PHYSICAL REVIEW B VOLUME 32, NUMBER 11 1 DECEMBER 1985

Hydrodynamics of icosahedral quasicrystals

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(Received 20 June 1985)

The equations governing long-wavelength, low-frequency excitations in icosahedral quasicrystals are derived. It is found that while the speeds of the propagating modes are isotropic, the attenuations are not, implying that purely macroscopic experiments can in principle distinguish quasicrystals from crystals, glasses, or conventional incommensurate systems. The coefficient of the anisotropy is, regrettably, quite small. The complete spectrum consists of three diffusive phasons, two pairs of transverse and one pair of longitudinal sound modes, a vacancy diffusion mode, a heat diffusion mode, and, in a material with n atomic species, $n-1$ additional particle diffusion modes. The diffusion times of the vacancy and phason modes are expected to be comparable and very long. It is shown that propagating phasons, even at short wavelength, are an unlikely prospect. The static, equilibrium elastic properties are also anisotropic, but are approached very slowly, and in many situations, the elastic response is isotropic on experimentally accessible time scales. Our results also imply that nonlinear fluctuation corrections to the linearized hydrodynamics presented here are finite as $\mathbf{q}$ and $\omega \to 0$, i.e., there is no breakdown of conventional hydrodynamics in icosahedral quasicrystals.

I. INTRODUCTION

Since the observation[1] of icosahedral symmetry in rapidly cooled Al_6Mn, quasicrystals[2] have been the subject of considerable experimental[3] and theoretical[4–9] activity. Much of this work has focused on the unique microscopic properties of the icosahedral phase. It would be pleasing, however, to find that a (monodomain) cubic centimeter, say, of quasicrystal could be distinguished from a similar block of crystalline or amorphous material without a detailed study of its internal structure. We show here that, in principle, it can be.

While some purely macroscopic differences between quasicrystals and ordinary materials, e.g., the presence of "phasons"[4–6] and some properties of dislocations[6] have been noted in earlier work, the hydrodynamic equations have not hitherto been presented. Since most of the unusual features which concern us here and some (regarding dislocation motion) to be discussed elsewhere[10] are obtained by means of these equations, we shall derive them here in some detail. In addition, if we wish to study the growth of quasicrystals from the melt and in particular the possibility of growing macroscopic domains, we must understand the manner in which the hydrodynamic fields relax. The dynamics of dislocations is also likely to play an important role in these processes.

A long-wavelength, low-frequency description of a macroscopic system requires only the slow or hydrodynamic variables[11]—those which oscillate or relax at a rate which vanishes as the wave number goes to zero. These slow variables and their equations of motion are determined by conservation laws (e.g., momentum, particle number) and spontaneously broken symmetries (e.g., translation invariance). Our discussion will concentrate on single-component icosahedral quasicrystals for which there are five conserved variables (mass, energy, and momentum) and six broken symmetry variables associated with the phases of the six mass-density waves needed to specify the icosahedral state.

Let us summarize the results of our analysis: first the dynamics, then the statics. The hydrodynamic spectrum consists of two pairs of transverse and one pair of longitudinal sound modes, a heat diffusion mode, a vacancy diffusion mode, and three diffusive "phasons." Of course, if there are n atomic species in the material, there are as many independently conserved particle numbers and hence $n-1$ additional diffusive modes. We shall not discuss these hereafter since they do not affect the structure of the propagating modes where most of the novel dynamical properties of quasicrystals arise. The speeds of the sound modes are isotropic while their attenuations are *anisotropic,* reflecting, even at asymptotically low frequencies, the presence of icosahedral order. The coefficient of the anisotropy is, unfortunately, likely to be exceedingly small for typical materials[12] (of order $D/\eta \approx 10^{-10}$, where D is a vacancy diffusion constant $\approx 10^{-10}$ cm^2/sec and $\eta \approx 1$ cm^2/sec is a typical kinematic viscosity in metals). The general features of the mode structure have been noted in Refs. 4–6. Reference 5, however, entertains the possibility of propagating phasons. Our analysis, which includes dissipation, rules out this possibility in the long-wavelength, low-frequency limit. Furthermore, as discussed in Sec. V, it appears unlikely that propagating phasons exist even at higher wave numbers in quasicrystals derived from metallic alloys. In more detail, the dispersion relations for the sound modes as a function of

wave number $\mathbf{q}\equiv q\hat{\mathbf{q}}$ ($q=|\mathbf{q}|$) are

$$\omega=c_L q-\frac{i}{2}\left[\eta_L q^2+\frac{\Gamma_w K_3^2}{\rho_0 c_L^2}(10q^{-4}I_6-9q^2)\right] \quad (1.1)$$

for longitudinal sound and

$$\omega_\alpha=c_T q-\tfrac{1}{2}i\gamma_\alpha(\hat{\mathbf{q}})q^2\ ,\quad \alpha=1,2 \quad (1.2)$$

for the two polarizations of transverse sound, where the dampings γ_1 and γ_2 separately involve icosahedral invariants of sixth, eighth, and higher order but satisfy

$$\tfrac{1}{2}[\gamma_1(\hat{\mathbf{q}})+\gamma_2(\hat{\mathbf{q}})]=\Gamma_u\rho_0 c_T^2+\eta_T+\frac{\Gamma_w K_3^2}{2\rho_0 c_T^2}(13-9q^{-6}I_6)\ . \quad (1.3)$$

Here c_L and c_T are the longitudinal and transverse sound velocities, η_L and η_T longitudinal and transverse kinematic viscosities, K_3 an elastic constant, and Γ_u and Γ_w kinetic coefficients for vacancies and phasons, respectively. I_6 is one form of the single icosahedral invariant, apart from q^6, of sixth order in q. In the coordinate system shown in Fig. 1,

$$I_6=q_x{}^6+q_y{}^6+q_z{}^6+\frac{15\tau^4}{1+\tau^6}(q_x{}^4q_y{}^2+q_y{}^4q_z{}^2+q_z{}^4q_x{}^2)+\frac{15\tau^2}{1+\tau^6}(q_x{}^2q_y{}^4+q_y{}^2q_z{}^4+q_z{}^2q_x{}^4)\ , \quad (1.4)$$

where $\tau=(1+\sqrt{5})/2$ is the golden mean. The ratio of the coefficients of the anisotropic and isotropic pieces of the damping is less than or of order D/η because $K_3\sim\rho_0 c_T^2<\rho_0 c_L^2$ and $\Gamma_w K_3\sim D$.

Vacancies and phasons are coupled—indeed they involve fundamentally the same process, so that it is not really meaningful to speak of them separately. Their motion is governed by an anisotropic 4×4 diffusivity tensor. Typical diffusion constants for them in, for example, Al_6Mn, are expected to be comparable to those for vacancies in ordinary phases of the same alloy. The heat mode decouples and will, therefore, diffuse isotropically.

The static elastic properties were for the most part discussed in Refs. 6 and 5(b), but one point is worth noting. The effective elasticity for response to external stresses, which is obtained by integrating over the phason degrees of freedom, is anisotropic: i.e., different shear moduli will in general be measured for shears applied in different directions. The anisotropy arises because the phasons, whose elastic energy is anisotropic, relax in the presence of applied stress to minimize the internal energy. Clearly, this process is relevant only if the time t over which the external stress is applied is longer than the relaxation time for the phasons. The phasons, however, are expected to relax on the same long time scale $\tau_R\sim D_v^{-1}L_{\min}^2$ as the vacancies, where D_v is a vacancy diffusion constant and $L_{\min}$ the smallest linear dimension of the sample. For typical vacancy diffusion constants,[12] $D_v\approx10^{-8}$ to 10^{-13} cm^2/sec and in a sample with $L_{\min}\approx1$ cm, τ_R is long indeed: from 3 to 300 000 years. This slow relaxation compels the experimentalist interested in measuring these equilibrium elastic properties to work at the highest possible temperatures (since D_v is thermally activated) and to work with the thinnest practical samples to make τ_R manageably small; for $L_{\min}\approx0.1$ mm, τ_R may be as small as a few hours. In any case, for $t\ll\tau_R$, the phason field will be essentially zero, and the response to the stress will be isotropic. For $t\gg\tau_R$ the anisotropic elastic energy described in the appendix applies. For intermediate times, the stress will be time dependent. What if the sample manages to acquire, in the process of preparation, a nonzero phason field? Such a distortion would also persist for times of order τ_R and would, therefore, affect the results of elastic measurements. However, the initial elastic response would still be isotropic. This may be seen by taking the elastic energy of Eqs. (2.6)—(2.11), determining the ordinary displacement field $\mathbf{u}_0$ that would arise to accommodate the phason deformation (which one takes to be static since τ_R is so large), and then determining the effective elastic energy for phonon fluctuations about $\mathbf{u}_0$. This effective elasticity will be found to be identical to that which obtains in the absence of phasons and, hence, isotropic. Moreover, the relaxation of the phasons to the equilibrium value in the presence of an applied stress should proceed qualitatively in the same way whether the phason field is initially zero or nonzero, although it may differ in detail. The exact form of this time dependence will be discussed elsewhere.[13]

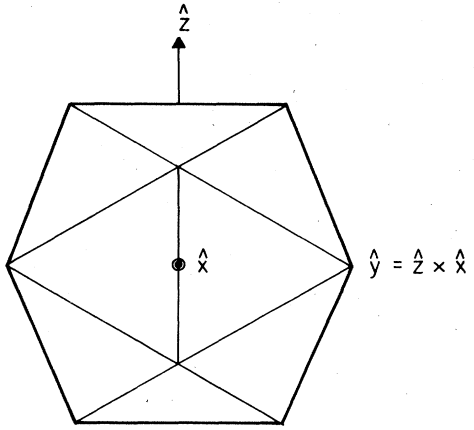

FIG. 1. This figure shows the coordinate system used in this paper with the x, y, and z axes normal to the edges of an icosahedron. In Ref. 6, the z axis passed through one of the vertices of an icosahedron.

In more familiar incommensurate systems—e.g., the mercury chain salts[14–16]—the phason mode (which in that case corresponds to the relative motion of sublattices) can propagate at intermediate wave number,[14(b)–17] although at sufficiently small wave number, it is always diffusive.[15,17] In quasicrystals, such estimates as can be made of viscosities, elastic constants, and vacancy mobilities suggest that there is *no* wave-number regime where phasons propagate. It should also be noted that in a fundamental sense the phason variable does *not* couple reactively to a conserved velocity field and is, therefore, always diffusive at hydrodynamic frequencies as first noted by Finger and Rice[15] for conventional incommensurate systems.

The slowness of the phason mode has remarkable consequences for dislocation motion. The fact[6] that any

dislocation has both a phonon and a phason part means that a moving dislocation must carry both fields with it. This adds tremendously to the drag on the dislocation, making it of order 10^{10} larger than in pure conventional crystals. This suggests that a perfect quasicrystal would be impervious to plastic deformation—dislocations would behave as though pinned in a work-hardened material. We shall discuss this in another article.[10]

The complete zero-temperature phonon and electron[2,8] spectra are expected to have densely spaced gaps. Calculations by Ostlund[18] for one-dimensional quasicrystals indicate, however, that the gap size for phonons tends to zero faster than q, justifying the traditional hydrodynamic treatment presented here.

One piece of speculation is in order. In a one-dimensional incommensurate system (the discrete Frenkel-Kontorova model[19]), Aubry[20,21] and Peyrard[21] have shown that anharmonic effects can pin the sliding mode (analogous to the phasons here), rendering it non-hydrodynamic. If a similar pinning transition were to take place for the phasons in a quasicrystal, the dynamics, especially of dislocations, would change dramatically. We do not know if such a transition is possible. All statements in this paper apply only when this transition has not occurred.

The remainder of this paper is organized as follows: Section II presents elasticity theory in a coordinate system more tractable than that used in Ref. 6. Section III derives the hydrodynamic equations, and Sec. IV discusses the mode structure. Section V ventures beyond the hydrodynamic domain to discuss the possibility of propagating phasons in quasicrystals. The Appendix discusses the effective elastic energy. The detailed calculation of the icosahedral elastic energies using group theory will be presented in a forthcoming paper.[22]

II. ELASTICITY THEORY FOR QUASICRYSTALS

We review briefly the construction of the elasticity theory discussed in Ref. 6. Consider a quasicrystal Q in d dimensions with density

$$\rho(\mathbf{r})=\sum_{\mathbf{G}\in L_R}\rho_{\mathbf{G}}e^{i\mathbf{G}\cdot\mathbf{r}}=\sum_{\mathbf{G}\in L_R}|\rho_{\mathbf{G}}|\,e^{-i\phi_{\mathbf{G}}+i\mathbf{G}\cdot\mathbf{r}}\,, \tag{2.1}$$

where L_R is the reciprocal lattice for Q and $|\rho_{\mathbf{G}}|$ and $\phi_{\mathbf{G}}$ are, respectively, the amplitude and phase of the density wave at $\mathbf{G}$. Let k be the number of independent incommensurate lengths in Q. L_R can then be constructed by taking linear combinations with integer coefficients of the vectors $\mathbf{G}_n$ in a minimal set B (the basis) containing kd elements. The kd phases ϕ_n of the complex order parameters $\rho_{\mathbf{G}_n}$ of Q provide a complete description of the long-wavelength, low-energy mechanical deformations of Q, as can be seen, for example, from Landau theory.[4–8]

For the three-dimensional icosahedral quasicrystal, to which we will hereafter restrict our attention, the basis B consists of six vectors $\mathbf{G}_n$, $n=1,\ldots,6$, which are conveniently taken as the vectors pointing to the six vertices in the upper half plane of an icosahedron. In Ref. 6, a coordinate system was chosen with the z axis along the vector $\mathbf{G}_1$. We have found that algebraic manipulations are much simpler in a coordinate system shown in Fig. 1 in which the z axis bisects one of the edges of an icosahedron. The new coordinate system is obtained from that used in Ref. 6 by a rotation through an angle $\alpha=31.72°$ about the y axis. The coordinates of the basis vectors are

$$\begin{aligned}
&\mathbf{G}_1=G(-\sin\alpha,0,\cos\alpha)\,,\\
&\mathbf{G}_2=G(\sin\alpha,0,\cos\alpha)\,,\\
&\mathbf{G}_3=G(0,\cos\alpha,\sin\alpha)\,,\\
&\mathbf{G}_4=G(-\cos\alpha,\sin\alpha,0)\,,\\
&\mathbf{G}_5=G(-\cos\alpha,-\sin\alpha,0)\,,\\
&\mathbf{G}_6=G(0,-\cos\alpha,\sin\alpha)\,,
\end{aligned} \tag{2.2}$$

where $\sin\alpha=(1+\tau^2)^{-1/2}$ with $\tau=(1+\sqrt{5})/2$ the golden mean. The six independent phases ϕ_n can be parametrized by two three-component fields $\mathbf{u}$ and $\mathbf{w}$ according to

$$\phi_n=\mathbf{G}_n\cdot\mathbf{u}+\mathbf{H}_n\cdot\mathbf{w}\,, \tag{2.3}$$

where

$$\begin{aligned}
&\mathbf{H}_1=-\mathbf{G}_1\,,\ \mathbf{H}_2=\mathbf{G}_2\,,\ \mathbf{H}_3=\mathbf{G}_5\,,\\
&\mathbf{H}_4=\mathbf{G}_3\,,\ \mathbf{H}_5=\mathbf{G}_6\,,\ \mathbf{H}_6=\mathbf{G}_4\,.
\end{aligned} \tag{2.4}$$

$\mathbf{u}$ is the familiar displacement field, and $\mathbf{w}$ represents relative motion of the constituent density waves. The fields $\mathbf{u}$ and $\mathbf{w}$ transform under different three-dimensional representations of the symmetry group of the icosahedron with $\mathbf{u}$ transforming like a vector. The orthogonality properties of the vectors $\mathbf{G}_n$ and $\mathbf{H}_n$ can be used to express $\mathbf{u}$ and $\mathbf{w}$ in terms of the ϕ_n's:

$$\mathbf{u}=G^{-2}\sum_n\mathbf{G}_n\phi_n\,, \tag{2.5a}$$

$$\mathbf{w}=G^{-2}\sum_n\mathbf{H}_n\phi_n\,. \tag{2.5b}$$

The transformation properties of $\mathbf{u}$ and $\mathbf{w}$ under the icosahedral group, along with constraints of translational and rotational invariance and the fact that the ϕ_n's are hydrodynamic allow us to construct[6,22] the harmonic elastic free energy $F(\mathbf{u},\mathbf{w})$ for the quasicrystal. We obtain

$$F(\mathbf{u},\mathbf{w})=F_u+F_w+F_{uw} \tag{2.6}$$

where, in terms of the Fourier-transformed fields $u(\mathbf{q})$ and $w(\mathbf{q})$, the various pieces of the elastic energy are

$$\begin{aligned}
F_u=\int\frac{d^3q}{(2\pi)^3}[&\tfrac{1}{2}(\lambda+2\mu)q_iq_j\\
&+\tfrac{1}{2}\mu(q^2\delta_{ij}-q_iq_j)]u_i(\mathbf{q})u_j(-\mathbf{q})\,,
\end{aligned} \tag{2.7}$$

$$F_w=\int\frac{d^3q}{(2\pi)^3}[K_1q^2\delta_{ij}+K_{ij}(\mathbf{q})]w_i(\mathbf{q})w_j(\mathbf{q})\,, \tag{2.8}$$

$$F_{uw}=\int\frac{d^3q}{(2\pi)^3}C_{ij}(\mathbf{q})u_i(\mathbf{q})w_j(\mathbf{q}) \tag{2.9}$$

with

$$K_{ij}(\mathbf{q})=K_2\tau^{-2}\begin{bmatrix}\tau^4q_x^2+q_y^2+\tau^2q_z^2 & -2\tau^2q_yq_z & -2\tau^2q_xq_z\\ -2\tau^2q_yq_z & q_x^2+\tau^2q_y^2+\tau^4q_z^2 & 2\tau^2q_xq_y\\ -2\tau^2q_xq_z & 2\tau^2q_xq_y & \tau^2q_x^2+\tau^4q_y^2+q_z^2\end{bmatrix}, \tag{2.10}$$

and

$$C_{ij}(\mathbf{q})=K_3\tau^{-2}\begin{bmatrix}2\tau q_xq_z & 2\tau^3q_xq_y & -\tau^2q_x^2-\tau q_y^2+\tau^3q_z^2\\ -2\tau^3q_yq_z & \tau^3q_x^2-\tau^2q_y^2-\tau q_z^2 & -2\tau q_xq_y\\ \tau q_x^2-\tau^3q_y^2+\tau^2q_z^2 & -2\tau q_yq_z & 2\tau^3q_xq_z\end{bmatrix}. \tag{2.11}$$

Stability requires $\mu>0$, $\lambda>0$, $K_1>0$, $K_1+K_2>0$, and $2\mu(K_1+K_2)-K_3^2>0$.

Equations (2.7)–(2.11) give the elastic free energy for distortions in both $\mathbf{u}$ and $\mathbf{w}$. The equilibrium response to external stresses is controlled by an effective elastic energy $F_{\rm eff}(\mathbf{u})$ obtained from $F(\mathbf{u},\mathbf{w})$ by allowing $\mathbf{w}$ to relax to its equilibrium value determined by the Euler-Lagrange equation $\delta F/\delta\mathbf{w}(\mathbf{x})=0$ in the presence of a nonzero $\mathbf{u}$. Alternatively, $F_{\rm eff}$ can be obtained by integrating out the $\mathbf{w}$ degrees of freedom:

$$\exp[-F_{\rm eff}(\mathbf{u})]=\int d(\mathbf{w})\exp[-F(\mathbf{u},\mathbf{w})]\,. \tag{2.12}$$

The form of $F_{\rm eff}$ and its implications for certain simple mechanical experiments are given in Appendix A. It is important to note that $F_{\rm eff}$ is not isotropic, i.e., it is not characterized simply by a bulk and a shear modulus. The coupling between shear strains has a complicated angular dependence. In order for this anisotropic form to be realized, it is necessary for the $\mathbf{w}$ field to be able to relax to its equilibrium value in the time of an experiment. As discussed earlier and in Sec. III, $\mathbf{w}$ relaxes very slowly, at a rate comparable to that of vacancy diffusion (diffusion constant[12] $<10^{-10}$ cm^2/sec). Thus for many (perhaps most) experimental systems, there will be a wide range of time scales over which the observed elastic response is isotropic. In samples that are thin enough, however, a moderately patient experimentalist should be able to observe the anisotropic response in single crystals.

III. DERIVATION OF HYDRODYNAMIC EQUATIONS

The hydrodynamic variables[11,23] for quasicrystals are the conserved mass and momentum densities, ρ and $\mathbf{g}$, and the "broken-symmetry" fields $\mathbf{u}$ and $\mathbf{w}$. The first two relax slowly at small wave number because their zero-wave-number components are constants of the motion, and the second two because the restoring force for spatially uniform changes in them is zero. The Hamiltonian governing the dynamics of these variables is, in the harmonic approximation,

$$H=\int d^dx[\tfrac{1}{2}\rho v^2+\tfrac{1}{2}A(\delta\rho/\rho_0)^2+B(\delta\rho/\rho_0)\nabla\cdot\mathbf{u}]+F(\mathbf{u},\mathbf{w}) \tag{3.1}$$

where $\mathbf{v}=\mathbf{g}/\rho$ is the hydrodynamic velocity field, ρ_0 the quiescent density, $\delta\rho=\rho-\rho^0$, A^{-1} is the compressibility, B couples the density to quasilattice dilations, and $F(\mathbf{u},\mathbf{w})$ is given by Eqs. (2.7)–(2.11). Note that there is no term coupling $\delta\rho$ to $\nabla\cdot\mathbf{w}$ since $\nabla\cdot\mathbf{w}$ is not a scalar under the icosahedral group.

For a general set $\{\psi_\alpha\}$ of classical hydrodynamic variables governed by a Hamiltonian H, the equations of motion are

$$\frac{\partial\psi_\alpha}{\partial t}+\int\{\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{x}')\}\frac{\delta H}{\delta\psi_\beta(\mathbf{x}')}d^dx'-\int d^dx'\frac{\delta}{\delta\psi_\beta(\mathbf{x}')}\{\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{x}')\}+\Gamma_{\alpha\beta}\frac{\delta H}{\delta\psi_\beta(\mathbf{x})}=0\,, \tag{3.2}$$

where $\{\ ,\ \}$ is the classical Poisson bracket[24] and $\Gamma_{\alpha\beta}$ is the matrix of kinetic coefficients. To obtain the complete set of hydrodynamical equations for quasicrystals from Eq. (3.2), we need the Poisson brackets among the variables ρ, $\mathbf{g}$ (or $\mathbf{v}=\mathbf{g}/\rho$), $\mathbf{u}$, and $\mathbf{w}$. Those involving the first two are known from the hydrodynamics of simple fluids, while the derivation of those for $\mathbf{u}$ and $\mathbf{w}$ is presented below.

Since the Poisson brackets depend only on the definition of variables and not on the form of the Hamiltonian, we can state that

$$\{\phi_n(\mathbf{x}),\mathbf{g}(\mathbf{x}')\}=[-\mathbf{G}_n+\nabla\phi_n(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}')\,, \tag{3.3}$$

exactly as in the case of a single density wave such as encountered in a smectic liquid crystal.[25,26] The equations $\{\phi_n(\mathbf{x}),\phi_m(\mathbf{x}')\}=\{\phi_n(\mathbf{x}),\rho(\mathbf{x}')\}=0$ are trivially satisfied since they involve only the coordinates and not the momenta of the constituent particles. The expressions (2.5) for $\mathbf{u}$ and $\mathbf{w}$ in terms of $\{\phi_n\}$ then imply

$$\begin{aligned}\{u_i(\mathbf{x}),g_j(\mathbf{x}')\}&=[-\delta_{ij}+\nabla_iu_j(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}')\,,\\ \{w_i(\mathbf{x}),g_j(\mathbf{x}')\}&=[\nabla_iw_j(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{3.4}$$

since $\sum_n G_{ni}H_{nj}=0$ and $\sum_n G_{ni}G_{nj}=\sum_n H_{ni}H_{nj}=G^2\delta_{ij}$. The hydrodynamic equations obtained from these relations are

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0 , \tag{3.5a}$$

$$\partial_t g_i + \nabla \cdot (\mathbf{v} g_i) - \nabla_j (\eta_{ijkl} \nabla_k g_l) = -(\delta_{ij} - \nabla_i u_j) \frac{\delta H}{\delta u_j} + (\nabla_i w_j) \frac{\delta H}{\delta w_j} - \rho \nabla_i \frac{\delta H}{\delta \rho} , \tag{3.5b}$$

$$(\partial_t + \mathbf{v} \cdot \nabla) u_i + \Gamma_u \frac{\delta H}{\delta u_i} - v_i = 0 , \tag{3.5c}$$

$$(\partial_t + \mathbf{v} \cdot \nabla) w_i + \Gamma_w \frac{\delta H}{\delta w_i} = 0 . \tag{3.5d}$$

That $\mathbf{u}$ and $\mathbf{w}$ play very different roles in the dynamics is evident from the absence of a reactive ($\partial_t \mathbf{w} \sim \mathbf{v}$) coupling of $\mathbf{w}$ to the hydrodynamic velocity field. This is because $\mathbf{w}$ is insensitive to spatial translations. Equations (2.1) and (2.3) tell us that a rigid shift ($\mathbf{x} \rightarrow \mathbf{x} + \text{const}$) of an undistorted quasicrystal gives rise to a uniform $\mathbf{u}$ but to no $\mathbf{w}$ field. We should expect that the time dependence of $\mathbf{w}$ will be diffusive, not oscillatory, at long wavelengths, and we shall see in the next section that this is so. With these preliminaries, we now note some features of the equations of motion and estimate the magnitudes of some of the parameters therein. Equation (3.5a) is simply the continuity equation expressing conservation of mass. Equation (3.5b) is a generalized Navier-Stokes equation or a local version of Newton's law equating the rate of change of momentum to the sum of the forces. The forces, from left to right, are viscous drag, elastic forces due to distortions of the quasilattice, and pressure. The viscosity is constrained by icosahedral symmetry to be isotropic:

$$\eta_{ijkl} = (\eta_L - \tfrac{4}{3}\eta_T)\delta_{ij}\delta_{kl} + \eta_T(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) , \tag{3.6}$$

where η_L and η_T are the longitudinal and transverse viscosities. The somewhat unfamiliar $\nabla u(\delta H/\delta u)$ and $\nabla w(\delta H/\delta w)$ terms in the elastic forces arise from the same pieces of the $\{\mathbf{u},\mathbf{v}\}$ and $\{\mathbf{w},\mathbf{v}\}$ Poisson brackets as do the convective ($\mathbf{v}\cdot\nabla\mathbf{u}$ and $\mathbf{v}\cdot\nabla\mathbf{w}$) terms in the $\mathbf{u}$ and $\mathbf{w}$ equations. The terms $\Gamma_u(\delta H/\delta u)$ and $\Gamma_w(\delta H/\delta w)$ represent a diffusive relaxation of the phase variables without a macroscopic velocity field. Since they have the same physical origin, it is reasonable to assume that Γ_w is of the same order as Γ_u. We already know (see Ref. 11) that Γ_u is simply a vacancy diffusion mobility. The diffusion constant for vacancies, given by Γ_u times a shear modulus, is of order 10^{-10} cm^2/sec in typical metals.[12] The $\mathbf{w}$ field should thus also relax with a diffusion constant of that order. This implies a relaxation time of 10^{10} sec for a 1-cm sample. The consequences of this small phason diffusion constant for dislocation motion, however, appear to be quite dramatic and will be discussed in a separate paper.[10]

IV. HYDRODYNAMIC MODE STRUCTURE

The total number of hydrodynamic variables in icosahedral quasicrystals is eleven ($\mathbf{g}$, $\mathbf{u}$, $\mathbf{w}$, ρ, and the energy density ϵ which we will not discuss in detail). There should, therefore, be a total of eleven peaks in the hydrodynamic correlation functions. Eight of these are the familiar[11] peaks of an ordinary solid: two pairs of transverse and one pair of longitudinal sound peaks, a vacancy diffusion, and a heat diffusion peak. The remaining three are diffusive and arise from the rearrangements[4–6] of the icosahedral structure described by $\mathbf{w}$. The positions and widths of these peaks at long wavelengths are determined from the equations of motion (3.5) (ignoring nonlinearities). These linearized equations of motion when Fourier-transformed in space and time read

$$-i\omega\delta\rho + i\mathbf{q}\cdot\mathbf{g} = 0 , \tag{4.1}$$

$$-i\omega\mathbf{u}_T = \mathbf{g}_T/\rho_0 - \Gamma_u[\mu q^2 \mathbf{u}_T + \underline{P}(\hat{\mathbf{q}})\underline{C}(\mathbf{q})\mathbf{w}] , \tag{4.2}$$

$$-i\omega u_L = g_L/\rho_0 - \Gamma_u[(\lambda+2\mu)q^2 u_L + \hat{\mathbf{q}}\cdot\underline{C}(\mathbf{q})\mathbf{w} - iBq(\delta\rho/\rho_0)] , \tag{4.3}$$

$$-i\omega\mathbf{g}_T = -\eta_T q^2 \mathbf{g}_T - \mu q^2 \mathbf{u}_T - \underline{P}(\hat{\mathbf{q}})\underline{C}(\mathbf{q})\mathbf{w} , \tag{4.4}$$

$$-i\omega g_L = -\eta_L q^2 g_L - (\lambda+2\mu-B)q^2 u_L - \hat{\mathbf{q}}\cdot\underline{C}(\mathbf{q})\mathbf{w} - i(A-B)q\delta\rho/\rho_0 , \tag{4.5}$$

$$-i\omega\mathbf{w} = -\Gamma_w[\underline{M}(\mathbf{q})\mathbf{w} + \underline{C}_T(\mathbf{q})\mathbf{u}] , \tag{4.6}$$

where here and hereafter an underline represents a matrix, the *superscript* T denotes the transpose matrix, and the *subscripts* T and L denote parts parallel and perpendicular to $\mathbf{q}$. $\underline{P}$ is a transverse projection operator with components

$$P_{ij} = \delta_{ij} - Q_{ij} = \delta_{ij} - \frac{q_i q_j}{q^2} , \tag{4.7a}$$

and

$$\underline{M} = K_1 \underline{I} + \underline{K} , \tag{4.7b}$$

where $\underline{I}$ is the unit tensor. Note that $\underline{M}$ and $\underline{C}$ are both of order q^2.

The speeds of the sound modes are isotropic at small wave number and can be calculated by setting to zero all dissipative coefficients and the fields $\mathbf{w}$ in the equations of motion. This is because on the time scales corresponding to propagating sound (that is frequency $\omega \sim$ wave number q), $\mathbf{w}$, being diffusive, does not respond at all to lowest order in wave number. It does have an effect at the next order in wave number: It contributes to the damping of sound waves, rendering it anisotropic despite the isotropy of the viscosity tensor (3.4). To see this, we simply solve the equations of motion (4.1)–(4.6) for $\omega \sim q$. In this regime, the term $\Gamma_w \underline{K}(\mathbf{q})\mathbf{w}$ in Eq. (4.6) is negligible in comparison to $-i\omega\mathbf{w}$ at small q so that

$$\mathbf{w} = \frac{-i\Gamma_w \underline{C}^T(\mathbf{q})}{\omega}\mathbf{u} . \tag{4.8}$$

Inserting this expression for $\mathbf{w}$ into Eqs. (4.1)–(4.6) leads after some manipulations (keeping in mind $\omega \sim q$) to the effective Fourier-transformed equations of motion:

$$(-i\omega + \Gamma_u \mu q^2)\mathbf{u}_T = \mathbf{g}_T/\rho_0 , \tag{4.9a}$$

$$(-i\omega + \eta_T q^2)\mathbf{g}_T + [\mu q^2 \underline{I} + (-i\omega)^{-1}\Gamma_w \underline{P}\underline{C}\underline{C}^T]\mathbf{u}_T = 0 \tag{4.9b}$$

for transverse sound and

$$-i\omega\delta\rho+iqg_L=0\ , \tag{4.10a}$$

$$(-i\omega+\eta_L q^2)g_L+i(A+2\mu+\lambda-2B)q(\delta\rho/\rho_0)$$
$$-(\Gamma_w/\omega q)\mathrm{Tr}(\underline{Q}\underline{C}\underline{C}^T)(\delta\rho/\rho_0)=0 \tag{4.10b}$$

for longitudinal sound. Dissipative cross couplings of $\mathbf{g}_T$ to ρ and of g_L to $\mathbf{u}_T$, which the substitution (4.8) will generate, have been neglected since they do not affect the attenuations to leading order in q. Note that the term $\underline{P}\underline{C}\underline{C}^T/i\omega$ in Eq. (4.9b) is of order q^3 (recall $\underline{C}\sim q^2$) and is consequently of one order higher in q than the $\mu q^2\underline{I}$ term. Likewise, $[\mathrm{Tr}(\underline{Q}\underline{C}\underline{C}^T)/\omega q]\sim q^2$ is one higher order in q than $i(A+\lambda-2B)q$ in Eq. (4.10b). As a result, both of these terms can affect the dispersion relations only at *second* nontrivial order in q; i.e., they alter the leading damping terms, not the propagating ones. Note that without these terms ($\underline{P}\underline{C}\underline{C}^T$ and $\underline{Q}\underline{C}\underline{C}^T$) the equations of motion are those of an isotropic solid. This implies that the sound speeds of the full equations are isotropic. Solving for the eigenvalues $\omega(\mathbf{q})$, we find for $q\to 0$

$$\omega=c_L q-\frac{i}{2}\left[\eta_L q^2+\frac{\Gamma_w}{\rho_0 c_L^2}\frac{\mathrm{Tr}(\underline{Q}\underline{C}\underline{C}^T)}{q^2}\right] \tag{4.11a}$$

for longitudinal sound, with

$$\rho_0 c_L^2=A+\lambda+2\mu-2B \tag{4.11b}$$

and

$$\omega_\alpha=c_T q-\tfrac{1}{2}iq^2\left[(\eta_T+\Gamma_u\mu)+\frac{\Gamma_w}{\mu q^4}\lambda_\alpha\right]\ ,\quad \alpha=1,2 \tag{4.12a}$$

for the two polarizations of transverse sound with

$$\rho_0 c_T^2=\mu \tag{4.12b}$$

where the λ_α's ($\sim q^4$) are the two nonzero eigenvalues of $\underline{P}\underline{C}\underline{C}^T\underline{P}$. [Equation (4.12a) is of course just Eq. (1.2a) with $\gamma_\alpha(\hat{\mathbf{q}})=\eta_T+\Gamma_u\mu+(2\Gamma_w/\mu q^4)\lambda_\alpha$.] The attenuations of both transverse and longitudinal sound are sensitive to the presence of icosahedral order through the terms λ_1, λ_2, and $\mathrm{Tr}\underline{Q}\underline{C}\underline{C}^T$. In fact, λ_1 and λ_2 contain invariants of higher than sixth order in $\hat{\mathbf{q}}$, while their sum $\lambda_1+\lambda_2=\mathrm{Tr}\underline{P}\underline{C}\underline{C}^T$ as well as $\mathrm{Tr}\underline{Q}\underline{C}\underline{C}^T$ are of sixth order in $\hat{\mathbf{q}}$. Since there exist *anisotropic* icosahedral invariants of sixth order in $\mathbf{q}$, one might expect the attenuations to be anisotropic, and indeed they are. There are precisely two independent icosahedral invariants of sixth order in $\mathbf{q}$. They can be taken to be $|\mathbf{q}|^6$, which is isotropic, and

$$I_6=(1+\tau^6)^{-1}\sum_n(\mathbf{G}_n\cdot\mathbf{q})^6, \tag{4.13}$$

which is not. In terms of these,

$$q^2\mathrm{Tr}(\underline{Q}\underline{C}\underline{C}^T)=K_3^2(-9q^6+10I_6)\ . \tag{4.14}$$

The anisotropic part of the viscous damping (from estimates in Sec. III) is probably smaller in magnitude by a factor of 10^{-10} than the isotropic part. It is nonetheless worth remarking that quasicrystals should, therefore, have sound speeds which are isotropic but sound-wave attenuations which even at small $\mathbf{q}$ reflect the icosahedral symmetry of the quasicrystal.

The diffusive modes come from $\mathbf{w}$, the energy density ϵ, and the longitudinal part of $\mathbf{u}$. Similar power counting can be used here, with $\omega\sim q^2$, to simplify the equations somewhat. One is still left with a fairly complicated 4×4 matrix to diagonalize since ϵ essentially decouples from $\mathbf{u}$ and $\mathbf{w}$. We, therefore, expect thermal diffusion to be isotropic; vacancy and $\mathbf{w}$ diffusion will reflect icosahedral symmetry in a more complicated way than does sound attenuation. The $\mathbf{w}$ field is expected to relax very slowly, as we remarked, with a diffusivity comparable to that for vacancies. The effective equations of motion for phason and vacancy diffusion, as well as the slow relaxation of the phasons in response to an applied stress, will be discussed in Ref. 13.

V. BEYOND HYDRODYNAMICS

In the previous sections, we found that the modes associated with the phason field $\mathbf{w}$ are diffusive rather than propagating. This is a result of friction that opposes relative motion of the incommensurate mass-density waves. If this friction were turned off, then the modes associated with $\mathbf{w}$ would be propagating. Alternatively, if the friction coefficient coupling the incommensurate waves is small, there may exist a regime of wave numbers where the $\mathbf{w}$ modes are effectively propagating modes as has been observed in mercury chain salts.[14–17]

We will, therefore, derive dynamical equations for the icosahedral quasicrystal that are valid in the limit when friction is zero and that reduce in the hydrodynamical regime to the equations derived in the previous sections when friction is nonzero. Our analysis will follow that of Ramaswamy and Mazenko[26] for the analogous case of longitudinal sound in a fluid adsorbed on a substrate. In the absence of dissipation, the icosahedral quasicrystal can be viewed as a superposition of six mass density waves that can slide frictionlessly across each other. This means that there are six independently conserved momenta P_n, $n=1,\ldots,6$ with associated densities $g_n(\mathbf{x})$. The $g_n(\mathbf{x})\equiv\rho v_n(\mathbf{x})$ are then the hydrodynamic fields satisfying the Poisson bracket relation,

$$\{g_n(\mathbf{x}),\phi_m(\mathbf{x}')\}=[G\delta_{nm}-\hat{\mathbf{G}}_n\cdot\nabla\phi_m(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}')\ , \tag{5.1}$$

where $\hat{\mathbf{G}}_n=\mathbf{G}_n/G$. When dissipation is included, the linearized equations for ϕ_n and v_n in the incompressible limit become

$$\partial_t\phi_n+\Gamma_{nm}\frac{\delta H}{\delta\phi_m}-Gv_n=0\ , \tag{5.2a}$$

$$\partial_t g_n+G\frac{\delta H}{\delta\phi_n}+\gamma_{nm}(\mathbf{q})g_m=0\ , \tag{5.2b}$$

where Γ_{nm} and $\gamma_{nm}(\mathbf{q})\equiv\gamma_{nm}+\lambda_{nm}q^2$ are dissipative coefficients. These equations clearly contain three extra momentum densities not appearing in the hydrodynamical equations of Sec. III. Of these six, however, only the vector

$$\mathbf{g} \equiv G^{-1} \sum_n \mathbf{G}_n g_n \tag{5.3}$$

(the hydrodynamic momentum density) is a slow variable. The remainder, packaged most conveniently as

$$\mathbf{g}_w \equiv G^{-1} \sum_n \mathbf{H}_n g_n \ , \tag{5.4}$$

must decay with a lifetime that remains finite for $\mathbf{q}=0$. Clearly $\mathbf{v}_w \equiv \mathbf{g}_w/\rho_0$ is the velocity associated with the rearrangements coming from the $\mathbf{w}$ field and involves internal relative motions even at $\mathbf{q}=0$. It is thus reasonable that it decay non-hydrodynamically (and indeed rather quickly, as we shall see). The fact that $\mathbf{v}$ cannot decay at $\mathbf{q}=0$, together with the observation that $\mathbf{u}$ and $\mathbf{w}$ carry irreducible representations of the icosahedral group, implies that Γ_{nm} is of the form $\Gamma_u \hat{\mathbf{G}}_n \cdot \hat{\mathbf{G}}_m + \Gamma'_w \hat{\mathbf{H}}_n \cdot \hat{\mathbf{H}}_m$ and γ_{nm} at $\mathbf{q}=0$ is of the form $\gamma \hat{\mathbf{H}}_n \cdot \hat{\mathbf{H}}_m$. From this, we obtain the equations of motion:

$$\partial_t \mathbf{u} - \mathbf{v} + \Gamma_u \frac{\delta H}{\delta \mathbf{u}} = 0 \ , \tag{5.5}$$

$$\partial_t \mathbf{w} - \mathbf{v}_w + \Gamma'_w \frac{\delta H}{\delta \mathbf{w}} = 0 \ , \tag{5.6}$$

$$\partial_t \mathbf{g} - (\eta \nabla \nabla \mathbf{g}) + \frac{\delta H}{\delta \mathbf{u}} = 0 \ , \tag{5.7}$$

$$\partial_t \mathbf{g}_w + \gamma \mathbf{g}_w + \frac{\delta H}{\delta \mathbf{w}} = 0 \ . \tag{5.8}$$

In the limit of low frequencies, Eq. (5.8) reduces to

$$\mathbf{g}_w = -\frac{1}{\gamma(\mathbf{q})} \frac{\delta H}{\delta \mathbf{w}} \ , \tag{5.9}$$

the effect of which is to modify Eq. (5.6), giving

$$\partial_t \mathbf{w} + \left[\Gamma'_w + \frac{1}{\gamma} \right] \frac{\delta H}{\delta \mathbf{w}} = 0 \ , \tag{5.10}$$

which is identical to Eq. (3.5d) with

$$\Gamma_w = \Gamma'_w + \frac{1}{\gamma} \ . \tag{5.11}$$

If, however, we can make observations at frequencies and wave numbers such that $\omega \gg \gamma(\mathbf{q}) \sim \gamma + \lambda q^2$ [so that $\gamma(\mathbf{q})$ can be neglected in Eq. (5.8)], then the three diffusive $\mathbf{w}$ modes should turn into three pairs of propagating phasons. In fact, in this regime, the $\mathbf{u}$ and $\mathbf{w}$ modes are highly coupled and do not separate cleanly into phonons and phasons. They are instead six pairs of propagating modes with anisotropic velocities. Does this regime exist? Probably not, since the physics of the situation suggests that γ is much too large. The argument is analogous to that which determines whether a fluid adsorbed on a substrate can display sound modes.[26] Let the speed of the phasons in the absence of dissipation be c, which is of order $(K/\rho)^{1/2}$ where K is a typical shear modulus in a crystal. The frequency ω is then cq. In order that the mode be underdamped, we require that $cq > \gamma + \lambda q^2$, but also that $\lambda q^2 < cq$. This is *never* possible if $4\gamma > c^2/\lambda$. If we put in numbers, estimating $c^2\gamma^{-1}$ to be of order of typical vacancy diffusion constants, of order 10^{-10} cm^2/sec, and λ to be of order of typical viscosities, of order 1 cm^2/sec, we find that normal systems are unlikely to be anywhere near the underdamped regime.

ACKNOWLEDGMENTS

We are grateful to Stellan Ostlund for helpful discussions regarding the phonon spectrum and to S. Pantelides for providing us with useful information on vacancy diffusion in metals. We also thank D. Levine and Paul Steinhardt for useful conversations. This work was supported in part by the NSF under Grants Nos. DMR 82-19216 and 82-16178.

APPENDIX: EFFECTIVE ELASTIC ENERGY

In this appendix, we derive and discuss the effective elastic energy for $\mathbf{u}$ alone. This energy describes the static response of the system to externally applied stresses on time scales long enough for the phason field $\mathbf{w}$ and the density ρ to have relaxed to their equilibrium values in the presence of a nonzero $\mathbf{u}$. These equilibrium values are, of course, those which minimize the elastic Hamiltonian of Eqs. (2.6)–(2.11). Performing this minimization, we have, in Fourier space

$$\mathbf{w}(\mathbf{q}) = -\underline{M}^{-1}(\mathbf{q}) \underline{C}^T(\mathbf{q}) \mathbf{u}(\mathbf{q}) \tag{A1}$$

where $\underline{M}$ is defined in Eq. (4.7b), and

$$\delta\rho = -\rho_0 \frac{B}{A} \nabla \cdot \mathbf{u} \ . \tag{A2}$$

Note that $\delta\rho$ relaxes to this equilibrium on the same time scale as $\mathbf{w}$ since the relevant process is vacancy diffusion. Inserting the results (A1) and (A2) into (3.1), we obtain an effective Hamiltonian for $\mathbf{u}$ alone:

$$H_{\text{eff}}(\mathbf{u}) = \tfrac{1}{2} \mathbf{u}^T \underline{C}^{\text{eff}} \mathbf{u} \tag{A3a}$$

with the effective elastic matrix

$$\underline{C}^{\text{eff}} = \mu q^2 \underline{P}(\hat{\mathbf{q}}) + (\lambda + 2\mu - B^2/A) q^2 \underline{Q}(\hat{\mathbf{q}}) - \underline{C}(\mathbf{q}) \underline{M}^{-1}(\mathbf{q}) \underline{C}^T(\mathbf{q}) \ . \tag{A3b}$$

The explicit expression for $\underline{C}^{\text{eff}}$ is so complicated as to be useless. For practical purposes, it is far more efficient to calculate $\underline{C}$ and $\underline{M}$ for the particular values of $\mathbf{q}$ needed and then perform the matrix operations in (A3) numerically.

A number of important observations about the effective elasticity can be made

(1) It is proportional to $|\mathbf{q}|^2$ as in conventional solids with an angle-dependent prefactor that does not vanish in any direction provided the stability conditions following Eq. (2.11) hold. This implies that all the power counting one is familiar with for conventional crystals applies here: Stresses are proportional to strains, $\langle |\mathbf{u}(\mathbf{q})|^2 \rangle \sim q^{-2}$ (where $\langle \ \rangle$ denotes a thermal average) and dislocation energies are linear in the size of the system in three dimensions, logarithmic in two. No surprises are likely in either the effect of the phasons or dislocations on long-ranged order or in the nonlinear coupling of hydrodynamic modes to thermal fluctuations.

(2) The effective elasticity is, nonetheless, non-analytic

in $\mathbf{q}$. This is because $\underline{M}^{-1}$ contains an overall uncanceled factor of $[\det\underline{M}(\mathbf{q})]^{-1}$ which is the reciprocal of a polynomial of sixth order in $\mathbf{q}$. This nonanalyticity prevents us from interpreting the effective elastic energy as an elastic tensor times two strains; i.e., we *cannot* write $H^{\text{eff}}=\frac{1}{2}C_{ijkl}\nabla_i u_j \nabla_k u_l$. Thus the symmetry arguments which implied that the $\mathbf{u}-\mathbf{u}$ part of the full elastic energy was isotropic do not apply to the effective energy for $\mathbf{u}$ alone.

(3) The effective elastic energy is anisotropic. For example, for $\mathbf{q}=q\hat{\mathbf{z}}$,

$$\underline{C}^{\text{eff}}(q\hat{\mathbf{z}})=q^2\begin{bmatrix}\mu_x & 0 & 0\\ 0 & \mu_y & 0\\ 0 & 0 & \lambda_{\text{eff}}\end{bmatrix} \tag{A4a}$$

with

$$\mu_x=\mu-K_3^2\tau^2(K_2+K_1\tau^2)^{-1}\,, \tag{A4b}$$

$$\mu_y=\mu-K_3^2[\tau^2(K_1+K_2\tau^2)]^{-1}\,, \tag{A4c}$$

$$\lambda_{\text{eff}}=\lambda+2\mu-B^2/A-K_3^2(K_1+K_2)^{-1}\,. \tag{A4d}$$

The anisotropy is now manifest in the fact that $\mu_x\neq\mu_y$. The effective elastic energy cannot be characterized by a single shear modulus. Analysis of other directions of $\mathbf{q}$ reveals that for a general $\mathbf{q}$, bulk and shear distortions are coupled.

(4) The total number of elastic constants needed to characterize the effective elastic energy is four: namely, μ, $\lambda+2\mu-B^2/A$, K_3^2/K_2, and K_1/K_2.

Because of the unusual nature of this effective elastic energy—in particular its nonanalyticity—its interpretation may seem obscure to those familiar with conventional elasticity. To elucidate its operational significance as well as to give a straightforward experimental prescription for measuring the four independent elastic constants, we consider the case of mechanical measurements done on a monodomain sample of quasicrystal in the thin-slab geometry of Fig. 2. The width l of the slab is taken to be much smaller than all other linear dimensions (though still macroscopic, of course); and the slab normal $\hat{\mathbf{s}}$ can point in any direction relative to the quasicrystalline coordinate axes. We now imagine statically applying a uniform force $\mathbf{f}$ per unit area to one side of the slab while holding the other fixed, and ask for the ultimate displacement $\mathbf{u}(l)$ of the surface of the slab in response to $\mathbf{f}$. As discussed in the text, for times $t\ll\tau_R$, the phason relaxation time, the response is isotropic and characterized by the bulk and shear moduli λ and μ of Eq. (2.7). Thus these two parameters can be determined from the $t\ll\tau_R$ response just as they would be for an isotropic crystal. For $t\gg\tau_R$, we can calculate the response from the effective elastic energy by noting that $\mathbf{u}(\mathbf{r})$ should, by virtue of the experimental geometry, depend only on s, the distance along the slab normal. As a result, the spatial Fourier transform $\mathbf{u}(\mathbf{q})$ will be nonzero only for $\mathbf{q}$ along $\hat{\mathbf{s}}$. Furthermore, by the power counting discussed above, $\underline{C}_{uu}^{\text{eff}}(q\hat{\mathbf{s}})=\underline{C}_{uu}^{\text{eff}}(\hat{\mathbf{s}})q^2$. Thus the effective elastic energy reduces in this geometry to

$$H=\tfrac{1}{2}\alpha C_{ij}^{\text{eff}}(\hat{\mathbf{s}})\int\frac{dq_s}{2\pi}q_s^2u_i(q_s)u_j(q_s)-\alpha\mathbf{f}\cdot\mathbf{u}(l) \tag{A5}$$

where α is the area of the slab orthogonal to $\hat{\mathbf{s}}$. Fourier transforming back to real space, we have

$$H=\tfrac{1}{2}\alpha\int_0^l ds\left[\frac{d\mathbf{u}}{ds}\right]^T\underline{C}^{\text{eff}}(\hat{\mathbf{s}})\left[\frac{d\mathbf{u}}{ds}\right]-\alpha\mathbf{f}\cdot\mathbf{u}(l)\,. \tag{A6}$$

The static $\mathbf{u}$ field which minimizes this Hamiltonian is trivially shown to be a linear function of s: $\mathbf{u}(s)=\mathbf{u}(l)s/l$ for which the total energy is

$$E=\alpha\left[\frac{1}{2l}\mathbf{u}^T(l)\underline{C}^{\text{eff}}(\hat{\mathbf{s}})\mathbf{u}(l)-\mathbf{f}\cdot\mathbf{u}(l)\right]. \tag{A7a}$$

Minimization of this equation with respect to $\mathbf{u}(l)$ gives the desired displacement in response to $\mathbf{f}$:

$$\mathbf{u}(l)=[\underline{C}^{\text{eff}}(\hat{\mathbf{s}})]^{-1}\mathbf{f}l\,. \tag{A7b}$$

Again, the simplest way to apply this formula in practice for arbitrary directions of $\hat{\mathbf{s}}$ is to calculate $\underline{C}^{\text{eff}}(\hat{\mathbf{s}})$ and $\underline{K}(\hat{s})$ numerically from Eqs. (2.10) and (2.11) and then numerically perform both the matrix operations in Eq. (A3) and the inversion in Eq. (A6). For the special directions discussed earlier, namely $\hat{\mathbf{s}}=\hat{\mathbf{z}}$, the matrices are simple enough to calculate analytically. We find

$$u_{x,y}(l)=\mu_{x,y}^{-1}f_{x,y}l\,, \tag{A8a}$$

$$u_z(l)=\lambda_{\text{eff}}^{-1}f_z l \tag{A8b}$$

where the effective shear and bulk moduli $\mu_{x,y}$ and λ_{eff} are given by Eq. (A4). Equations (A6) justify our referring to them as bulk and shear moduli.

It is now apparent that mechanical measurements done in this single experimental geometry are sufficient to determine all four of the elastic constants needed to characterize fully the effective elastic energy. Recall that the initial response for $t\ll\tau_R$ is isotropic with shear and bulk Lamé coefficients μ and λ given by Eq. (2.7). Thus measurements of this response determine μ and λ. Now, by waiting until $t\gg\tau_R$, μ_x, μ_y, and λ_{eff} can be deter-

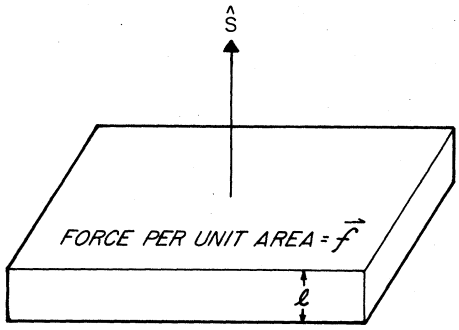

FIG. 2. The thin-slab geometry for determination of elastic constants.

mined via Eq. (2.8a). μ_x and μ_y together with the knowledge of μ from measurements for $t \ll \tau_R$ fix K_3^2/K_2 and K_1/K_2; λ_{eff} then fixes $\lambda+2\mu-B^2/A$. Thus the elastic parameters are now completely determined, and the elastic response for any other slab orientation is given by Eq. (A7), which now contains no further parameters needing adjustment.

The detailed time dependence of the relaxation of **u** and **w** to their final equilibrium values will be discussed in Ref. 13. We note here that for short times ($t \ll \tau_R$), changes in **u** and **w** from their initial values are proportional to $(t/\tau_R)^{1/2}$, whereas for long times deviations from their equilibrium values are proportional to $\exp(-t/\tau_R)$.

Note added: The basic diffusive hydrodynamics for the **w** modes and the result that the longitudinal and transverse sound velocities are isotropic whereas their dampings reflect the icosahedral anisotropy were obtained independently by P.A. Kalugin, A. Yu Kitayev, and L.S. Levitov and reported in an article [J. Phys. Lett. (Paris) **46**, L-601 (1985)] that appeared after this article was submitted for publication.

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