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Static and dynamic real-space renormalization through series expansions for correlation functions

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The renormalization-group analysis of correlation-function series expansions introduced by Stella *et al.* for computing the static critical properties of lattice-spin systems is refined by employing an additional interaction variable, and extended to dynamical critical phenomena. The static approach is applied to the square-lattice Ising model with nearest-neighbor and diagonal interactions, with use of the original high-temperature series to ninth order for pair-correlation functions. The critical point, the thermal and magnetic exponent, and the leading correction-to-scaling exponent are computed. Applications of the method to the spin-$\frac{1}{2}$ Ising and XY models in two and three dimensions are also reviewed. The possibility of basing a dynamical renormalization approach on this type of analysis is shown. The new dynamical method, which avoids proliferation of interactions and memory effects, is applied to the square-lattice Glauber model. The study of original series for pair relaxation times (to eigth order in the nearest-neighbor interaction and to fourth order in the diagonal one) gives estimates of the dynamic exponent z.

The purpose of this paper is to present a rich variety of applications as well as systematic improvements of a recently proposed renormalization-group (RG) approach[1] to static critical phenomena in lattice-spin systems. The method implements RG maps through the scaling law for the pair-correlation functions, in a way which is close in spirit to the original field-theoretic RG procedure.[2]

In its first applications, the approach made use of high-temperature-series expansions in one interaction variable for the correlation functions. In the present paper, the analysis is extended to make use of two interactions. As a consequence, better accuracy is obtained as the introduction of an additional interaction helps to suppress disturbing corrections to scaling at the fixed point. Furthermore, in the space of two interactions the fixed point and critical surface are found, much like in standard real-space renormalization.[3] The universality concept then arises naturally and the leading correction-to-scaling exponent can be computed. In this respect, the method is more flexible than the well-known phenomenological renormalization proposed by Nightingale,[4] although the latter could be developed to a higher level of accuracy.

The RG analysis is now also extended to dynamical critical phenomena, using scaling laws for relaxation times of pair correlations. These relaxation times are computed in the form of high-temperature series in two interaction variables. The implementation of RG maps then yields, in addition to the static critical quantities, the dynamic exponent describing the critical slowing down of the spin relaxation.

The paper is divided into two parts. The static RG is treated first. The basic equations pertaining to the scaling behavior of the correlation functions are discussed, and a motivation is given for working with more than one type of spin interactions. The RG technique is outlined with emphasis on the group structure properties of the transformations as a criterion for selecting the appropriate scaling behavior. It is explained how estimates for the critical point K_c and for the critical exponents y_T and y_H can be computed. An application is given to the square Ising model with nearest-neighbor and diagonal interactions. Further applications to the cubic Ising model and the quantum XY model in two and three dimensions, which have already been presented in the earlier work,[1] are discussed in more detail.

In the second part of the paper a new method is proposed to perform a RG analysis of the relaxation times of pair correlations. An application to the square Glauber model with nearest-neighbor and diagonal interactions follows and estimates for K_c, y_T, and the dynamical exponent z are given.

I. A STATIC RENORMALIZATION-GROUP ANALYSIS OF PAIR CORRELATIONS

A. Correlation-function scaling

Consider a ferromagnetic nearest-neighbor Ising model on a d-dimensional lattice of N sites. Every site i carries a spin $s_i = \pm 1$. The reduced Hamiltonian is written as

$$H(\{s\}) \equiv -\mathscr{H}(\{s\})/k_B T = K \sum_{\langle ij \rangle} s_i s_j + h \sum_{i=1}^{N} s_i$$

and gives the reduced energy of a spin configuration $\{s\}$. Here, $\langle ij \rangle$ denotes a sum over nearest-neighbor pairs, K is the reduced nearest-neighbor coupling ($K = J/k_B T > 0$), and h is the reduced external magnetic field.

The partition function is given by

$$Z_N = \sum_{\{s\}} \exp H(\{s\})$$

and the free energy per spin by

$$f(K,h) = -(k_B T/N)\ln Z_N .$$

The pair-correlation function is written as

$$G(\mathbf{r}) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle ,$$

where $\langle \cdots \rangle$ denotes a thermal average with respect to $H(\{s\})$ and $\mathbf{r}$ is the lattice vector connecting sites i and j. The correlation length ξ measuring the range of coherence in the system is defined through the second moment of G:

$$\xi^2 = \sum_{\mathbf{r}} r^2 G(\mathbf{r}) \Big/ \sum_{\mathbf{r}} G(\mathbf{r}) ,$$

where $r = |\mathbf{r}|$, and for $r \to \infty$ and r/ξ constant, G is expected to have the form (for magnetic field $h=0$)

$$G(\mathbf{r}) \to \widetilde{G}(r/\xi)/r^{d-2+\eta} , \tag{1.1}$$

where η is the critical exponent of the correlation function.

We remark that this asymptotic form for $G(\mathbf{r})$ is expected to hold with an exponent different from $d-2+\eta$ when, instead of the ratio r/ξ, the temperature (and therefore ξ) is kept constant.[5] A renormalization-group transformation rescales $\mathbf{r}$ while keeping r/ξ constant, and therefore (1.1) is the relevant asymptotic form for our purposes.

When the critical point ($K = K_c$, $h=0$) is approached, ξ diverges, and therefore G becomes long ranged in space. Within RG theory there exist several derivations of the scaling law for the correlation function: See, for example, Kadanoff's original work,[6] the extensive analysis of Niemeijer and van Leeuwen,[3] and Wilson's review,[7] where the specific example of a "spin decimation" RG is treated. In the following we will briefly discuss some aspects of correlation-function scaling in order to motivate our calculations.

Consider a generalized reduced Ising Hamiltonian $H(\{s\})$. The "even" interactions $K_1, K_2, \ldots, K_m$ and "odd" interactions $h_1, h_2, \ldots, h_n$ are represented by a vector

$$\mathbf{K} = (K_1, K_2, \ldots, K_m,\ h_1, h_2, \ldots, h_n)$$

which transforms under the RG to

$$\mathbf{K}' = \mathbf{R}_L(\mathbf{K}) ,$$

where L is a length-rescaling factor.

A fixed point $\mathbf{K}^*$ of $\mathbf{R}_L$ represents a universality class of "critical Hamiltonians" whose $\mathbf{K}$ vectors are "renormalized" toward $\mathbf{K}^*$. Within the framework of real-space RG,[3] the standard approach to construct the map $\mathbf{R}_L$ makes use of a weight function $P_L(\{s'\},\{s\})$ which associates configurations $\{s'\}$ of "block spins" to configurations $\{s\}$ of spins. One then derives a recursion relation for the free energy:

$$f(\mathbf{K}) = L^{-d} f(\mathbf{K}') + g(\mathbf{K}) , \tag{1.2}$$

where g is a suitable function. Assuming analyticity of the map $\mathbf{R}_L$ and of g, the familiar scaling law for the singular part of the free energy is derived on the basis of (1.2).

Generalizing the concepts of critical-point scale invariance to systems with inhomogeneous interactions $\mathbf{K}(\mathbf{r})$, one may derive,[3] by differentiating (1.2) twice with respect to *local* magnetic fields, the following asymptotic scaling law for G:

$$G(\mathbf{K}', r/L) = L^{2(d-y_H)} G(\mathbf{K}, r) , \tag{1.3}$$

for $r \to \infty$, and for $\mathbf{K}$ sufficiently close to the fixed point $\mathbf{K}^*$. The "magnetic exponent" y_H describes the renormalization of the "relevant" magnetic scaling field[3] h:

$$h' = L^{y_H} h .$$

The behavior of systems close to the "critical surface" (i.e., the domain of attraction of $\mathbf{K}^*$) is governed by $\mathbf{K}^*$. In this way, it follows from (1.3) that, on the critical surface [where $\xi(\mathbf{K}) = \infty$],

$$G(\mathbf{K}, \mathbf{r}) \to r^{2(y_H - d)} \quad \text{for } r \to \infty ,$$

implying an isotropic algebraic decay. Comparison with (1.1) also yields the familiar relation

$$\eta = d + 2 - 2y_H .$$

A further consequence of (1.3) is the following scaling law for G in the nearest-neighbor model (with $h_i = 0$ for all i, $K_i = 0$ for $i > 1$, and $K_1 = K$):

$$G(K', r/L) = L^{d-2+\eta} G(K, r) \quad \text{with } K' = K_c + L^{y_T}(K - K_c) , \tag{1.4}$$

for $r \to \infty$, and for K sufficiently close to the critical point $K_c \equiv K_{1c}$, which is renormalized toward $\mathbf{K}^*$. The "thermal exponent" y_T is the inverse of the correlation length exponent ν.

So far, our discussion has been general. When employing a special type of weight function P_L for constructing the RG map, one can obtain a stronger scaling in the transformation law of the correlation function than the one implied by (1.3). With a "linear" weight function[3,8]

$$P_L(q;\ \{s'\},\{s\}) = \prod_{\alpha} \left[\tfrac{1}{2}(1 + q s'_\alpha s_\alpha)\right] ,$$

where α denotes the sites of the new ("decimated") lattice which coincides with a sublattice of the old one and q is a free parameter, it follows that

$$G(\mathbf{K}',\mathbf{r}')=q^2G(\mathbf{K},\mathbf{r}) , \tag{1.5}$$

with $r'=r/L$, and both $\mathbf{r}$ and $\mathbf{K}$ arbitrary (the relative orientation of $\mathbf{r}$ and $\mathbf{r}'$ depends on the actual value of the rescaling L).

The RG map with the linear weight function has peculiar properties which are discussed in detail by Subbarao,[8] Wilson,[7] Bell and Wilson,[9] and van Leeuwen.[10] For our purposes, the interesting property is the role of the parameter q. In order that (1.5) be a meaningful transformation law, in accordance with the asymptotic scaling law (1.3), we must have

$$q^2=L^{d-2+\eta} . \tag{1.6}$$

Therefore, in the RG map, one expects to find nontrivial fixed points $\mathbf{K}^*$ only for this particular choice of q.

In the following we will assume that RG maps with a linear weight function actually possess fixed points capable of describing correctly the entire critical surface, if the parameter q is chosen to satisfy (1.6). As a consequence, in the neighborhood of such fixed points, the correlation function scales for *all distances* $\mathbf{r}$, and not only for large r's as in the general case of Eq. (1.3)

This assumption represents the following idea. Properties of G at asymptotically long distances in systems on the critical surface are mapped, under repeated iteration of the RG map, onto short-distance properties of G in systems close to $\mathbf{K}^*$. Starting from a Hamiltonian with only nearest-neighbor interactions, the RG iterations generate more-distant-neighbor and multispin interactions which may gradually wash out the anisotropy (due to the lattice structure) and suppress deviations from scaling behavior, which certainly occur at finite distances. A quantitative test hereof will be presented in the following sections, where the RG analysis of correlation functions, introduced by Stella *et al.*,[1] is generalized to incorporate more than one interaction.

B. The renormalization-group analysis

As in previous work,[1] our starting point is the scaling equation (1.4), interpreted as the implicit definition of a RG map $K\to K'(K)$, with a parametrical dependence on the (*a priori* unknown) exponent η. With the extension of this scheme to more than one interaction, it becomes possible to study the "universality" of critical behavior and to compute corrections to scaling. As far as we know, this has not previously been accomplished within RG approaches of "phenomenological" character.

In order to construct a RG map for n even interactions,

$$\mathbf{K}=(K_1,K_2,\ldots,K_n)\to\mathbf{K}'(\mathbf{K}) ,$$

we choose $2n$ lattice vectors $\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n$, $\mathbf{r}'_1,\mathbf{r}'_2,\ldots,\mathbf{r}'_n$ such that $r_i=Lr'_i$, $i=1,\ldots,n$, and calculate an approximation to the map

$$\mathbf{K}'=\mathbf{R}_L^{(n)}(\mathbf{K},\eta)$$

on the basis of the following system of equations:

$$\begin{aligned} &G(\mathbf{K}',\mathbf{r}'_1)=L^{d-2+\eta}G(\mathbf{K},\mathbf{r}_1) ,\\ &G(\mathbf{K}',\mathbf{r}'_2)=L^{d-2+\eta}G(\mathbf{K},\mathbf{r}_2) ,\\ &\ldots ,\\ &G(\mathbf{K}',\mathbf{r}'_n)=L^{d-2+\eta}G(\mathbf{K},\mathbf{r}_n) . \end{aligned} \tag{1.7}$$

As n is increased, one may hope that the implied RG map $\mathbf{R}_L^{(n)}$ becomes less dependent on the actual choice of lattice vectors and possesses adequate fixed points only for η closer to its exact value. In practical computations, it will be necessary to use *short* distances r_i and r'_i. As suggested by our discussion in Sec. I A, we may expect the error involved to become smaller as we work with more interactions.

For any finite n, the system (1.7) yields a fixed-point *curve* $K_L^*(\eta)$, parametrized by η. We are therefore forced, just as in the one-parameter approach,[1] to select an "optimal" value for η. For this purpose we employ a previously developed technique[11–13] of "group-structure optimalization." Let us review briefly how this works.

Suppose we choose two rescalings, L and L', and compute the corresponding fixed-point curves $\mathbf{K}_L^*(\eta)$ and $\mathbf{K}_{L'}^*(\eta)$. We can then look for a minimum of the Euclidean distance in $\mathbf{K}$ space

$$\Delta^{(L,L')}(\eta)=|\mathbf{K}_L^*(\eta)-\mathbf{K}_{L'}^*(\eta)|$$

by varying η. The value η^* for which the minimum occurs is selected as the optimal one, and $\mathbf{K}_L^*(\eta^*)$ and $\mathbf{K}_{L'}^*(\eta^*)$ are chosen as the optimal fixed points for the corresponding two RG maps with rescalings L and L', respectively. In practice, these fixed points are often very close.

To either of both optimal fixed points, the following applies. A critical nearest-neighbor coupling $K_c\equiv K_{1c}$ is computed as the intersection of the K_1 axis with the critical surface of $\mathbf{K}^*$. The temperature exponent y_T and correction-to-scaling exponents are computed from the $n\times n$ matrix $\nabla_{\mathbf{K}}\mathbf{R}_L^{(n)}(\mathbf{K},\eta)$ evaluated in $(\mathbf{K}^*,\eta^*)$, by standard methods.

As already mentioned, we will work with correlations at short distances. Before describing the computations, it is instructive to illustrate in the case of the square Ising model where exact information is available, to which extent the behavior of short-distance correlations obeys the asymptotic scaling law. In Table I we quote values of η computed from ratios of exact values[5,14] for G at different

TABLE I. Values of the critical exponent η which result from imposing the correlation-function scaling law on critical correlations in the square Ising model at distances $\mathbf{r}=(x,y)$ and $\mathbf{r}'=(x',y')$. The rescaling factor is $L=|\mathbf{r}|/|\mathbf{r}'|$.

(x',y')	(x,y)	L	$\eta(\mathbf{r},\mathbf{r}')$
(1,0)	(2,0)	2	0.250
(1,0)	(3,0)	3	0.253
(1,0)	(1,1)	$\sqrt{2}$	0.303
(1,0)	(2,1)	$\sqrt{5}$	0.261
(1,1)	(2,2)	2	0.236
(2,0)	(3,0)	$\frac{3}{2}$	0.257

short distances, at criticality: Let $\mathbf{r}=(x,y)$ and $\mathbf{r}'=(x',y')$ be vectors on the square lattice, with $r=Lr'$; then $\eta(\mathbf{r},\mathbf{r}')$ is defined as

$$\ln[G(K_c,\mathbf{r}')/G(K_c,\mathbf{r})]/\ln L \ .$$

The exact asymptotic value of η is $\frac{1}{4}$. From Table I it appears clear that if one can approximate correlations reasonably well, albeit at short distances, there is a good chance of determining meaningful η values already when working with just nearest-neighbor interactions.

C. Application to the square Ising model with nearest-neighbor and diagonal interactions

In this section we combine the technique of high-temperature-series expansions with the renormalization group. Suppose we have at our disposal M terms in the high-temperature series for G in the nearest-neighbor model:

$$G(K,\mathbf{r})\cong\sum_{m=1}^{M}a_m(\mathbf{r})K^m\equiv G_M(K,\mathbf{r}) \quad (K\sim 1/T) \ .$$

Consider, then, for given $\mathbf{r}$, $\mathbf{r}'$, and L ($r'=r/L$), the equation

$$G_M(K',\mathbf{r}')=L^{d-2+\eta}G_M(K,\mathbf{r}) \tag{1.8}$$

as the implicit definition of an approximate RG. The fixed-point curve $K_L^*(\eta)$ follows directly from

$$G_M(K^*,\mathbf{r}')/G_M(K^*,\mathbf{r})=L^{d-2+\eta} \ .$$

Analogously, by considering $\mathbf{r}$ and $\mathbf{r}'$ with $r'=r/L'$, we obtain a different curve $K_{L'}(\eta)$ and, by group-structure optimalization, estimates for K_c and η are computed. In fact, in this simple case with only one coupling K, the fixed-point curves typically have one nontrivial point of intersection, which is then identified with (K_c,η^*). The exponent y_T is computed from

$$L^{y_T}=(\partial_K K')_{K_c,\eta^*}$$
$$=L^{d-2+\eta^*}[\partial_K G_M(K,\mathbf{r})]_{K_c}/[\partial_{K'}G_M(K',\mathbf{r}')]_{K_c} \ .$$

One has to be careful, however. For finite M, both G_M and $K'(K,\eta)$ are analytic. For $M\to\infty$, G_M develops a singularity in $K=K_c$ (e.g., its derivative with respect to K diverges logarithmically in the two-dimensional Ising model[5]). In order that our estimates of K_c, η, and y_T converge to the correct answer, it is necessary that the map K' remains analytic as $M\to\infty$. Although there is no guarantee, evidence for this lies in the fact that Padé approximants, which we will compute for the renormalized interaction K', typically indicate that K' behaves regularly at and around K_c.

In order to extract information efficiently from the series expansions, we apply the technique of Padé approximants and compute Padé approximants not only for the truncated series G_M but also for the renormalized interactions. Indeed, it is possible, with some algebraic manipulation, to derive high-temperature series for K' itself:

$$K'\cong\sum_{m=1}^{M}c_m(\eta)K^m\equiv K'_M(K,\eta) \ . \tag{1.9}$$

The $c_m(\eta)$ turn out to be polynomials in the variable $L^{d-2+\eta}$ and are obtained by direct substitution of (1.9) in (1.8), such that (1.8) is an identity in every order up to (and including) M. For our RG maps the polynomials $c_m(\eta)$ are unique and have real coefficients (which need not be so, in general). We note that, after substitution of (1.9), (1.8) itself will no longer be an identity because the left-hand side will contain terms of higher order than M. Therefore, the map denoted by K'_M is different from the RG map which is *implicitly* defined through (1.8). The difference should become irrelevant for $M\to\infty$. We call the map K'_M a "RG-map series" and the map defined through (1.8) an "implicit RG map." Padé approximants for implicit RG maps involve Padé approximants for the G_M (cf. Ref. 1), whereas the Padé approximants for the RG-map series are simply Padé approximants for the K'_M.

An interesting property of this approach is that we can construct high-temperature series directly for the renormalized couplings without being hindered by proliferation of interactions. This advantage is lacking in an earlier high-temperature RG approach, introduced by Betts *et al.*,[15] which has been applied to both static[15] and, recently, dynamic,[16] critical phenomena.

In a first approximation we work with one coupling K and choose $\mathbf{r}_1=(1,0)$, $\mathbf{r}_2=(2,0)$, and $\mathbf{r}_3=(3,0)$, all along a principal axis on the square lattice. For the G_M we use expansions of eleventh order in the variable v ($=\tanh K$). These series have been published by Fisher and Burford.[17] The following equation then defines the *implicit* RG map (for $d=2$):

$$G_M(v',\mathbf{r}_1)=L^{\eta}G_M(v,\mathbf{r}_L) \ , \tag{1.10}$$

for length rescalings $L=2$ or 3, and with $M\leq 11$ in our computations. As suggested by the data in Table I, choosing all lattice vectors $\mathbf{r}_i$ of the same orientation may help to minimize short-distance deviations from scaling behavior. When computing Padé approximants for the G_M, implying (1.10) on them, and analyzing the implicit RG maps, we obtain the following results.

In Table II we present estimates for K_c (to be compared with the exact value $K_c\cong 0.441$). In Table III estimates are given for y_H ($=2-\eta/2$). The exact y_H is 1.875. Finally, in Table IV estimates of y_T (with exact value 1) are shown: Below one another are displayed the values of y_T corresponding to the RG maps with $L=2$ and 3 and the y_T which can be associated in a unique way to both RG

TABLE II. Padé table for the critical point K_c in the square Ising model from implicit RG maps with one interaction. The exact value is 0.441. Here and in the following tables, double bars (=) indicate cases where the fixed-point curves have only trivial intersections. Slashes (/) denote cases where Padé approximants have defects.

$N \backslash D$	0	2	4	6
5	0.576	0.389	0.455	0.436
6	=	0.392	/	
7	0.499	0.436	/	
8	=	=		
9	0.471	0.453		
10	=			
11	0.462			

TABLE III. Padé table for the exponent y_H in the square Ising model from implicit RG maps with one interaction. The exact value is 1.875.

$N \backslash D$	0	2	4	6
5	1.79	1.64	1.84	1.80
6	=	1.66	/	
7	1.84	1.79	/	
8	=	=		
9	1.85	1.86		
10	=			
11	1.85			

maps according to an interpolation procedure developed in previous work.[12]

The following remarks apply to Tables I—IV. The degree D of the denominator of the Padé approximants takes only even values because in the series expansion for a given correlation function the coefficients of either all even or all odd powers of v $(=\tanh K)$ vanish. For Padé approximants with numerators of degree $N \leq 5$ and for those entries where double bars are displayed, the fixed-point curves $v_2^*(\eta)$ and $v_3^*(\eta)$, corresponding to $L=2$ and 3, respectively, do not have a nontrivial intersection. Entries with slashes correspond to Padé approximants with defects. In all entries where numbers are displayed, the criterion of group-structure optimalization has been used successfully to select *unique* values for K_c, y_H, and y_T. The previous remarks apply, until further notice, also to forthcoming tables.

We now proceed to use RG-map series for the renormalized coupling in (1.10). It is straightforward to obtain

TABLE IV. Padé table for the exponent y_T in the square Ising model from implicit RG maps with one interaction. The exact value is 1. Below one another for given N and D: $y_T(L=2)$, $y_T(L=3)$, and the self-consistent y_T.

$N \backslash D$	0	2	4	6
5	0.54 0.65 1.50	1.01 1.13 1.70	0.70 0.75 1.09	0.80 0.87 1.29
6	=	1.03 1.12 1.55	/	
7	0.72 0.77 1.11	0.85 0.91 1.25	/	
8	=	=		
9	0.80 0.83 1.01	0.77 0.80 1.00		
10	=			
11	0.83 0.85 0.98			

TABLE V. Padé table for the critical point K_c in the square Ising model from RG-map series in one interaction. The exact value is 0.441.

$N \backslash D$	0	2	4	6
5	=	0.426	0.454	=
6	=	0.385	/	
7	=	=	/	
8	=	0.434		
9	=	0.442		
10	0.424			
11	0.448			

them on the basis of the series expansions[17] for the correlation functions. With $v=\tanh K$ and $x=L^{\eta}$, for $L=2$,

$$v'=xv^2[1+6v^2+(16-2x^2)v^4+2(23-18x^2)v^6 +2(79-156x^2+4x^4)v^8+\cdots]$$

and, for $L=3$,

$$v'=xv^3[1+12v^2+48v^4+2(76-x^2)v^6 +2(253-36x^2)v^8+\cdots] .$$

Applying Padé approximants to these series, we obtain the following results for K_c (Table V), y_H (Table VI), and y_T (Table VII).

Apparently, these results of RG-map series are, at least for y_T, of lower quality than the results of the previous implicit RG maps. This is possibly due to the fact that, when (1.9) is extracted from (1.8), some high-order terms in the G_M are not involved. Therefore, information is left unexploited. This effect is relatively more important for smaller M.

Finally, as a last application using only one interaction K, we have examined a different choice of lattice vectors in which two orientations are mixed. We choose $\mathbf{r}_1=(1,0)$ and $\mathbf{r}_{\sqrt{2}}=(1,1)$ and compute the corresponding RG-map series for $L=\sqrt{2}$ $(x=2^{\eta/2})$:

$$v'=2xv^2[1+2v^2+(5-8x^2)v^4+16(1-3x^2)v^6 +(59-216x^2+128x^4)v^8+\cdots] .$$

This transformation is now combined with the RG-map series for $L=2$ [involving $\mathbf{r}_1=(1,0)$ and $\mathbf{r}_2=(2,0)$]. The

TABLE VI. Padé table for the exponent y_H in the square Ising model from RG-map series in one interaction. The exact value is 1.875.

$N \backslash D$	0	2	4	6
5	=	2.02	2.07	=
6	=	1.77	/	
7	=	=	/	
8	=	1.99		
9	=	2.02		
10	1.88			
11	2.02			

results[18] for K_c, y_H, and y_T are somewhat poorer than in the previous calculations where all three lattice vectors were parallel. In conclusion, choosing the lattice vectors parallel to a principal axis on the lattice appears to help reduce the errors. However, one can readily check that such a choice generally cuts down the number of terms available in the series expansions.

Proceeding toward our second approximation, we introduce, in addition to a nearest-neighbor coupling K_1, a diagonal (or next-nearest-neighbor) coupling K_2. Our first step is the derivation of two-coupling high-temperature series for an appropriate set of correlation functions. The technique we have employed is very similar to the one outlined by Oitmaa,[19] using graphs with multiple edges. In Appendix A we present the coefficients which we calculated, up to ninth order (i.e., the coefficients of $v^m w^n$, with $v=\tanh K_1$, $w=\tanh K_2$, and $1 \le m+n \le 9$). We denote the series expansions by

$$G_M(\mathbf{v},\mathbf{r}) = \sum_{m+n=1}^{M} a_{mn}(\mathbf{r}) v^m w^n ,$$

where $\mathbf{v}=(v,w)$.

Choosing the lattice vectors $\mathbf{r}_1$, $\mathbf{r}_{\sqrt{2}}$, and $\mathbf{r}_2$ for $L=\sqrt{2}$ [and $\mathbf{r}_1$, $\mathbf{r}_{\sqrt{2}}$, $\mathbf{r}_2$, and $\mathbf{r}_{2\sqrt{2}}=(2,2)$ for $L=2$], we define a RG map through the following set of equations:

$$\begin{aligned} G_M(\mathbf{v}',\mathbf{r}_1) &= L^{\eta} G_M(\mathbf{v},\mathbf{r}_L) \\ G_M(\mathbf{v}',\mathbf{r}_{\sqrt{2}}) &= L^{\eta} G_M(\mathbf{v},\mathbf{r}_{\sqrt{2}L}) , \end{aligned} \tag{1.11}$$

with $M \le 9$. The choice of lattice vectors is a compromise: giving up the possible advantage of parallel lattice vectors in favor of having more terms in the series expansions.

Working with two sets of equations like (1.11), one for $L=\sqrt{2}$ and one for $L=2$, we apply the criterion of group-structure optimalization, as outlined in Sec. I B. We then compute K_c, y_H, y_T, and also, in some cases, the "irrelevant" exponent y_2 (<0). Our computational scheme allows us to describe the ferromagnetic regime ($v \ge 0$ and $w \ge 0$) of the Ising model with two interactions. In principle, the analysis could be extended to study the entire phase diagram, including antiferromagnetic and layered antiferromagnetic phases.[20]

At this point, implicit RG maps can be constructed directly through (1.11) where we replace G_M, in general, by its Padé approximant of order M. Remark that this time we are using Padé approximants in two variables. For constructing these, there exist several techniques.[21] We have found most convenient the generalization of the standard "ϵ algorithm"[22] which serves to evaluate a Padé approximant *numerically* in a chosen point (v_0,w_0). In our cases, the ϵ algorithm is initialized by computing the partial sums of the series in triangular form [i.e., $\sum_{m+n=1}^{p} a_{mn}(\mathbf{r}) v_0^m w_0^n$ for a partial sum of order p].

In the following tables we present our results of the implicit RG analysis with two interactions. Table VIII shows the estimates for the critical point $K_c \equiv K_{1c}$: Below one another are K_{1c}'s for $L=\sqrt{2}$ and 2. These critical points are renormalized towards the optimal fixed points $\mathbf{v}^*_{\sqrt{2}}$ and $\mathbf{v}^*_2$, respectively, which have been determined by the group-structure—optimalization procedure. Explicitly, they have been found by computing

$$\Delta_{\min} \equiv \min_{\eta} \{ \, | \mathbf{v}^*_{\sqrt{2}}(\eta) - \mathbf{v}^*_2(\eta) | \, \} .$$

In order to measure how well the group-structure criterion is satisfied, we calculate the percentage

$$\delta_{\min} \equiv 200[\Delta_{\min}/(\,|\mathbf{v}^*_{\sqrt{2}}(\eta^*)| + |\mathbf{v}^*_2(\eta^*)|\,)]\,\% .$$

As suggested by previous work,[13] $\delta_{\min} < 10\%$ corresponds to a successful optimalization. Table IX displays $\delta_{\min}$ for our implicit RG maps. We note that in several cases $\delta_{\min}=0$. This means that the two fixed-point curves in the three-dimensional space (v,w,η) actually have a nontrivial *intersection,* leading to complete satisfaction of the group-structure criterion. These lucky accidents are due to the fact that the two curves are *a priori* constrained to lie in a two-dimensional subspace when the lattice vectors which enter in (1.11) are chosen in the manner we used. Table X shows our results for y_H and Table XI those for y_T: Below one another are y_T's for $L=\sqrt{2}$ and 2.

A second method of analysis consists of computing the RG-map series for v' and w':

$$v' \cong \sum_{i+j=1}^{9} c_{ij}(\eta) v^i w^j ,$$

$$w' \cong \sum_{i+j=1}^{9} d_{ij}(\eta) v^i w^j .$$

In Appendix B we describe how we have calculated the

TABLE VII. Padé table for the exponent y_T in the square Ising model from RG-map series in one interaction. The exact value is 1. Below one another for given N and D: $y_T(L=2)$, $y_T(L=3)$, and the self-consistent y_T.

$N \backslash D$	0	2	4	6
5	 = 	1.99 1.87 1.30	1.83 1.73 2.15	 =
6	 = 	1.65 1.69 /	 / 	
7	=	=	/	
8	 = 	1.81 1.73 1.34		
9	 = 	1.83 1.73 1.18		
10	1.51 1.57 1.40			
11	1.74 1.66 1.94			

TABLE VIII. Padé table for the critical point K_c in the square Ising model from implicit RG maps with two interactions. The exact value is 0.441. Below one another for given N and D: $K_c(L=\sqrt{2})$ and $K_c(L=2)$.

$N \backslash D$	0	1	2	3	4	5
4	/	/	0.42 /	0.39 0.39	/	/
5	/	/	0.42 /	/	/	
6	/	0.44 0.50	0.46 /	/ 0.45		
7	/	/	0.45 0.45			
8	0.38 /	/ 0.54				
9	0.48 0.50					

coefficients with the help of a symbol-manipulation computer program. The coefficients themselves, for $L=\sqrt{2}$ and 2, are given in Appendix C.

After applying Padé approximants with two variables to the RG-map series, we obtain the following results. Table XII shows the critical point K_{1c} (cf. Table VIII). Table XIII gives y_H (cf. Table X) and Table XIV displays y_T (cf. Table XI). In Table XV $\delta_{\min}$ is presented (cf. Table IX). Note that in all cases $\delta_{\min} < 6\%$.

The present RG-map series yield Padé-approximant tables of better quality (i.e., greater stability and smoother trend of convergence as N and D increase while $N \cong D$) than the corresponding implicit RG maps. Further evidence, therefore, can be found by comparing the stability of the estimates for the fixed-point coordinates K_1^* and K_2^* (Ref. 18).

The present analysis is also yields a table of estimates for the irrelevant exponent y_2 which describes corrections to scaling.[23] Table XVI shows y_2 for $L=2$. The exact value is -1 (cf. Refs. 5 and 23). The results for $L=\sqrt{2}$ are worse (around -3).[18]

At the end of this section of applications to the square Ising model, it appears that the results obtained with two interactions K_1 and K_2, and using ninth-order series expansions, are more rewarding than those of the eleventh-order analysis with one interaction. This supports our assumptions pertaining to scaling at short distances and suggests that further improvement is possible, in the first place, by adding interactions and, next, by increasing the order of the series expansions. Considering the calculational effort involved, however, it is a lot easier to work with just one interaction, and already then qualitatively satisfactory results can be obtained.

D. Application to the cubic Ising model and to the quantum (spin-$\frac{1}{2}$) XY model in two and three dimensions

A summary of the results presented in this section has been given in earlier work.[1] For the three-dimensional Ising model on a simple-cubic lattice, we choose the lattice vectors $\mathbf{r}_i=(i,0,0)$, $i=1$, 2, and 3. We use high-T series of tenth order in $v=\tanh K$, calculated by Fisher and Burford,[17] and perform an analysis of the implicit RG maps for $L=2$ and 3.

Table XVII shows the critical point K_c (cf. the value 0.222 from conventional high-T-series analysis[24]). Table XVIII gives y_H (cf. the value 2.48 from field-theoretical ϵ expansions[25]), and Table XIX displays y_T (cf. the value 1.59 from field theory[25]): Below one another are the y_T's

TABLE IX. Padé table for $\delta_{\min}(\%)$ in the implicit RG maps for the square Ising model with two interactions.

$N \backslash D$	0	1	2	3	4	5
4	/	/	0	33	0	0
5	/	/	37	17	0	
6	2.5	0.1	0	0		
7	/	/	0.1			
8	47	0.1				
9	42					

TABLE X. Padé table for the exponent y_H in the square Ising model from implicit RG maps with two interactions. The exact value is 1.875.

$N \backslash D$	0	1	2	3	4	5
4	/	/	1.81	1.68	1.87	1.82
5	/	/	1.79	1.84	1.72	
6	2.07	1.74	1.86	1.86		
7	/	/	1.86			
8	1.60	1.89				
9	1.90					

TABLE XI. Padé table for the exponent y_T in the square Ising model from implicit RG maps with two interactions. The exact value is 1. Below one another for given N and D: $y_T(L=\sqrt{2})$ and $y_T(L=2)$.

$N \backslash D$	0	1	2	3	4	5
4	/	/	0.96 0.73	0.91 0.77	0.52 0.37	0.80 0.62
5	/	/	0.88 0.38	0.96 /	1.07 0.80	
6	/	0.84 0.78	0.68 0.76	0.40 0.69		
7	/	/	0.66 0.62			
8	0.90 /	0.63 0.57				
9	0.74 0.62					

TABLE XII. Padé table for the critical point K_c in the square Ising model from RG-map series in two interactions. The exact value is 0.441. Below one another for given N and D: $K_c(L=\sqrt{2})$ and $K_c(L=2)$.

$N \backslash D$	0	1	2	3	4	5	6
3	/	0.261 /	/	/	/	0.415 /	0.417 /
4	/ 0.371	/	0.419 0.383	/	0.443 0.431	0.442 0.427	
5	0.500 0.476	/	0.421 0.438	0.421 0.399	0.443 0.430		
6	0.501 0.473	/	0.415 0.358	0.415 0.350			
7	0.397 0.390	/	0.417 0.355				
8	0.490 0.488	/					
9	0.495 0.496						

TABLE XIII. Padé table for the exponent y_H in the square Ising model from RG-map series in two interactions. The exact value is 1.875.

$N \backslash D$	0	1	2	3	4	5	6
3	/	1.54	/	/	/	1.82	1.83
4	1.83	/	1.81	/	1.82	1.82	
5	2.27	/	1.82	1.81	1.83		
6	2.38	/	1.82	1.83			
7	1.78	/	1.82				
8	2.34						
9	2.40						

TABLE XIV. Padé table for the exponent y_T in the square Ising model from RG-map series in two interactions. The exact value is 1. Below one another for given N and D: $y_T(L=\sqrt{2})$ and $y_T(L=2)$.

$N \backslash D$	0	1	2	3	4	5	6
3	/	1.56 1.76	/	/	/	1.30 1.23	1.27 1.17
4	1.58 1.56	/	1.13 1.27	/	1.28 1.18	1.28 1.18	
5	2.02 1.78	/	1.17 0.91	1.12 1.04	1.24 1.16		
6	2.19 1.88	/	1.18 1.17	1.26 1.11			
7	1.07 1.07	/	1.17 1.07				
8	2.22 1.87	/					
9	2.39 1.92						

TABLE XV. Padé table for $\delta_{\min}(\%)$ in the RG-map series for the square Ising model with two interactions.

$N \backslash D$	0	1	2	3	4	5	6
3	/	4.0	/	/	/	0.8	1.3
4	3.9	/	5.3	/	1.4	1.1	
5	1.5	/	5.6	1.1	0.9		
6	2.3	/	0.1	2.1			
7	2.2	/	1.4				
8	2.8	/					
9	0.9						

TABLE XVI. Padé table for the leading correction-to-scaling exponent y_2 in the square Ising model from the RG-map series with rescaling $L=2$, in two interactions. The exact value is -1.

$N \backslash D$	0	1	2	3	4	5	6
3	/	−1.05	/	/	/	−2.12	−1.27
4	−2.15	/	−1.61	/	−2.58	−1.19	
5	−3.36	/	−2.87	−1.30	−1.20		
6	−3.76	/	−1.19	−1.30			
7	−1.24	/	−1.46				
8	−6.10	/					
9	−3.79						

TABLE XVII. Padé table for the critical point K_c in the cubic Ising model from implicit RG maps with one interaction. The expected value is 0.222.

$N \backslash D$	0	2	4	6
3	0.475	0.178	0.198	=
4	=	0.181	0.196	=
5	0.393	0.196	=	
6	=	0.194	=	
7	0.346	0.219		
8	=	0.223		
9	0.313			
10	=			

TABLE XVIII. Padé table for the exponent y_H in the cubic Ising model from implicit RG maps with one interaction. The expected value is 2.48.

$N \backslash D$	0	2	4	6
3	2.00	1.97	2.16	=
4	=	2.03	2.14	=
5	2.39	2.16	=	
6	=	2.12	=	
7	2.62	2.34		
8	=	2.40		
9	2.70			
10	=			

TABLE XIX. Padé table for the exponent y_T in the cubic Ising model from implicit RG maps with one interaction. The expected value is 1.59. Below one another for given N and D: $y_T(L=2)$, $y_T(L=3)$, and the self-consistent y_T.

$N \backslash D$	0	2	4	6
3	0.09 0.43 5.83	1.14 1.65 3.39	1.31 1.65 2.80	=
4	=	1.29 1.72 3.13	1.27 1.62 2.82	=
5	0.21 0.44 3.20	1.33 1.68 2.85	=	
6	=	1.25 1.65 3.00	=	
7	0.34 0.48 1.93	1.18 1.41 2.38		
8	=	1.30 1.43 1.92		
9	0.44 0.53 1.37			
10	=			

TABLE XX. Padé table for the critical point K_c in the cubic quantum XY model from implicit RG maps with one interaction. The expected value is 0.496.

$N \backslash D$	0	2	4	6
3	1.53	0.411	0.429	=
4	=	0.451	0.423	
5	0.749	0.446	0.423	
6	=	0.420		
7	0.728	=		
8	=	/		
9	0.605			

for $L=2$ and 3, and the y_T from the interpolation scheme introduced in earlier work.[12] All entries have $\delta_{\min}=0$ (cf. Sec. I C), except where double bars are displayed.

The spin-$\frac{1}{2}$ XY model for planar magnets and quantum liquids (e.g., superfluid ^{4}He) has been studied extensively by means of high-T-series expansion in two and three dimensions.[26] The reduced Hamiltonian is given by

$$H(\{s\})=-\mathscr{H}(\{s\})/k_B T=2K\sum_{\langle ij\rangle}(s_i^x s_j^x+s_i^y s_j^y)\ ,$$

where the spin components s_i^x, s_i^y, and s_i^z satisfy the angular-momentum commutation relations.

In three dimensions the model displays conventional long-range order of the xy component of the spin below a critical temperature T_c. We perform a RG analysis using xx correlations on a simple-cubic (sc) and a body-centered-cubic (bcc) lattice. The lattice vectors are chosen to be $\mathbf{r}_i=(i,0,0)$, $i=1$, 2 and 3, and $\mathbf{r}_{i\sqrt{3}}=(i,i,i)$, $i=1$, 2, and 3, respectively. We use high-T series of ninth order in K, calculated by Grundke.[27] The results obtained from the implicit RG maps are the following.

For the sc lattice, Table XX shows K_c (cf. the value 0.496 from conventional high-T-series analysis[28]), Table XXI gives y_H (cf. the field-theoretical estimate[25] 2.48), and Table XXII displays y_T (cf. 1.49 from field theory[25]): y_T for $L=2$, for $L=3$, and the self-consistent interpolation are below one another.

For the bcc lattice, the Padé tables are of the same quality as those for the sc lattice. They are reported in Ref.

TABLE XXI. Padé table for the exponent y_H in the cubic quantum XY model from implicit RG maps with one interaction. The expected value is 2.48.

$N \backslash D$	0	2	4	6
3	2.68	2.08	2.18	=
4	=	2.41	2.14	
5	2.49	2.33	2.13	
6	=	2.12		
7	2.83	=		
8	=	/		
9	2.62			

18. We limit ourselves here to present the case $N=7$ and $D=2$, where best agreement is found with the expected values. We obtain $K_c=0.343$ (0.344 is expected[28]), $y_H=2.33$ (cf. 2.48), and y_T $(L=2)=1.21$, y_T $(L=3)=1.30$, and y_T (self-consistent) $=1.67$ (cf. 1.49).

In two dimensions, due to the continuous symmetry of the Hamiltonian, the XY model does not display conventional long-range order at $T>0$.[29] Nevertheless, the classical XY (plane-rotator) model exhibits a topological long-range order at low T.[30–32] A line of critical points extends from $T=0$ to $T=T_v$, the vortex-unbinding temperature. Along this line the correlation length ξ is infinite and the correlation functions decay algebraically, i.e., $G\sim r^{2(y_H-d)}$, with an exponent y_H which varies continuously between $y_H=2$ for $T=0$ and $y_H=1.875$ (the Ising value) for $T=T_v$. The thermal exponent y_T is constant and takes the value $y_T=0$ ("temperature marginality").

For the quantum model, on the other hand, an analogous topological phase transition and critical line are expected.[33] Quantitative evidence, therefore, from RG calculations[34] and high-T-series expansions,[35] is very limited.

We perform a RG analysis on the triangular and the square spin-$\frac{1}{2}$ XY model. In both cases the lattice vectors $\mathbf{r}_i=(i,0,0)$, $i=1$, 2, and 3, are chosen along a principal axis. We use high-T series of ninth order in K, calculated by Grundke.[27] Our results of the implicit RG maps are the following.

For the triangular lattice, Table XXIII shows the fixed point K^*. In two cases, indicated with double asterisks, *two* fixed points are found. This means that the fixed-point curves $K_2^*(\eta)$ and $K_3^*(\eta)$ have two nontrivial intersections rather than one. We have interpreted the lower K^* value as an estimate for K_v $(=J/kT_v)$ and the occurrence of a second K^* as an indication in favor of the existence of a line of fixed points. Table XXIV describes these two cases with double fixed points. One sees that the exponent y_H corresponding to the lower K^* is not far from the value 1.875 expected at T_v. The y_T exponents are poor (cf. expected value zero). At the upper K they are imaginary, except when the rescaling is taken to be $\frac{3}{2}$; we present the corresponding value as the self-consistent y_T.

Table XXV shows our results for y_H at the fixed point. The expected range is $1.875\le y_H\le 2$. Table XXVI presents y_T: Below one another are the estimates for $L=2$, $L=3$, and the self-consistent value.

For the square lattice, analogous results are found.[18] Again, in two cases, i.e., for $(N,D)=(5,2)$ and $(7,0)$ double fixed points occur. Table XXVII shows the corresponding K^*, y_H, and y_T values. The y_H exponent associated with the lower K^* is again not far from 1.875.

Finally, we remark that in cases where two fixed points K_1^* and K_2^* are found, the corresponding y_T's need not satisfy $y_{T,1}y_{T,2}\le 0$ (i.e., a consistency requirement for the RG flow between two neighboring fixed points) because K_1^* and K_2^* belong to *different* RG maps (corresponding to different values of the parameter η).

TABLE XXII. Padé table for the exponent y_T in the cubic quantum XY model from implicit RG maps with one interaction. The expected value is 1.49. Below one another for given N and D: $y_T(L=2)$, $y_T(L=3)$, and the self-consistent y_T.

N \ D	0	2	4	6
3	0.59	1.40	1.50	
	0.74	1.78	1.79	=
	1.76	2.99	2.71	
4		2.30	1.38	
	=	2.22	1.72	
		1.99	2.83	
5	0.48	1.94	1.34	
	0.68	1.97	1.70	
	2.17	2.06	2.89	
6		1.33		
	=	1.71		
		2.96		
7	0.82			
	0.82	/		
	0.82			
8	=			
9	0.93			
	0.96			
	1.11			

TABLE XXIII. Padé table for the fixed point K^* in the triangular quantum XY model from implicit RG maps with one interaction. In cases where a double asterisk is displayed, two fixed points are found.

N \ D	0	1	2	3	4	5	6
3	/	/	/	0.186	=	0.215	/
4	=	/	=	0.506	=	0.446	
5	=	=	2.91	0.601	0.449		
6	0.470	0.456	**	0.626			
7	0.475	=	**				
8	=						
9	=						

TABLE XXIV. Lower and upper fixed points and the corresponding critical quantities for the entries with a doule asterisk in Table XXIII.

	K^*	y_H	$y_T(L=2)$	$y_T(L=3)$	y_T(self-consistent)
$N=6, D=2$	0.71	1.83	1.22	1.16	0.88
	2.00	2.95	/	/	0.52
$N=7, D=2$	0.66	1.76	1.26	1.23	1.10
	1.58	3.40	/	/	2.12

TABLE XXV. Padé table for the exponent y_H in the triangular quantum XY model from implicit RG maps with one interaction. The expected range is $1.875 \leq y_H \leq 2$.

$N \backslash D$	0	1	2	3	4	5	6
3	/	/	/	0.53	=	0.66	/
4	=	/	=	1.51	=	1.28	
5	=	=	2.43	1.68	1.28		
6	1.45	1.39	**	1.69			
7	1.40	/	**				
8	=	/					
9	=						

TABLE XXVI. Padé table for the exponent y_T in the triangular quantum XY model from implicit RG maps with one interaction. The expected value is zero. Below one another for given N and D: $y_T(L=2)$, $y_T(L=3)$, and the self-consistent y_T.

$N \backslash D$	0	1	2	3	4	5	6
3				1.09		1.10	
	/	/	/	1.83	=	1.97	/
				4.2		4.6	
4				1.19		/	
	=	/	=	1.32	=	1.14	
				1.85		1.14	
5			0.21	1.21	/		
	=	=	0.24	1.24	1.14		
			0.85	1.37	1.14		
6	1.28	1.23		1.24			
	1.37	1.45	**	1.21			
	1.73	2.28		1.07			
7	0.88						
	1.16	/	**				
	2.44						
8	=	/					
9	=						

TABLE XXVII. Lower and upper fixed points and the corresponding quantities for the cases with double fixed points in the square quantum XY model.

	K^*	y_H	$y_T(L=2)$	$y_T(L=3)$	y_T(self-consistent)
$N=5, D=2$	1.15	1.88	1.88	1.62	0.67
	1.94	2.66	/	/	1.10
$N=7, D=0$	0.97	1.37	−4.78	−2.62	/
	1.09	1.14	/	/	2.42

II. DYNAMIC RENORMALIZATION-GROUP ANALYSIS OF PAIR-CORRELATION RELAXATION TIMES

A. The dynamical critical exponent in the Glauber model

Consider a nearest-neighbor Glauber model (or kinetic Ising model).[36,37] The time evolution of the probability $P(\{s\},t)$ of a spin configuration $\{s\}$ at time t is given by the master equation

$$\partial_t P(\{s\},t) = -\sum_j W_j(\{s\})P(\{s\},t) + \sum_j W_j(\{s\},-s_j)P(\{s\},-s_j,t) .$$

The transition rates W obey, as usual, "detailed balance:"

$$W_j(\{s\})P(\{s\}) = W_j(\{s\},-s_j)P(\{s\},-s_j) ,$$

where $P(\{s\})$ is the equilibrium probability distribution $\exp H(\{s\})/Z$. Because of the detailed-balance condition, $P(\{s\})$ is stationary. In spite of this condition, a lot of freedom is left for choosing the form of W. In one dimension *and* for a special choice of W, the Glauber model is exactly solvable (Glauber's solution[36]). No solutions have been obtained in higher dimensions. Much attention has been devoted to the study of the critical slowing down of the relaxation of the order parameter, characterized by the dynamical exponent z (or $\Delta \equiv z\nu$, where ν is the static correlation-length exponent). Exact calculations of z have been possible only in the one-dimensional model[36–38] and at the level of the mean-field theory.[39]

Scaling hypotheses for dynamical critical phenomena have been formulated by Halperin and Hohenberg.[40] Later, renormalization-group theory for critical dynamics has been developed, combining space rescaling $(r \to r' = r/L)$ and time rescaling $(t \to t' = t/L^z)$.[41–43] For continuous-spin systems the dynamical RG program has yielded quantitative predictions for the dynamical exponent near four dimensions. For lattice-spin systems in two and three dimensions less progress has been made. One of the major problems in a microscopic approach is the generation of memory effects in the renormalization of the master equation. Only in the most simple cases, such as the one-dimensional Glauber model and the infinite-range Glauber model, has an "exact" dynamical renormalization (reproducing the exact exponents) been performed.[44–46]

There have been numerous RG approaches for calculating the dynamic exponent in the two-dimensional Glauber model.[16,47] The accuracy has been rather limited, however, due to uncontrolled approximations. Critical analysis of some of these approaches has shed some light on the problems involved[48,49] and alternative methods, avoiding memory effects and proliferation of interactions, have recently been proposed.[50] Most RG approaches to the two-dimensional Glauber model have yielded values for z in the range 1.75 (i.e., the exact lower bound[51]) $\le z \le 2.3$.

For comparison, we mention some of the recent results from high-temperature series[52] ($z = 2.125 \pm 0.01$), Monte Carlo simulations[53] ($z \cong 2.1$), finite-size scaling[54] ($z = 2.2 \pm 0.1$), and Monte Carlo RG ($z \cong 2.23$).[55] In three dimensions the need for accurate RG determinations of z is less pronounced, because the field-theoretical estimate[41,56] $z = 2.0$ is believed to be reliable. Estimates from high-T series (Ref. 52), Monte Carlo RG (Ref. 57), and real-space RG (Refs. 16 and 50) more or less agree with this number.

B. Renormalization-group analysis of relaxation times

Consider the dynamic correlation function

$$G(K,\mathbf{r},t) \equiv \langle s_i(t)s_j(0)\rangle ,$$

where $\mathbf{r}$ is the lattice vector connecting sites i and j, and the average is defined as

$$\langle s_i(t)s_j(0)\rangle = \sum_{\{s\}} s_j P(\{s\}) \sum_{\{s'\}} s_i' P(\{s'\},t \mid \{s\},0) .$$

In this expression, $P(\{s'\},t \mid \{s\},0)$ is the probability of the configuration $\{s'\}$ at time t, if $\{s\}$ is the configuration at $t=0$.

For the dynamic correlation function, the following scaling law applies:[58]

$$G[L^{y_T}(K-K_c), r/L, t/L^z] = L^{d-2+\eta} G(K-K_c, r, t) , \tag{2.1}$$

for $K \cong K_c$, $r \to \infty$, and $t \to \infty$. This law also holds for $t=0$ where it reduces to the static form (1.4). A time-independent scaling law can be obtained by taking the integral over t of (2.1), taking into account that the dominant contribution comes from the long-time domain provided $K \cong K_c$ (critical slowing down). One obtains

$$\Lambda[L^{y_T}(K-K_c), r/L] = L^{d-2+\eta-z}\Lambda(K-K_c, r) \tag{2.2}$$

for $K \cong K_c$ and $r \to \infty$. Here, we have defined a relaxation time Λ as

$$\Lambda(K,\mathbf{r}) \equiv \int_0^\infty G(K,\mathbf{r},t)dt . \tag{2.3}$$

Finally, a scaling law for a (normalized) relaxation time

$$\tau(K,\mathbf{r}) \equiv \Lambda(K,\mathbf{r})/G(K,\mathbf{r})$$

follows from (2.3) and (1.4):

$$\tau[L^{y_T}(K-K_c), r/L] = L^{-z}\tau(K-K_c, r) \tag{2.4}$$

for $K \cong K_c$ and $r \to \infty$.

Now we proceed to interpret (2.2) or (2.4) as the implicit definition of a RG map $K \to K'$, which, just as in the static RG, takes the following form in the neighborhood of the critical point:

$$K' - K_c = L^{y_T}(K-K_c) .$$

The only difference with the static RG procedure is that the role of the exponent η is now played by the combination $\eta - z$ (working with Λ) or by $-z$ (working with τ).

In analogy with the statics, we proceed to compute Λ and τ at short distances in a high-temperature-series expansion using one or two couplings. Using the standard formalism for the Glauber model,[37,59] one derives

$$G(K,\mathbf{r},t) = \sum_{\{s\}} s_j \{e^{-\mathscr{L}t}s_i\} P(\{s\})$$

and

$$\Lambda(K,\mathbf{r})=\sum_{\{s\}}s_j\{\mathscr{L}^{-1}s_i\}P(\{s\}) ,$$

where

$$\mathscr{L}\equiv\sum_j W_j(\{s\})(1-p_j) ,$$

with spin-flip "operators" p_j which act as follows on a function A of the spins:

$$p_jA(s_1,s_2,\ldots,s_{j-1},s_j,s_{j+1},\ldots)=A(s_1,s_2,\ldots,s_{j-1},-s_j,s_{j+1},\ldots) .$$

Consequently, Λ can be expressed as a sum of static (multi)spin correlations.

C. Application to the square Glauber model with nearest-neighbor and diagonal interactions

Our first step is to derive two-coupling high-temperature-series expansions for an appropriate set of relaxation times, in the form

$$\Lambda(\mathbf{v},\mathbf{r})\cong\sum_{m=0}^{8}\sum_{n=0}^{4}a_{mn}(\mathbf{r})v^m w^n ,$$

where $m+2n\le 8$ and, as before, $v=\tanh K_1$ and $w=\tanh K_2$. We expand to eighth order in v, but may restrict the expansion to fourth order in w, because at the ferromagnetic fixed point w is typically of order v^2. This can be checked, for example, by inspection of the fixed points in our static RG calculation, where we worked consistently to ninth order in *both* v and w (for details, see Ref. 18).

The technique we employ for calculating the high-T series makes use of the formalism outlined by Yahata and Suzuki.[59] For the transition rate W, we choose a standard form:

$$W_j(\{s\})=\frac{1}{2}\left[1-s_j\tanh\left[K_1\sum_k^{\mathrm{NN}}s_k\right]\right]\times\left[1-s_j\tanh\left[K_2\sum_l^{\mathrm{NNN}}s_l\right]\right] ,$$

where $k(l)$ labels the (next-) nearest neighbors of s_j. The calculation relies on the high-T expansion for the operator $\mathscr{L}^{-1}$:

$$\mathscr{L}^{-1}=\sum_{k=0}^{\infty}\mathscr{L}_0^{-1}(V\mathscr{L}_0^{-1})^k ,$$

where $\mathscr{L}_0=\mathscr{L}(v=w=0)$ and $V=\mathscr{L}_0-\mathscr{L}$ ($V\sim 1/T$ for $T\to\infty$). When $\mathscr{L}^{-1}$ acts on a spin s_j, numerous "clusters" of spins are generated through the action of V and its powers. Finally, the thermal average of $s_i\mathscr{L}^{-1}s_j$ is obtained in the form

$$\Lambda(\mathbf{v},\mathbf{r})\cong\sum_{j=1}^{m}r_jf_j(\mathbf{v})\sum_{\{s\}}\left[\prod_{i=1}^{n_j}s_{j_i}\right]P(\{s\}) ,$$

where m is the number of terms relevant to eighth order in v and fourth order in w, r_j is a rational number, f_j a function of v and w which results from the action of powers of V, and $n_j=0$, 2, 4, or 6 is the number of spins in the static multispin correlation ($n_j=0$ corresponds to a factor 1). The calculation of these multispin correlations has to be done to seventh order (for $n_j=2$), fifth order (for $n_j=4$), and third order only (for $n_j=6$), using standard techniques.[18,19] To carry out the final summation which yields Λ, we have employed the REDUCE (Ref. 60) program for symbol manipulation.

The coefficients of the relaxation-time series expansions are tabulated in Appendix D. Proceeding toward the construction of RG transformations, we work in a first approximation with one interaction (K) only.

Both scaling laws (2.2) and (2.4) are suitable for implementing the RG. We start with (2.2) and define implicit RG maps through

$$\Lambda(K',\mathbf{r}_1)=L^{\eta-z}\Lambda(K,\mathbf{r}_L)$$

for $L=\sqrt{2}$ and 2, and using eighth-order series expansions and the associated Padé approximants for Λ.

In complete analogy with the static methods in Sec. I C, we also construct RG-map series. These are explicitly given, for $L=\sqrt{2}$, by

$$v'=3xv^2[1+10v^2+(77-99x^2)v^4+(\tfrac{1696}{3}-2970x^2)v^6+\cdots] ,$$

and, for $L=2$, by

$$v'=\tfrac{3}{2}xv^2[1+50v^2/3+(\tfrac{436}{3}-99x^2/4)v^4+(\tfrac{3250}{3}-2475x^2/2)v^6+\cdots] ,$$

where $x=L^{\eta-z}$ and $v=\tanh K$.

The analysis of the implicit RG maps yields qualitatively reasonable results for K_c and y_T, but unphysical (negative) values for z. (Actually, z is obtained indirectly from $\eta-z$ after substituting the exact value $\frac{1}{4}$ for η.) The analysis of the RG-map series, however, is more satisfactory. Taking $\eta=\frac{1}{4}$, we obtain the estimates for z shown in Table XXVIII (cf. the expected value $z\cong 2$). The results for the critical point K_c range between 0.221 and 0.351 (cf. the exact value 0.441). The estimates for y_T are poor, except for $N=8$, $D=0$, and $L=\sqrt{2}$, where $y_T=0.94$ is found (the exact value is 1).

The scaling law (2.4) provides different RG maps defined through

$$\tau(K',\mathbf{r}_1)=L^{-z}\tau(K,\mathbf{r}_L) ,$$

TABLE XXVIII. Padé table for the exponent z in the square Glauber model from RG-map series in one interaction. The expected value is around 2.

N \ D	0	2	4	6
2	/	0.85	/	/
4	/	6.65	/	
6	/	2.67		
8	1.43			

TABLE XXIX. Padé table for the exponent z in the square Glauber model from implicit RG maps with two interactions. The expected value is around 2.

$N \backslash D$	0	2	4
4	<0	1.06	/
5	/	0.18	
6	0.24	0.55	
7	/		
8	1.45		

for $L=\sqrt{2}$ and 2, and using high-T-series expansions for τ derived from series for Λ and G. Explicitly, we obtain

$$\tau(K,\mathbf{r}_1)=2(1+9v^2+65v^4+1387v^6/2+\cdots),$$

$$\tau(K,\mathbf{r}_{\sqrt{2}})=3(1+8v^2+56v^4+1192v^6/3+\cdots),$$

and

$$\tau(K,\mathbf{r}_2)=3+32v^2+196v^4+1424v^6+\cdots .$$

Due to the constant terms in the expansions, we cannot construct RG-map series and perform only the analysis of the implicit RG maps. The estimates for z are systematically too small (ranging between 0.57 and 1.05). The results for K_c are reasonable (ranging between 0.212 and 0.362, and improving as N and D increase). The estimates for y_T are poor, and in many cases unphysical ($y_T<0$).

In a second approximation we introduce an additional next-nearest-neighbor interaction. Starting from scaling law (2.2), the RG transformations are defined through the system of equations

$$\Lambda(\mathbf{v}',\mathbf{r}_1)=L^{\eta-z}\Lambda(\mathbf{v},\mathbf{r}_L),$$
$$\Lambda(\mathbf{v}',\mathbf{r}_{\sqrt{2}})=L^{\eta-z}\Lambda(\mathbf{v},\mathbf{r}_{\sqrt{2}L}), \tag{2.5}$$

for $L=\sqrt{2}$ and 2, using series expansions and Padé approximants to eighth order in $v=\tanh K_1$ and fourth order in $w=\tanh K_2$.

The analysis of the implicit RG maps yields Table XXIX for z (assuming $\eta=\frac{1}{4}$). Although these estimates are not yet close to the expected value $z\cong 2$, they are certainly much better than the unphysical ones ($z<0$) found in the corresponding analysis using only one interaction. The results for K_c and y_T are poor, except for $N=8$ and $D=0$, where also z is best. We find in this case $K_c=0.495$ ($L=\sqrt{2}$) and 0.462 ($L=2$), and $y_T=1.38$ ($L=\sqrt{2}$) and 1.09 ($L=2$). The values for $\delta_{\min}$ are quite large (e.g., 27% in the case $N=8$, $D=0$).

TABLE XXX. Padé table for the exponent z in the square Glauber model from RG-map series in two interactions. The expected value is around 2.

$N \backslash D$	0	2	4
4	/	0.33	/
6	0.90	1.05	
8	1.73		

TABLE XXXI. Padé table for $\delta_{\min}(\%)$ in the RG-map series for the square Glauber model with two interactions.

$N \backslash D$	0	2	4
4	/	3.4	/
6	6.2	4.5	
8	5.6		

Next, we construct, following the method outlined in Appendix B, the RG-map series compatible with the system (2.5) in the form

$$v'\cong\sum_{m+2n=1}^{8}c_{mn}(\eta-z)v^m w^n ,$$

and similarly a series for w' with coefficients $d_{mn}(\eta-z)$. The coefficients are listed in Appendix E.

The estimates for z (assuming $\eta=\frac{1}{4}$) are presented in Table XXX. The group-structure optimalization was quite successful, as the low values for $\delta_{\min}$ in Table XXXI illustrate. We have checked[18] at the fixed points in our calculations that w is indeed of order v^2. Therefore, our approximate expansion to eighth order in v and only fourth order in w is self-consistent. Results for the critical point $K_{1c}\equiv K_c$ are given in Table XXXII: below one another are K_c for $L=\sqrt{2}$ and 2. The estimates for y_T are systematically too large (ranging between 1.56 and 2.22). Altogether, the present analysis is better than the corresponding one with only one interaction, in spite of the poor values for y_T.

In a last application, we implement the RG maps using the scaling law (2.4) for the relaxation time τ. The RG equations now read

$$\tau(\mathbf{v}',\mathbf{r}_1)=L^{-z}\tau(\mathbf{v},\mathbf{r}_L),$$
$$\tau(\mathbf{v}',\mathbf{r}_{\sqrt{2}})=L^{-z}\tau(\mathbf{v},\mathbf{r}_{\sqrt{2}L})$$

for $L=\sqrt{2}$ and 2. Here, τ is a *ratio* of series expansions (in two couplings) for Λ and G. This ratio can, in general, not be written as an expansion. Instead of the Padé approximants for τ, we use the ratios of Padé approximants

TABLE XXXII. Padé table for the critical point K_c in the square Glauber model from RG-map series in two interactions. The exact value is 0.441. Below one another for given N and D: $K_c(L=\sqrt{2})$ and $K_c(L=2)$.

$N \backslash D$	0	2	4
4	/	0.241 0.255	/
6	0.442 0.293	0.326 0.302	
8	0.421 0.331		

for Λ and G. This leads to a complicated analysis with ill-behaved RG maps. A few reasonable values for z can be obtained.[18]

III. CONCLUDING REMARKS

The methods presented in this work constitute an original and efficient way of combining the modern renormalization-group strategy with the technique of series expansions already in use for decades in the study of critical phenomena. Even if these new methods of series analysis cannot yet compete in accuracy with the more traditional approaches, for many problems (such as the two-dimensional XY model discussed here) they certainly can play an important complementary role.

Besides the practical advantages, the results presented here have some general implications worth mentioning. They suggest that the success of a real-space renormalization-group strategy is connected with the existence of nearly asymptotic scaling properties of the correlation functions at short distances (see, e.g., Table I). The absence of such properties in a given model would make a renormalization-group approach (based as usual on drastic truncation of the interaction space) completely powerless. It is the scaling at short distances which entitles us to represent the exact fixed point approximately in a low-dimensional parameter space.

The dynamic renormalization approach presented here has appealing features due to its phenomenological character: memory effects as well as proliferation of interactions are avoided. Approximations sensibly improve when an additional interaction is included. However, the degree of accuracy of the results is not yet satisfactory, indicating that short-distance scaling is less pronounced in the dynamic than in the static properties of the correlation functions in the Ising model.

Systematic improvement of numerical accuracy for both statics and dynamics should be expected when the number of interactions on the lattice is increased. In particular, the inclusion of a third interaction being either the four-spin coupling or the third-nearest-neighbor coupling would be meaningful and could perhaps be technically feasible still. Another direction of systematic improvement would, of course, be to go to higher orders in the high-temperature-series expansions, but our experience in the static renormalization indicates that this may be less rewarding than including additional interactions. One should realize, however, that both ways of improving accuracy would considerably increase computational burden.

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APPENDIX A: HIGH-TEMPERATURE SERIES FOR CORRELATION FUNCTIONS IN TWO INTERACTION VARIABLES

The following Tables XXXIII through XXXIX show the series coefficients calculated for seven pair-correlation functions $G(\mathbf{v},\mathbf{r})$ in the square Ising model with two interactions (nearest-neighbor and diagonal) which are denoted by $\mathbf{v}=(v,w)=(\tanh K_1, \tanh K_2)$.

APPENDIX B: ALGEBRAIC CONSTRUCTION OF RG SERIES EXPANSIONS

Here we present the algebraic manipulations which underlie the construction of the series expansions for the renormalized couplings. We restrict ourselves to the case of two variables (v and w).

Consider the equations

$$\sum_{m+n=1}^{\infty} a'_{mn} v'^m w'^n = x \sum_{m+n=1}^{\infty} a_{mn} v^m w^n \,,$$
$$\sum_{m+n=1}^{\infty} b'_{mn} v'^m w'^n = x \sum_{m+n=1}^{\infty} b_{mn} v^m w^n \,, \tag{B1}$$

where all coefficients are known and real, and x is an unknown parameter [e.g., $x = L^{d-2+\eta}$ as in (1.11)]. The following solution is proposed:

TABLE XXXIII. Coefficients of $G(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(1,0)$.

	1	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8	w^9
1	0	0	0	0	0	0	0	0	0	0
v	1	4	10	28	72	188	482	1236	3140	
v^2	0	0	0	0	0	0	0	0		
v^3	2	12	58	244	946	3476	12234			
v^4	0	0	0	0	0	0				
v^5	4	44	328	1988	10598					
v^6	0	0	0	0						
v^7	12	196	1988							
v^8	0	0								
v^9	42									

TABLE XXXIV. Coefficients of $G(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(2,0)$.

	1	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8	w^9
1	0	0	2	0	4	0	10	0	32	0
v	0	0	0	0	0	0	0	0	0	0
v^2	1	12	48	172	558	1732	5160	15012		
v^3	0	0	0	0	0	0	0			
v^4	6	56	318	1536	6672	27040				
v^5	0	0	0	0	0					
v^6	16	200	1620	10712						
v^7	0	0	0							
v^8	46	816								
v^9	0									

TABLE XXXV. Coefficients of $G(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(3,0)$.

	1	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8	w^9
1	0	0	0	0	0	0	0	0	0	0
v	0	0	6	24	66	192	486	1272	3204	
v^2	0	0	0	0	0	0	0	0		
v^3	1	24	174	808	3248	11824	40534			
v^4	0	0	0	0	0	0				
v^5	12	180	1332	7508	36756					
v^6	0	0	0	0						
v^7	48	756	6816							
v^8	0	0								
v^9	152									

TABLE XXXVI. Coefficients of $G(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(4,0)$.

	1	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8	w^9
1	0	0	0	0	6	0	24	0	76	0
v	0	0	0	0	0	0	0	0	0	
v^2	0	0	12	120	504	1896	6236	19440		
v^3	0	0	0	0	0	0	0			
v^4	1	40	484	3128	15532	66984				
v^5	0	0	0	0	0					
v^6	20	456	4564	31200						
v^7	0	0	0							
v^8	118	2432								
v^9	0									

TABLE XXXVII. Coefficients of $G(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(6,0)$.

	1	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8	w^9
1	0	0	0	0	0	0	20	0	120	0
v	0	0	0	0	0	0	0	0	0	
v^2	0	0	0	0	90	840	3720	14760		
v^3	0	0	0	0	0	0	0			
v^4	0	0	30	840	8640	54120				
v^5	0	0	0	0	0					
v^6	1	84	2220	27828						
v^7	0	0	0							
v^8	42	1872								
v^9	0									

$$v'=\sum_{k+l=1}^{\infty} c_{kl}(x)v^k w^l\,,$$
$$w'=\sum_{k+l=1}^{\infty} d_{kl}(x)v^k w^l\,. \tag{B2}$$

The coefficients c_{kl} and d_{kl} are determined in order that (B1) may be an identity in every order in v and w, as follows. After substitution of (B2) in (B1), the first equation in (B1) can be written as

$$\sum_{m+n=1}^{\infty} a'_{mn} \sum_{k,l=0}^{\infty} \tilde{c}^{\,m}_{kl}(x)v^k w^l \sum_{p,q=0}^{\infty} \tilde{d}^{\,n}_{pq}(x)v^p w^q = x\sum_{m+n=1}^{\infty} a_{mn}v^m w^n\,. \tag{B3}$$

The second equation in (B1) can be written similarly, replacing a by b. Here, the $\tilde{c}^{\,m}_{kl}$ and $\tilde{d}^{\,m}_{kl}$ are defined as follows: For $k+l>0$,

$$\tilde{c}^{\,m}_{kl}(x)=\sum_{\{(i_t,j_t)\}^m_{kl}} \left[\prod_{t=1}^{m} c_{i_t j_t}(x)\right] m!\Big/\prod_{(i_t,j_t)}(n_{i_t,j_t}!)\,, \tag{B4}$$

and $\tilde{d}^{\,m}_{kl}(x)$ is analogous, replacing c by d. The summation in (B4) is over all sets $\{(i_t,j_t)\}^m_{kl}$ of m couples (i_t,j_t) of natural numbers, which satisfy

$$\{(i_t,j_t)\}^m_{kl}=\left\{(i_t,j_t)\in\{0,1,\ldots,k\}\times\{0,1,\ldots,l\};\ t=1,2,\ldots,m;\ \sum_{t=1}^{m} i_t=k;\ \sum_{t=1}^{m} j_t=l \text{ and } i_t+j_t>0\right\}.$$

The integer $n_{(i_t,j_t)}$ is the number of times that the couple (i_t,j_t) occurs in the set $\{(i_t,j_t)\}^m_{kl}$. On the other hand, for $k=l=0$,

$$\tilde{c}^{\,m}_{00}(x)=\tilde{d}^{\,m}_{00}(x)=\delta_{m,0}\,. \tag{B5}$$

An equation like (B3) can be rearranged to become

$$\sum_{s+t=1}^{\infty} v^s w^t \sum_{m+n=1}^{\infty} a'_{mn}\sum_{k=0}^{s}\sum_{l=0}^{t}\tilde{c}^{\,m}_{kl}(x)\tilde{d}^{\,n}_{s-k,t-l}(x)=x\sum_{s+t=1}^{\infty} v^s w^t a_{st}\,,$$

where $s-k\geq 0$ and $t-l\geq 0$.

Finally, we obtain a nonlinear system in the coefficients c_{kl} and d_{kl} by writing the equations

$$xa_{st}=\sum_{m+n=1}^{\infty} a'_{mn}\sum_{k,l=0}^{\infty}\tilde{c}^{\,m}_{kl}(x)\tilde{d}^{\,n}_{s-k,t-l}(x),\quad xb_{st}=\sum_{m+n=1}^{\infty} b'_{mn}\sum_{k,l=0}^{\infty}\tilde{c}^{\,m}_{kl}(x)\tilde{d}^{\,n}_{s-k,t-l}(x)$$

for all s and t, after substituting (B4) and (B5). The following equations then result:

$$xa_{10}=a'_{01}d_{10}(x)+a'_{10}c_{10}(x),\quad xa_{01}=a'_{01}d_{01}(x)+a'_{10}c_{01}(x),\quad xb_{10}=\cdots,\quad xb_{01}=\cdots\,,$$

$$xa_{11}=a'_{01}d_{11}(x)+a'_{10}c_{11}(x)+2a'_{01}d_{10}(x)d_{01}(x)+2a'_{20}c_{10}(x)c_{01}(x)+a'_{11}[c_{10}(x)d_{01}(x)+c_{01}(x)d_{10}(x)]\,,$$

etc. These equations can be generated by, e.g., a FORTRAN program. Then, the unknowns c_{kl} and d_{kl} which are polynomials in x are found analytically with the help of the REDUCE (Ref. 60) program for symbol manipulation. After finding the solutions up to a given order M, they are substituted in (B1) as a check. Equation (B1) should then be identically satisfied *up to order M*.

It is important to note that in our cases several a'_{kl} and b'_{kl} are zero. As a consequence, the equations above can be solved one by one and the solutions are real and unique.

TABLE XXXVIII. Coefficients of $G(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(1,1)$.

	1	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8	w^9
1	0	1	0	2	0	4	0	12	0	42
v	0	0	0	0	0	0	0	0	0	
v^2	2	10	40	134	428	1302	3856	11 114		
v^3	0	0	0	0	0	0	0			
v^4	4	34	200	984	4376	18 056				
v^5	0	0	0	0	0					
v^6	10	132	1120	7612						
v^7	0	0	0							
v^8	32	596								
v^9	0									

TABLE XXXIX. Coefficients of $G(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(2,2)$.

	1	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8	w^9
1	0	0	1	0	6	0	16	0	46	0
v	0	0	0	0	0	0	0	0	0	
v^2	0	6	36	166	584	1912	5856	17 376		
v^3	0	0	0	0	0	0	0			
v^4	6	76	518	2644	11 548	46 104				
v^5	0	0	0	0	0					
v^6	24	342	2780	17 510						
v^7	0	0	0							
v^8	76	1328								
v^9	0									

TABLE XL. Coefficients of $\Lambda(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=\mathbf{0}$ (autorelaxation time).

	1	w	w^2	w^3	w^4
1	1	0	8	0	64
v^2	8	64	$\frac{1120}{3}$	$\frac{17\,528}{9}$	
v^4	64	912	$\frac{74\,368}{9}$		
v^6	$\frac{1480}{3}$	$\frac{87\,952}{9}$			
v^8	$\frac{91\,808}{27}$				

TABLE XLI. Coefficients of $\Lambda(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(1,0)$.

	1	w	w^2	w^3
v	2	12	56	228
v^3	22	252	$\frac{5530}{3}$	
v^5	174	$\frac{8936}{3}$		
v^7	$\frac{3842}{3}$			

TABLE XLII. Coefficients of $\Lambda(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(2,0)$.

	1	w	w^2	w^3	w^4
1	0	0	6	0	60
v^2	3	48	312	$\frac{5192}{3}$	
v^4	50	816	$\frac{205\,910}{27}$		
v^6	436	$\frac{27\,424}{3}$			
v^8	3250				

TABLE XLIII. Coefficients of $\Lambda(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(1,1)$.

	1	w	w^2	w^3	w^4
1	0	2	0	22	0
v^2	6	56	$\frac{1094}{3}$	$\frac{5464}{3}$	
v^4	60	884	$\frac{72\,232}{9}$		
v^6	462	$\frac{256\,808}{27}$			
v^8	3392				

TABLE XLIV. Coefficients of $\Lambda(\mathbf{v},\mathbf{r})$ for $\mathbf{r}=(2,2)$.

	1	w	w^2	w^3	w^4
1	0	3	0	0	50
v^2	0	24	204	$\frac{12\,116}{9}$	
v^4	30	624	$\frac{175\,790}{27}$		
v^6	360	$\frac{222\,064}{27}$			
v^8	$\frac{79\,976}{27}$				

APPENDIX C: HIGH-TEMPERATURE SERIES FOR STATIC RG TRANSFORMATIONS

Presented here are the *nonzero* coefficients c_{kl} and d_{kl} of the series expansions for the renormalized couplings

$$v' \cong \sum_{k+l=1}^{9} c_{kl}(x) v^k w^l \quad \text{and} \quad w' \cong \sum_{k+l=1}^{9} d_{kl}(x) v^k w^l ,$$

where

$$v = \tanh K_1, \quad w = \tanh K_2, \quad \text{and } x = L^{\eta} \; (d=2) .$$

For $L=\sqrt{2}$,

$$c_{01}=x,\ c_{20}=2x,\ d_{20}=x,\ d_{02}=2x(1-x),\ c_{21}=2x(5-2x),$$

$$c_{03}=2x(3x^2-4x+1),\ d_{21}=4x(3-2x),\ c_{40}=4x(1-2x),\ c_{22}=4x(9x^2-16x+10),$$

$$d_{40}=2x(3-4x),\ d_{22}=2x(3x^2-20x+24),\ d_{04}=4x(-2x^3+3x^2-2x+1),$$

$$c_{41}=2x(39x^2-80x+17),\ c_{23}=2x(-14x^3+102x^2-140x+67),$$

$$c_{05}=4x(6x^4-14x^3+15x^2-8x+1),\ d_{41}=8x(3x^2-12x+7),$$

$$d_{23}=4x(-16x^3+30x^2-44x+43),\ c_{60}=2x(30x^2-32x+5),$$

$$c_{42}=8x(-21x^3+123x^2-160x+25),\ c_{24}=4x(60x^4-168x^3+300x^2-284x+107),$$

$$d_{60}=2x(11x^2-16x+8),\ d_{42}=2x(-98x^3+240x^2-328x+159),$$

$$d_{24}=2x(17x^4-168x^3+264x^2-308x+279),\ d_{06}=2x(-15x^5+34x^4-40x^3+28x^2-12x+5)$$

$$c_{61}=4x(-85x^3+357x^2-270x+33),\ c_{43}=6x(169x^4-620x^3+1308x^2-1224x+164),$$

$$c_{25}=2x(-58x^5+708x^4-1620x^3+2310x^2-1932x+651),$$

$$c_{07}=4x(15x^6-58x^5+114x^4-120x^3+72x^2-26x+3),\ d_{61}=8x(-34x^3+81x^2-59x+25),$$

$$d_{43}=16x(17x^4-138x^3+234x^2-219x+96),$$

$$d_{25}=4x(-90x^5+238x^4-480x^3+546x^2-516x+433),\ c_{80}=8x(-29x^3+57x^2-33x+4),$$

$$c_{62}=4x(561x^4-2344x^3+4257x^2-2456x+280),$$

$$c_{44}=8x(-145x^5+1584x^4-3976x^3+5790x^2-4428x+547),$$

$$c_{26}=8x(105x^6-464x^5+1326x^4-2026x^3+2181x^2-1580x+482),$$

$$d_{80}=2x(-72x^3+102x^2-56x+23),\ d_{62}=2x(409x^4-2592x^3+3816x^2-2064x+810),$$

$$d_{44}=4x(-467x^5+1618x^4-4140x^3+5277x^2-4048x+1668),$$

$$d_{26}=4x(39x^6-518x^5+1162x^4-1848x^3+1875x^2-1610x+1290),$$

$$d_{08}=4x(-21x^7+78x^6-158x^5+174x^4-112x^3+51x^2-20x+8),$$

$$c_{81}=4x(643x^4-2452x^3+2829x^2-1286x+149),$$

$$c_{63}=4x(-1175x^5+11\,327x^4-28\,536x^3+35\,169x^2-16\,916x+1903),$$

$$c_{45}=4x(1298x^6-7514x^5+27\,885x^4-50\,496x^3+57\,756x^2-38\,230x+4514),$$

$c_{27}=2x(-6x^7+2404x^6-9524x^5+22\,192x^4-29\,932x^3+28\,866x^2-19\,440x+5557)$,

$c_{09}=2x(-61x^8-12x^7+724x^6-1632x^5+1832x^4-1276x^3+588x^2-184x+21)$,

$d_{81}=8x(137x^4-608x^3+603x^2-266x+102)$,

$d_{63}=8x(-668x^5+2803x^4-6852x^3+7395x^2-3502x+1339)$,

$d_{45}=16x(117x^6-1408x^5+3509x^4-5966x^3+6288x^2-4259x+1690)$,

$d_{27}=4x(-336x^7+1404x^6-4060x^5+6174x^4-7120x^3+6210x^2-4896x+3753)$.

For $L=2$,

$c_{20}=x$, $c_{02}=2x$, $d_{02}=x$, $c_{21}=12x$, $d_{21}=6x$, $c_{40}=6x$, $c_{22}=4x(12-x)$,

$c_{04}=4x(1-2x)$, $d_{40}=2x(3-x)$, $d_{22}=4x(9-2x)$, $d_{04}=2x(3-4x)$,

$c_{41}=8x(7-3x)$, $c_{23}=4x(43-24x)$, $d_{41}=4x(19-12x)$, $d_{23}=2x(83-48x)$,

$c_{60}=2x(3x^2-12x+8)$, $c_{42}=6x(6x^2-84x+53)$, $c_{24}=6x(13x^2-84x+93)$,

$c_{06}=2x(30x^2-32x+5)$, $d_{60}=24x(1-x)$, $d_{42}=2x(3x^2-264x+259)$,

$d_{24}=8x(3x^2-50x+73)$, $d_{06}=2x(11x^2-16x+8)$, $c_{61}=8x(27x^2-92x+25)$,

$c_{43}=8x(117x^2-547x+192)$, $c_{25}=4x(270x^2-600x+433)$, $d_{61}=2x(18x^2-256x+171)$,

$d_{43}=4x(72x^2-860x+661)$, $d_{25}=4x(99x^2-392x+478)$, $c_{80}=2x(54x^2-120x+23)$,

$c_{62}=4x(-7x^3+1044x^2-2334x+405)$, $c_{44}=12x(-14x^3+921x^2-2250x+556)$,

$c_{26}=4x(-85x^3+1242x^2-2174x+1290)$, $c_{08}=8x(-29x^3+57x^2-33x+4)$,

$d_{80}=4x(-2x^3+9x^2-34x+19)$, $d_{62}=4x(-16x^3+324x^2-1310x+695)$,

$d_{44}=4x(-49x^3+1026x^2-4434x+2887)$, $d_{26}=8x(-34x^3+249x^2-659x+732)$,

$d_{08}=2x(-72x^3+102x^2-56x+23)$, $c_{81}=24x(-7x^3+168x^2-253x+34)$,

$c_{63}=8x(-252x^3+6459x^2-9672x+1339)$, $c_{45}=8x(-765x^3+9753x^2-16\,946x+3380)$,

$c_{27}=4x(-1392x^3+5718x^2-7504x+3753)$, $d_{81}=8x(-48x^3+219x^2-364x+166)$,

$d_{63}=2x(-1176x^3+9954x^2-18\,944x+8755)$, $d_{45}=24x(-204x^3+1313x^2-3330x+1921)$,

$d_{27}=12x(-288x^3+797x^2-1424x+1448)$.

APPENDIX D: HIGH-TEMPERATURE SERIES FOR RELAXATION TIMES IN TWO INTERACTION VARIABLES

Tables XL through XLIV show the series coefficients calculated for five relaxation times $\Lambda(\mathbf{v},\mathbf{r})$ in the square Glauber model with two interactions (nearest-neighbor and diagonal) which are denoted by $\mathbf{v}=(v,w)=(\tanh K_1,\tanh K_2)$. Trivial zeros are not shown.

APPENDIX E: HIGH-TEMPERATURE SERIES FOR DYNAMIC RG TRANSFORMATIONS

Presented here are the *nonzero* coefficients c_{kl} and d_{kl} of the series expansions for the renormalized couplings

$$v'\cong\sum_{k+2l=1}^{8}c_{kl}(x)v^kw^l \quad\text{and}\quad w'\cong\sum_{k+2l=1}^{8}d_{kl}(x)v^kw^l ,$$

where $v=\tanh K_1$, $w=\tanh K_2$, and $x=L^{\eta-z}$ $(d=2)$.

For $L=\sqrt{2}$,

$c_{01}=x$, $c_{20}=3x$, $d_{20}=3x/2$, $d_{02}=3x(1-x)$, $c_{21}=x(28-9x)$,

$c_{03}=x(7x^2-18x+11)$, $d_{21}=6x(4-3x)$, $c_{40}=3x(10-9x)$, $c_{22}=x(63x^2-198x+\frac{547}{3})$,

$d_{40}=x(25-27x)$, $d_{22}=12x(x^2-14x+13)$, $d_{04}=6x(2x^3+4x^2-11x+5)$,

$c_{41}=x(207x^2-834x+442)$, $c_{23}=x(63x^3+660x^2-1539x+\frac{2732}{3})$,

$$d_{41}=12x(6x^2-57x+34),\quad d_{23}=4x(108x^3+252x^2-969x+649)/3\,,$$
$$c_{60}=3x(81x^2-240x+77),\quad c_{42}=x(5103x^3+44\,550x^2-103\,221x+36\,116)/9\,,$$
$$d_{60}=x(567x^2-4320x+1744)/8,\quad d_{42}=x(93\,474x^3+217\,863x^2-894\,888x+411\,820)/108\,,$$
$$c_{61}=2x(26\,244x^3+160\,704x^2-285\,525x+64\,202)/27\,,$$
$$d_{61}=x(7803x^3+11\,646x^2-43\,146x+13\,712)/3,\quad c_{80}=x(2430x^3+8010x^2-10\,503x+1696)\,,$$
$$d_{80}=x(11\,718x^3+8415x^2-27\,432x+6500)/4\,.$$

For $L=2$,

$$c_{20}=3x/2,\quad c_{02}=3x,\quad d_{02}=3x/2,\quad c_{21}=24x,\quad d_{21}=12x,\quad c_{40}=25x\,,$$
$$c_{22}=3x(52-9x/2),\quad c_{04}=3x(10-9x),\quad d_{40}=3x(5-9x/4),\quad d_{22}=3x(34-9x)\,,$$
$$d_{04}=x(25-27x),\quad c_{41}=12x(34-9x),\quad c_{23}=4x(649-324x)/3,\quad d_{41}=24x(13-9x)\,,$$
$$d_{23}=2x(3029-1944x)/9,\quad c_{60}=x(189x^2-1080x+1744)/8\,,$$
$$c_{42}=x(15\,309x^2-339\,228x+411\,820)/108,\quad d_{60}=45x(4-5x)\,,$$
$$d_{42}=x(729x^2-96\,714x+87\,895)/27,\quad c_{61}=2x(1701x^2-10\,152x+6856)/3\,,$$
$$d_{61}=8x(729x^2-24\,543x+13\,879)/27,\quad c_{80}=5x(945x^2-3096x+1300)/4\,,$$
$$d_{80}=x(6561x^3+29\,160x^2-414\,396x+159\,952)/108\,.$$

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