

$$\overline{dI(\omega)}=N(\omega)\frac{\hbar\omega^3}{\pi^2c^2}d\omega\ . \tag{5.2}$$

The condition of detailed balance implies that in thermal equilibrium the rate of emission and absorption of photons with energy $\hbar\omega$ are equal and hence

$$n_u[\overline{d\nu_\downarrow(\omega)}+d\nu_s(\omega)]=n_d\overline{d\nu_\uparrow(\omega)}\ , \tag{5.3}$$

where $\overline{d\nu}$ is the thermally averaged photoinduced transition rate and $d\nu_s(\omega)$ is the rate of spontaneous emission of photons with frequencies in the range $(\omega, \omega+d\omega)$. From Eqs. (5.2), (5.3), and the symmetry relation for $g(\omega)$, Eq. (4.8), an expression for the spontaneous decay rate can be derived

$$d\nu_s(\omega)=\nu_0 g(2\epsilon/\hbar-\omega)\frac{\hbar\omega\, d\omega}{\omega_c}\ , \tag{5.4}$$

where

$$\nu_0=\frac{\omega_c}{3c\hbar}\left[\frac{4S_0\Delta}{L\hbar c}\right]^2\ . \tag{5.5}$$

The total spontaneous decay rate can be obtained by integrating Eq. (5.4) with respect to ω,

$$\nu_s=\nu_0\int_0^\infty d\omega\frac{\hbar\omega}{\omega_c}g(2\epsilon/\hbar-\omega)\ . \tag{5.6}$$

At zero temperature, only the range of frequencies from 0 to $2\epsilon/\hbar$ contributes to the integral; that is only photons with energy less than 2ϵ can be emitted. For the case of ohmic dissipation, the integral can be evaluated approximately with the result that

$$\nu_s\approx\frac{\nu_0}{\Gamma(2\alpha+2)}\left[\frac{2\epsilon}{\hbar\omega_c}\right]^{2\alpha+1}\ . \tag{5.7}$$

VI. DISCUSSION AND CONCLUSION

We have shown in this paper that the phenomenon of photon-assisted tunneling, which is familiar from many studies of microscopic systems,[18] can also occur in a macroscopic system, namely a SQUID. To do this we first showed that, so long as

$$\hbar/\tau_0\gg 2\epsilon,\Delta\ , \tag{6.1}$$

it is possible to map the original two-well system into an equivalent two-state system. Here 2Δ is the tunnel splitting, 2ϵ is the asymmetry between the two wells, and τ_0 is the tunneling time defined in Eq. (3.34) which is greater than, but typically of order ω_0^{-1}. We then solved the two-state problem, in the limit $\Delta\ll 2\epsilon$, using standard methods. A similar strategy can be employed profitably to study a host of other dynamical problems of macroscopic quantum mechanics. For instance, it has been used to study the process of quantum resonant activation in SQUID's and Josephson junctions.[19] In addition, the model of a two-state system coupled to an ohmic heat bath has been used to study problems of macroscopic quantum coherence.[20] Thus, although it was explored in detail in the text, it seems worthwhile to conclude with a few comments about the mapping.

Let us first consider the conditions on the validity of our mapping. It is intuitively clear that we can truncate the problem by including only one state localized in each well only if the splitting between levels in the well is large compared to the other energies in the problem. Thus, it is clear that $\hbar\omega_0\gg 2\epsilon$ and Δ is a necessary condition. The slightly more restrictive condition in (6.1) arises from the requirement that the typical separation (in imaginary time) between hops, which is determined by the greater of 2ϵ and Δ, be large compared to the hop width, τ_0. This is necessary since in the two-level system, the hop width is effectively zero. This is also the reason that the heat-bath modes with frequencies $\omega_\alpha>1/\tau_0$ cannot appear explicitly in the two-state model; they merely contribute to the renormalization of the value of the hop fugacity, y, and hence of Δ. In this limit we were able to show that there is a one-to-one correspondence between the terms in the partition functions of the two systems. We believe, although we have not proven, that this implies that the two systems will have the same dynamical behavior. If the condition $h/\tau_0\gg 2\epsilon$ is violated, the mapping breaks down. Even then, we do not expect the nature of the photoinduced transitions to be qualitatively different from what we have calculated. However, we cannot, at present, support this assertion.

Finally, let us consider the accuracy of our calculations. We have chosen a rather simple straight line ansatz for the tunneling path, and have made only the crudest arguments concerning the effects of Gaussian fluctuations about these paths. From comparison with a careful calculation in the dissipationless case, we estimated that these crude approximations result in a 10% error in the computed exponent of Δ. Two points should be stressed concerning this error. The first is that it would not be hard to substantially improve the accuracy of our calculation by improving our choice of tunneling path. However, the

errors made by our choice of path are purely quantitative. The second point is that it is rarely true that the value of the potential, $\overline{V}$, and the capacitance, C, are known *a priori* to better than 10% accuracy. Thus, there is at present nothing to be gained by doing a more accurate calculation. It is our feeling that Δ should be treated as a phenomenological parameter whose value can be *estimated* theoretically.

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APPENDIX

When $\phi_{\rm ext}=\frac{1}{2}\phi_0$, and $\beta_L>1$, it is convenient to use a quartic fit to the potential $V(\phi)$ given in Eq. (2.4). First it is useful to define a shifted variable $\tilde{\phi}=\phi-\frac{1}{2}\phi_0$. It is now easy to fit $V(\tilde{\phi})$ to a quadratic plus a quartic potential. The fit that we shall describe is global, i.e., we do not expand $V(\tilde{\phi})$ locally around $\tilde{\phi}=0$ and determine the coefficients of the quadratic and the quartic terms, but rather fit it in such a way that the actual locations of the minima of $V(\tilde{\phi})$ are also the minima of the fitted potential, and also the actual barrier height of $V(\tilde{\phi})$ is also the barrier height of the fitted potential. We have checked that for all cases under consideration this procedure is more accurate than the other one mentioned above. We therefore write

$$u(\tilde{\phi})=V(\tilde{\phi})-V(\tilde{\phi}_m)=V_0(\tilde{\phi}^2-\tilde{\phi}_m^2)^2 , \tag{A1}$$

where $\pm\tilde{\phi}_m$ are the actual locations of the minima of $V(\tilde{\phi})$ and V_0 is given by

$$V_0\tilde{\phi}_m^4=V(0)-V(\tilde{\phi}_m) . \tag{A2}$$

Once $\tilde{\phi}_m$ and V_0 are determined, the small oscillation frequency in the wells is given by

$$\hbar\omega_0=\hbar\omega_J(2\sqrt{2}/\theta_m)[1-(\theta_m^2/2\beta_L+\cos\theta_m)]^{1/2} , \tag{A3}$$

where $\theta_m=2\pi\tilde{\phi}_m/\phi_0$ and $\omega_J=(2\pi I_c/\phi_0 C)^{1/2}$ is the plasma frequency. For each β_L, θ_m can be determined numerically, by a simple iterative procedure, from the equation

$$\frac{\theta_m}{\beta_L}=\sin\theta_m . \tag{A4}$$

The quantities entering in Eq. (3.18) can be expressed in terms of a convenient set of units as follows. Let us define $C=c\times10^{-13}F$, $I_c=i_c\times10^{-6}A$, then

$$\hbar\omega_J=1.33\left[\frac{i_c}{c}\right]^{1/2}K , \tag{A5}$$

$$\hbar\omega_0=\frac{3.76}{\theta_m}\left[\frac{i_c}{c}\right]^{1/2}[1-(\theta_m^2/2\beta_L+\cos\theta_m)]^{1/2}K , \tag{A6}$$

and

$$\Delta V=23.86 i_c[1-(\theta_m^2/2\beta_L+\cos\theta_m)]K . \tag{A7}$$

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