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Nonasymptotic critical behavior from field theory at $d=3$: The disordered-phase case

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The theoretical framework and the details of calculations of nonasymptotic critical behavior, above T_c, previously published for $n=1$, are presented and extended to the cases $n=2$ and 3. The complete description of the real preasymptotic critical domain $\mathscr{D}_{\text{preas}}$ needs only three adjustable parameters. We show that the ϕ^4 model at infinite cutoff is sufficient to obtain this description. The higher-transient and finite-cutoff effects in $\mathscr{D}_{\text{preas}}$ affect only the adjustable parameters. A precise nonperturbative treatment of the ϕ^4 field theory exactly for $d=3$, in the spirit of the Parisi work, yields nonasymptotic functions of temperature which, including all the quantitative universal characteristics of the critical behavior in $\mathscr{D}_{\text{preas}}$, are adapted to a suitable comparison with experiments. We emphasize that this work allows a determination, within the experimental accuracy, of the size of $\mathscr{D}_{\text{preas}}$ and an appreciation of the effects of the higher corrections to scaling. We also give the estimates for $n=1$, 2, and 3 of a new universal combination of amplitudes ($R_{B_{\text{cr}}}$) which concerns the specific heat alone.

I. INTRODUCTION

In two previous papers[1,2] we presented the theoretical results for precise nonasymptotic critical behavior obtained with the pure scalar ($n=1$) ϕ^4 model exactly for $d=3$. We proposed explicit theoretical functions of temperature for the complete set of measurable singular quantities (the correlation length ξ, the susceptibility χ, and the specific heat C) in the disordered phase along the critical isochore. These functions exhibit, as they must,[3] a crossover between the Wilson-Fisher (near the critical temperature T_c) and mean-field (very far from T_c) behaviors which seems like that observed in real systems.[4] However, it follows clearly that this crossover is not truly realistic because of the physical limitations of the model used. We emphasized, in agreement with the general statements of the Wilson renormalization-group (RG) approach,[5] the conditions of applicability of our nonasymptotic functions for an improved analysis of experimental data. The use of a minimal set of adjustable parameters, inherent to the physics of critical phenomena, associated with the great accuracy of the work, allow an estimate—within the experimental accuracy—of the temperature domains corresponding, respectively, to the asymptotic (pure lower law) and preasymptotic (first confluent correction) regimes. Indeed the control of the convergence of the Wegner expansion[6] (power-law expansion) that we have obtained, allows an illustration (see Fig. 4 of the present paper) of the cascade structure[5,7] induced by the progressive relevance of the various degrees of freedom as one moves far away from T_c. Within the ϕ^4 model used, only the first two steps are completely reproduced.

To a certain extent we think that we grant the experimentalist's wish which has been well expressed by Debye:[8] "I would like the theoretical people to tell me when I am so and so far away from the critical point then my curve should look so and so." Indeed, it is very difficult (approach of T_c, gravitational effects) to measure the critical singularities in the very vicinity of the critical point (CP) where the pure scaling behavior is dominant. Therefore, early on experimentalists[9] were led to introduce corrections to scaling[10] in fitting their data farther from T_c, where, in particular, the gravitational effects are negligible. However, a question then arises:[4] To what distance from T_c can such a power-law expansion be valid?

There is no general theoretical answer to this question owing to the great complexity of quantitatively describing the whole critical domain. Wilson and Kogut have shown[5] that this complete description would require considering an infinite set of degrees of freedom. Nevertheless, it seems now well established that the number of relevant degrees of freedom decreases as one approaches the CP and only one (the ϕ^4 coupling) is *essentially* needed to reproduce the critical behavior within the preasymptotic critical domain ($\mathscr{D}_{\text{preas}}$).

In the Wilson terminology, the other degrees of freedom (higher-transient and finite-cutoff effects) are called irrelevant variables in $\mathscr{D}_{\text{preas}}$ (in contrast to the relevance of the ϕ^4 coupling). In reality, this is true only if one limits the description, within $\mathscr{D}_{\text{preas}}$, to the universal features (critical and subcritical exponents, combinations and ratios of critical and subcritical amplitudes). If one tries to obtain the complete description of $\mathscr{D}_{\text{preas}}$, then the infinite set of degrees of freedom will contribute to the nonuniversal features (amplitudes) and we are faced with a very complicated problem.

We claim that the minimal set of adjustable parameters introduced in Refs. 1 and 2 contains all the nonuniversal features in $\mathscr{D}_{\rm preas}$. In the present paper we give the theoretical and technical details of our work and extend it to the cases $n=2,3$.

Before explaining the organization of this article, we will indicate the main differences between our work and those[11] which have also tried to get nonasymptotic critical behavior from the ϕ^4 model.

(1) Instead of working within the ϵ expansion of the μ-renormalized theory[12,13] (referred to, in the following, as theory I) we use the massive theory[13] directly for $d=3$ (theory II). In 1973, Symanzik[14] showed that theory I irremediably generates spurious infrared (ir) singularities which affect the critical amplitudes. Arguing that theory I is only valid infinitesimally close to $d=4$ and could never give information on critical behavior at $d=3$ without treating the above ir singularities, Parisi[15] proposed the use of theory II which has no such problems. The drawback is the necessity for a numerical nonperturbative treatment (the renormalized integrals of Parisi). Although this proposal was first used to obtain precise estimates[16,17] of the critical exponents, it takes on its full significance in a global determination of the critical behavior. Indeed, theory I also yields a precise knowledge of the critical exponents[18] but, as stated above, should fail when critical amplitudes are considered. It must be realized that the high-order perturbative series computed by Nickel *et al.*[19] at $d=3$, within theory II, which gave the first precise estimates of the critical exponents, contain all the information needed for a precise and complete determination of nonasymptotic critical behavior. We have achieved this and, using the resummation methods introduced by Le Guillou and Zinn-Justin,[18] we obtained functions more accurate than those obtained in theory I to small orders in ϵ.

(2) In addition to the above technical aspects, we endeavor to give a physical meaning to the explicit functions of temperature that we have obtained within the field-theory (FT) approach. This requires considering the effects of all the degrees of freedom usually neglected in the ϕ^4 model (higher-transient and finite-cutoff effects). We stress the fact that one may give a general meaning to the FT approach beyond the formal ϵ expansion. In particular, in Ref. 20 we showed that the specific-heat case could not be completely understood within a FT approach restricted to the vicinity of $d=4$. In Refs. 1 and 2 we described the critical behavior, inside $\mathscr{D}_{\rm preas}$, with only three free parameters. This description is valid because the introduction of the irrelevant operators leads simply to a multiplicative renormalization of our free parameters. We have recently shown that there exists a universal constraint ($R_{B_{\rm cr}}$) on the specific heat in a given phase which corroborates this description of $\mathscr{D}_{\rm preas}$. Our numerical study allows estimations of $R_{B_{\rm cr}}$ is the cases $n=1$, 2, and 3 (see Sec. V). Owing to the accuracy of our calculations and the precise consideration of the consequences of the fundamental hypothesis of the RG approach, we think that we supply experimentalists with a theoretical description of $D_{\rm preas}$ which will allow a better comparison between the theoretical predictions and experimental data.

At this point we summarize the developments presented below. In Sec. II we emphasize the finite-cutoff and higher-transient effects within the FT framework by extending the usual perturbative hypothesis to a nonperturbative form. We show that the ϕ^4 model at infinite cutoff is formally efficient in accounting for the critical behavior in $\mathscr{D}_{\rm preas}$ with only three free adjustable parameters. In Sec. III we point out that the massive FT at $d=3$ is well adapted to curing the ir singularities of the critical theory in the ϵ-expansion scheme. We also introduce explicitly the ϕ^4 model at infinite cutoff at $d=3$ and we show the necessity of a nonperturbative treatment. In Sec. IV we present the technical details of this treatment to get the nonasymptotic critical behavior of ξ, χ, and C from the series at high orders of Ref. 19. Finally, in Sec. V we discuss the interest of our results in relation to the experiments. Two appendixes contain general considerations dealing with the basic hypotheses of the RG approach (Appendix A) and with the physical relation between the renormalization of FT and the critical limit (Appendix B).

II. ϕ^4 MODEL, FINITE-CUTOFF, AND HIGHER-TRANSIENT EFFECTS

A. Preliminaries

In this section we show that the FT approach allows a complete determination of the nonuniversal features of the preasymptotic domain ($\mathscr{D}_{\rm preas}$). In other words, we want to justify completely the introduction of the single three free parameters of the Refs. 1 and 2.

In practice, the FT framework appears to be very efficient for calculating any universal features within $\mathscr{D}_{\rm preas}$ (i.e., up to the first confluent correction to scaling) where the higher transient (such as the ϕ^6 coupling and others) and the finite cutoff effects are *a priori* neglected.[13] The calculations are then greatly simplified. The main justifications[5,7,13] of this approach have essentially been made within the ϵ-expansion scheme. However, as said in the Introduction and recalled in more detail in Sec. III, this scheme presents spurious ir singularities[14] essentially due to the definition of the critical theory (zero-mass theory). Starting from the observation that the proposal of Parisi[15] to use the massive FT directly in $d=3$ gives good estimates of the critical exponents[16,17] we shall assume that the renormalization transformations of FT have a fundamental significance outside of any perturbative approach. The FT approach to critical phenomena does not seem to us to be justified only by the existence of a renormalized ϕ^4 FT at $d=4$.[13] Consequently, the infinite cutoff limit and the elimination of the higher transients must be justified independently of the situation at $d=4$. Our main assumption in this section is the following: Despite technical difficulties in introducing the critical theory, we assume that the RG equation for the μ-renormalized correlation functions[13] with a finite-cutoff, has a fundamental meaning for any d and beyond any perturbative or approximate scheme.

We show that from hypotheses consistent with the recent theoretical developments (Newman and Riedel[21] and Symanzik[22]), the higher transient and finite-cutoff effects in $\mathscr{D}_{\rm preas}$ may be well controlled. Let us first briefly recall the general results which seem now well established by considering, for example, the susceptibility χ.

B. Wegner's expansion and real systems

The general form of χ within the whole critical domain may be expressed in terms of a scaling field t through the formal Wegner expansion (WE):

$$\chi(t)=\chi_0 t^{-\gamma}\left[1+\sum_{n=1}^{\infty} a_n t^{\Delta_n}\right], \tag{2.1}$$

in which γ,Δ_1 (usually noted Δ or $\omega\nu$) are, respectively, the dominant critical and subcritical universal exponents and are given by the ϕ^4 model. Recently,[21] the exponent Δ_2 (also universal) has been determined using a truncation of the exact RG equations[5] of Wilson and Kogut including the higher-transient couplings, while the estimation of γ and Δ were found to be in agreement with the FT approach.[17]

Within the pure ϕ^4 model, the WE reads

$$\chi(t)=\widetilde{\chi}_0 t^{-\gamma}\left[1+\sum_{n=1}^{\infty} \widetilde{a}_n t^{n\Delta}\right], \tag{2.2}$$

for which the set $\{n\Delta\}$ is a subset of the set $\{\Delta_n\}$ of Eq. (2.1). One sees that, even if the ϕ^4 model reproduces an infinite power-law expansion, it cannot reproduce the complete Eq. (2.1) as soon as the first correction to scaling is no longer quantitatively dominant. As stated in the Introduction, the critical-to-classical crossover behavior exhibited by the ϕ^4 model, once the expansion (2.2) is resumed, is not realistic.

The definition of $\mathscr{D}_{\rm preas}$ coincides with the experimental values of τ $[\tau=(T-T_c)/T_c]$ for which the corresponding WE (in terms of τ) may be truncated after the first correction. The link between the expansions [(2.1) or (2.2)] and real critical behavior (described in terms of τ) comes from the hypothesis of analyticity[5] for the scaling field t as a function of the physical parameters (temperature T and the field H conjugated to the order parameter). In the vicinity of T_c, one thus has

$$t=a_0\tau+O(\tau^2,\tau H^2) \tag{2.3}$$

in which a_0 is some unknown constant characteristic of the physical system considered. Any scaling field which verifies Eq. (2.3) is called a linear scaling field. Substituting Eq. (2.3) into Eq. (2.1), one sees that the situation beyond the first correction, in real systems, is very complicated to describe quantitatively. Let us illustrate this by considering the three-dimensional Ising case.

Owing to the small value[17] of Δ ($\simeq$0.5), the first correction may be well distinguished from the others provided that τ is sufficiently small. Beyond this term, one has three corrections of a similar temperature dependence (but with different amplitudes): that controlled by Δ_2 [$\sim$1.05 (Ref. 21)], that controlled by 2Δ ($\simeq$1) and the first analytic correction coming from the $O(\tau^2)$ term in Eq. (2.3) (whose exponent is one). In an analysis of experimental data, it is unlikely that one can distinguish between these three higher correction terms. The excessive number of adjustable parameters (the nonuniversal amplitudes) needed then would prevent any improvement in the determination of the asymptotic pure power-law behavior.[23] Furthermore, the convergence of the expansion could never be completely controlled owing to the fundamental asymptotic nature of the RG approach expressed by Eq. (2.3).

C. Cutoff and higher-transient effects

It follows from the preceding subsection that the ϕ^4 model contains the universal characteristics of $\mathscr{D}_{\rm preas}$. In this part we show that the influence of the higher transients and the finite cutoff within the amplitudes does not fundamentally change the results obtained from the ϕ^4 model at infinite cutoff. These irrelevant degrees of freedom induce only a change in the definition of the adjustable parameters of Refs. 1 and 2 without increasing their number. This important point greatly simplifies the global quantitative description of the critical behavior in $\mathscr{D}_{\rm preas}$.

To account for the effects of the finite lattice in the computational calculations for quantum chromodynamics (QCD), Symanzik,[22] at a purely theoretical level, recently showed that the higher transients and the cutoff have similar effects. The higher-transient couplings can be adjusted to eliminate completely the nonzero-lattice (finite-cutoff) effects in the whole critical domain. Of course, this adjustment does not correspond to the physical situation of critical phenomena where the couplings and the cutoff characterize a given system and are independent. However, this means that one may study the nonuniversality in $\mathscr{D}_{\rm preas}$, at least qualitatively, by considering, in addition to the ϕ^4 coupling, only a finite cutoff which will mimic its own role and that of the higher transients.

We shall recall the main steps of the FT approach by retaining a finite cutoff Λ in the correlation functions, even in the renormalized theory. We shall consider the soft-mass renormalization scheme[12] (the μ renormalization), which requires defining the theory at the CP.[13] In this scheme the RG equation has a simple form, but problems occur when explicit calculations are attempted perturbatively, for some dimensions below four[14] (except infinitesimally close to four). We shall thus remain at the formal level and assume that this scheme expresses correctly the fundamental ideas of the RG approach beyond any approximate framework.

The FT framework admits,[24] at least formally, the possibility of dealing with a finite cutoff Λ.

The starting point is the ϕ^4 bare Hamiltonian which has the following form:

$$\mathscr{H}\{\phi_0\}=\int d^dx\,\{\tfrac{1}{2}[(\nabla\phi_0)^2(x)+r_0\phi_0^2(x)]+(g_0/4!)\phi^4(x)\}\ . \tag{2.4}$$

Since we shall not perform explicit calculations in this section, we formally suppose that the cutoff occurs in a sharp manner in the Feynman integrations on the loop variables q (i.e., $0\le q\le\Lambda$).

In Appendix A we recall the physical meaning of the bare Hamiltonian (2.1). In particular, all the bare quantities ϕ_0, r_0, and g_0 are measured in units of Λ [see Eqs. (A14)]. Moreover, asymptotically close to the CP the parameters r_0 and g_0 have Taylor expansions around T_c (along the critical isochore $H=0$) by the hypothesis of analyticity:

$$r_0=r_{0c}+b_0(T-T_c)+O((T-T_c)^2)\ , \tag{2.5a}$$

$$g_0=\text{const}+O(T-T_c)\ . \tag{2.5b}$$

The bare correlation functions $\Gamma_0^{(L,N)}(\{q,p\};\ r_0,g_0,\Lambda)$ will represent the physical correlation functions in the vicinity of the CP (see Appendix A). In the following we shall not repeat the superscript (L,N) of the correlation functions except when explicit values of L and N are considered. The renormalization transformations, which define the renormalized correlation functions $\Gamma_R(\{q,p\};\ t,u,\mu,\Lambda)$ are the following:

$$\phi_0(x)=[Z_3(u,\rho)]^{1/2}\phi_R(x)\ , \tag{2.6a}$$

$$u_0=\rho^\epsilon u Z_1(u,\rho)/[Z_3(u,\rho)]^2\ , \tag{2.6b}$$

$$r_0/\Lambda^2=r_{0c}/\Lambda^2+\rho^2 t Z_2(u,\rho)/Z_3(u,\rho)\ , \tag{2.6c}$$

in which $\rho=\mu/\Lambda$ and $u_0=g_0/\Lambda^\epsilon$ are dimensionless. It will be useful in the following to deal with dimensionless correlation functions that we shall denote by $\tilde\Gamma_0$ and $\tilde\Gamma_R$ with

$$\Gamma_0(\{q,p\};r_0,g_0,\Lambda)=\Lambda^{D_{LN}}\tilde\Gamma_0(\{q/\Lambda,p/\Lambda\};r_0/\Lambda^2,u_0)\ , \tag{2.7a}$$

$$\Gamma_R(\{q,p\};t,u,\mu,\Lambda)=\mu^{D_{LN}}\tilde\Gamma_R(\{q/\mu,p/\mu\};t,u,\rho)\ , \tag{2.7b}$$

in which $D_{LN}=d-2L-N(d-2)/2$. The bare and renormalized correlation functions (for $N>0$) are thus related as follows:

$$\rho^{D_{LN}}\tilde\Gamma_R(\{q/\mu,p/\mu\};t,u,\rho)=[Z_3(u,\rho)]^{(N/2-L)}[Z_2(u,\rho)]^L\times\tilde\Gamma_0(\{q/\Lambda,p/\Lambda\};r_0/\Lambda^2,u_0). \tag{2.8}$$

The case $N=0$ is presented in Appendix B (see also Ref. 20). The renormalization functions $Z_i(u,\rho)$ ($i=1$ to 3) are defined by subtraction conditions which may be chosen as[13]

$$[\partial\Gamma_R^{(0,2)}(p;0,u,\mu,\Lambda)/\partial p^2]_{p^2=\mu^2}=1\ , \tag{2.9a}$$

$$\Gamma_R^{(0,4)}(p_1,p_2,p_3,p_4;0,u,\mu,\Lambda)\,|_{p_ip_j=(4\delta_{ij}-1)\mu^2/3}=\mu^\epsilon u\ , \tag{2.9b}$$

$$\Gamma_R^{(1,2)}(q,p_1,p_2;0,u,\mu,\Lambda)\,|_{p_1^2=p_2^2=\mu^2;p_1p_2=-\mu^2/3}=1\ , \tag{2.9c}$$

while r_{0c} (the critical bare mass) is defined such that the inverse susceptibility vanishes:

$$\Gamma_0^{(0,2)}(0;r_{0c},g_0,\Lambda)=0\ . \tag{2.10}$$

The inverse length μ is arbitrary. Its introduction is a consequence of the hypothesis of scale invariance at the CP which is expressed by the relations (2.6). These relations express the fact that *at the critical point* the system may be described similarly whether considered at the length scale $1/\Lambda$ (the bare theory) or at any length scale $1/\mu$ greater than $1/\Lambda$ (the renormalized theory) through a change of the normalization of the physical parameters (the field ϕ_0, the coupling g_0 and the distance to the CP $\bar r_0=r_0-r_{0c}$). At the CP no relevant length other than $1/\Lambda$ is available and the choice of μ is arbitrary. In our notation this means that $\rho=\mu/\Lambda$ is arbitrary.

At this stage it seems that μ and Λ play a similar role.[24] However, this is not true since Λ has a physical origin while μ is really arbitrary (it has a theoretical origin). In particular, the bare theory cannot be considered as a particular renormalized theory but as a reference which defines the physical parameters and correlation functions (see Appendix A).

The left-hand side of Eq. (2.6b) contains the fixed physical parameters u_0 (and implicity Λ through the ρ^ϵ factor of the right-hand side). Hence, the two new parameters u and ρ are also fixed once μ (or ρ) is chosen (arbitrarily). The critical limit corresponds to $r_0\to r_{0c}$ at u_0 (and Λ) fixed. From Eq. (2.6c) it follows that for the renormalized theory the critical limit is $t\to 0$ at u and ρ fixed.

The problem solved by the RG theory is that of resummation of the ir divergences which occur order by order in the perturbative expansion in powers of u_0 (or u) when $\bar r_0=r_0-r_{0c}$ (or t) goes to zero. The definition of the renormalized theory, through Eqs. (2.6)–(2.9), does not, by itself, solve this problem but implicitly contains its solution owing to the arbitrariness of μ (or ρ). The differential RG equation is obtained by expressing the ρ (or μ) independence of the bare correlation functions. To simplify the notation, let us consider the correlation functions at zero momenta ($\{q,p\}=\{0\}$). The derivative of Eq. (2.8) with respect to ρ at fixed r_0, u_0, and Λ yields

$$\left[\rho\frac{\partial}{\partial\rho}+W(u,\rho)\frac{\partial}{\partial u}-\frac{t\partial}{\nu(u,\rho)\partial t}+d-\frac{L+N\beta(u,\rho)}{\nu(u,\rho)}\right]\times\tilde\Gamma_R(\{0\};t,u,\rho)=0\ , \tag{2.11}$$

in which

$$W(u,\rho)=\rho(\partial u/\partial\rho)_{u_0}\ , \tag{2.12}$$

and

$$\nu(u,\rho)=[2-\eta_3(u,\rho)+\eta_2(u,\rho)]^{-1}\ , \tag{2.13a}$$

$$\beta(u,\rho)=\tfrac{1}{2}\nu(u,\rho)[d-2+\eta_3(u,\rho)]\ , \tag{2.13b}$$

with

$$\eta_i(u,\rho)=\rho\partial\ln[Z_i(u,\rho)]/\partial\rho\,|_{u_0}\quad (i=2,3)\ . \tag{2.14}$$

Up to now no reference to any approximate framework, such as the ϵ expansion, is needed to obtain Eq. (2.11). If we assume that the bare correlation functions Γ_0 are known, then so are the renormalized, by using Eqs. (2.8) and (2.9). The reference to the ϵ-expansion scheme comes usually in the next steps.[13]

The infinite-cutoff limit that one usually considers within the renormalized theory ($\Lambda\to\infty$ at u and μ fixed) corresponds here to $\rho\to 0$ at u fixed. Knowing that the universal characters of the critical behavior are expected to be independent of Λ, one considers *a priori* this limit within the renormalized theory. The main argument is its nonobvious existence at $d=4$ which appears essential in the ϵ-expansion scheme owing to the necessity of calculating some (primarily) divergent Feynman integrals at $d=4$ and at infinite cutoff. However, it is not necessary to take *a priori* this limit. We shall show that the calculation at an effective infinite cutoff comes without having set $1/\Lambda=0$. As already stated, Λ is a finite parameter which characterizes the microscopic structure of a given physical system ($1/\Lambda$ is of the order of the range a of the molecular forces) and cannot be set equal to zero

Let us look at the solutions to Eq. (2.11). Equations (2.6) show that the three variables t, u, and ρ are not independent; for fixed bare parameters, t and u are functions of ρ. Hence, the correlation functions $\tilde{\Gamma}_0(\{0\};t,u,\rho)$ are defined on a plane and the characteristic curve method may be applied. We introduce the flow parameter λ and the functions $\bar{u}(\lambda)$, $\bar{t}(\lambda)$, and $\bar{\rho}(\lambda)$ such that

$$\lambda d\bar{\rho}/d\lambda=\bar{\rho}\ , \tag{2.15a}$$

$$\lambda d\bar{u}/d\lambda=W(\bar{u},\bar{\rho})\ , \tag{2.15b}$$

$$\lambda d\bar{t}/d\lambda=-\bar{t}/\nu(\bar{u},\bar{\rho})\ , \tag{2.15c}$$

with the initial conditions $\bar{u}(1)=u$, $\bar{t}(1)=t$, and $\bar{\rho}(1)=\rho$.

It is then simple to find that the solutions of Eq. (2.11) have the following property:

$$\tilde{\Gamma}_R(\{0\};t,u,\rho)=\exp\left[-\int_1^\lambda dx\{d-[L+N\beta(x)]/\nu(x)\}\right]\times\tilde{\Gamma}_R(\{0\};\bar{t},\bar{u},\bar{\rho}) \tag{2.16}$$

in which $\beta(x)$ and $\nu(x)$ stand for $\beta(\bar{u}(x),\bar{\rho}(x))$ and $\nu(\bar{u}(x),\bar{\rho}(x))$. The relation (2.16) shows that there is a family of solutions to Eq. (2.11) the members of which differ by the values of their arguments and by an exponential factor.

The flow parameter λ may then be chosen in such a way that $\bar{t}(\lambda)$ remains fixed when the critical limit $t\rightarrow0$ (at u and ρ fixed) is considered. The hope is that $\tilde{\Gamma}_R(\{0\};\bar{t},\bar{u},\bar{\rho})$ will remain free of singularity while the singular behavior of the left-hand side of Eq. (2.16) will be entirely reproduced by the exponential factor. To obtain this result one needs hypotheses on the *nonperturbative* properties of the various functions $W(\bar{u},\bar{\rho})$, $\beta(\bar{u},\bar{\rho})$, $\nu(\bar{u},\bar{\rho})$, and $\tilde{\Gamma}_R(\{0\};\bar{t},\bar{u},\bar{\rho})$. They are well known[13] in the case where the infinite-cutoff limit ($\rho\rightarrow0$) is considered *a priori*. The solution of Eq. (2.15a) with the initial condition $\bar{\rho}(1)=\rho$ gives

$$\bar{\rho}(\lambda)=\lambda\rho\ . \tag{2.17}$$

Hence these hypotheses concern the quantities listed above at $\bar{\rho}=0$ (i.e., $\rho=0$). Let us recall them.

Having fixed $\bar{t}(\lambda)$ by the matching condition,

$$\bar{t}(\lambda)=\sigma\ , \tag{2.18}$$

with σ a strictly positive arbitrary number, Eq. (2.15c) defines λ as a function of t which goes to zero as t^ν provided that $\nu=\nu(u^*,0)$ has a finite positive value. The fixed-point value u^* is the zero of the function $W(u,0)$ that $\bar{u}(\lambda)$ reaches when λ goes to zero. More precisely one assumes that

$$W(\bar{u}(\lambda),0)\sim\omega[\bar{u}(\lambda)-u^*]\ , \tag{2.19}$$

with a positive $\omega=[dW(u,0)/du]\,|_{u=u^*}$.

From this, the asymptotic critical behavior of $\tilde{\Gamma}_R(\{0\};t,u,0)$ is obtained from an expansion around u^* ($\lambda=0$) of the right-hand side of Eq. (2.16), the function $\tilde{\Gamma}_R(\{0\};\sigma,\bar{u},0)$ (and all of its derivative with respect to $\bar{u}$) being well defined at $\bar{u}=u^*$ and $\sigma\neq0$. The successive powers of $\bar{u}-u^*$ then generate confluent corrections to the scaling behavior of the form $t^{n\Delta}$ ($n=1,2,3,\ldots$) with $\Delta=\omega\nu$.

These correction terms, that we shall call "of the pure ϕ^4 model," do not involve all the corrections of a real critical behavior. The higher transients will also generate corrections to scaling. Let us call Δ_2 the exponent of the greatest one. Recent numerical estimates[21] show that $\Delta_2>\Delta$ and Δ is found to be in agreement with the estimates obtained within the FT approach.[17] In addition to the higher transient effects, there is that of the finite cutoff neglected *a priori* in considering $\rho=0$ (and thus also $\bar{\rho}$) in the renormalized theory.

Indeed ρ must be kept constant, with only $\bar{\rho}(\lambda)$ going to zero when the critical limit $t\rightarrow0$ (i.e., $\lambda\rightarrow0$) is taken [see Eq. (2.17)]. In order to appreciate the influence of the corrections generated by a nonzero $\bar{\rho}(\lambda)$, the only argument usually given is valid perturbatively and for a dimension d near four: the $\bar{\rho}$ dependence for small $\bar{\rho}$ is of order $\bar{\rho}^2\ln\bar{\rho}$ in the renormalized correlation functions at $d=4$.[13] In order to treat completely the finite-cutoff effects we must make some hypotheses, similar to those concerning the existence of u^* given above, without reference to perturbative theory or to the proximity of dimension four. The work of Symanzik[22] and the results of Newman and Riedel,[21] already mentioned, suggest the following hypotheses.

As $\bar{\rho}$ goes to zero, with $\bar{t}$ and $\bar{u}$ fixed, the renormalized correlation functions are such that

$$\tilde{\Gamma}_R(\{0\};\bar{t},\bar{u},\bar{\rho})=\tilde{\Gamma}_R(\{0\};\bar{t},\bar{u},0)+O(\bar{\rho}^{\omega_2})\ , \tag{2.20}$$

with $\omega_2=\Delta_2/\nu$. Hence, from Eq. (2.11) we obtain

$$W(\bar{u},\bar{\rho})\underset{\lambda\rightarrow0}{\sim}\omega(\bar{u}-u^*)+O(\sup[\bar{\rho}^{\omega_2},(\bar{u}-u^*)^2])\ , \tag{2.21}$$

with $\omega_2>\omega$.

The corrections to scaling generated by a finite cutoff are obtained from Eq. (2.16) when λ goes to zero by expanding the $\bar{\rho}$ dependence around $\bar{\rho}=0$. Equation (2.20) and the fact that $\lambda\sim t^\nu$ show that the correction terms are of the same kind as those generated by the higher transients of the form t^{Δ_2} as stated by Symanzik.[22]

The corrections to scaling induced by the finite cutoff ($\bar{\rho}\neq0$) may thus be neglected within $\mathscr{D}_{\text{preas}}$ since $\Delta_2>\Delta$. However, the cutoff ρ remains present inside the amplitudes which depend on the initial condition of Eqs. (2.15).

In particular, the approach of u^* by $\bar{u}(\lambda)$, when $\lambda\rightarrow0$, will also depend on ρ and not only on u. This can be seen from Eqs. (2.15b) and (2.21) and the initial conditions

$\bar{u}(1)=u, \bar{\rho}(1)=\rho$. One easily obtains

$$|\bar{u}(\lambda)-u^*| \underset{\lambda\to 0}{\sim} c(u,\rho)\lambda^{\omega} . \quad (2.22)$$

Usually, when ρ is considered to be zero, one obtains the coefficient $c(u,0)$ which, in the case where $u\sim u^*$, has the form

$$c(u,0)\sim |u-u^*| . \quad (2.23)$$

The constant $c(u,\rho)$ controls the amplitudes of the corrections of the pure ϕ^4 model. In the case of Eq. (2.23), there would exist a value of u (and thus of u_0) for which all these corrections vanish, in particular the first, whatever the value of ρ. In fact, Eq. (2.21) shows that the function $W(u,\rho)$ could not be zero at $u=u^*$ as long as ρ is different from zero, and hence $c(u^*,\rho)$ is in general different from zero. We recall that ρ represents here the role of an infinite set of degrees of freedom and it is very unlikely that nature arranges them in such a way that $c(u^*,\rho)=0$, although not impossible. In this latter case, the correction induced by the pure ϕ^4 model vanishes, but there will remain the analytic corrections and those induced by $\bar{\rho}$. The fixed-point theory corresponds to $u=u^*$ and all the other couplings equal to zero with ρ. In that case all the corrections to scaling vanish whatever the distance to the CP, and there remain only the analytic corrections. This is an ideal situation which cannot be thought of as being physical since the cutoff may never be infinite in a real system. Consequently, the fixed-point theory can be considered only within the renormalized (unphysical) theory.

The expression of the two first terms (leading and first correction) of the power-law expansion of the renormalized correlation functions is obtained by an expansion around $\bar{u}=u^*$ with $\bar{\rho}$ set to zero, of Eq. (2.16). Following the usual[25,26] considerations and notation we obtain

$$\tilde{\Gamma}_R(\{0\};t,u,\rho) \underset{t\to 0}{\sim} \rho^{D_{LN}} X^{*L} Y^{*N} (X^* t/\sigma)^{d\nu-L-N\beta} \tilde{\Gamma}_R(\{0\};\sigma,u^*,0)$$
$$\times\{1-c(u,\rho)(X^*t/\sigma)^{\Delta}[(\nu' d-N\beta')/\Delta+(\ln[\tilde{\Gamma}_R(\{0\};\sigma,\bar{u},0)])']_{\bar{u}=u^*}\} , \quad (2.24)$$

in which the notation $(f)'$ means $df/d\bar{u}$ and ν' stands for $d\nu(\bar{u},0)/d\bar{u}$ (similarly for β'). The quantities X^* and Y^* are the values at $\lambda=0$ of the functions $X(\lambda)$ and $Y(\lambda)$ defined as

$$X(\lambda)=\exp\left[\int_1^{\lambda} dx[1/\nu-1/\nu(x)]/x\right] , \quad (2.25a)$$

$$Y(\lambda)=\exp\left[\int_1^{\lambda} dx[\beta/\nu-\beta(x)/\nu(x)]/x\right] , \quad (2.25b)$$

with $\nu(x)=\nu(\bar{u}(x),\bar{\rho}(x))$ and $\beta(x)=\beta(\bar{u}(x),\bar{\rho}(x))$. The integrations from 1 to λ indicate that $X(\lambda)$ and $Y(\lambda)$ depend on the initial conditions of Eqs. (2.15), and thus on u and ρ. Hence X^* and Y^* depend also on u and ρ as $c(u,\rho)$. Apart from the dimensional factor $\rho^{D_{LN}}$, these three constants carry all the nonuniversality of the critical behavior.

The universality of combinations or ratios between asymptotic[25] and correction[26,27] amplitudes comes from this particular structure. One may easily check that the return to the bare (physical) theory corresponds only to a multiplicative change of these nonuniversal factors by using the inverse renormalization change indicated by Eqs. (2.6)—(2.8). The σ dependence explicitly written in Eq. (2.24) expresses only the arbitrariness introduced in deriving the solution to Eq. (2.11).

We recall that the dependence on the cutoff, in the nonuniversal factors, also summarizes the effects of all the higher degrees of freedom. Hence, we claim that all the nonuniversality within $\mathscr{D}_{\text{preas}}$ may be reproduced through four adjustable parameters related, respectively, to (a) a dimensional factor [$\rho^{D_{LN}}$ in Eq. (2.24)]; (b) a temperature scale (X^*); (c) a field scale (Y^*); (d) the strength of the confluent corrections [$c(u,\rho)$]. Let us make some remarks about $c(u,\rho)$.

(i) The corrections of the pure ϕ^4 model have the following general form:

$$\sum_{n=1}^{\infty} a_n(\sigma)[(X^*t/\sigma)^{\Delta}c(u,\rho)]^n ,$$

in which $a_n(\sigma)$ are numbers such that the combination $[a_{n+1}(\sigma)]^n/[a_n(\sigma)]^{n+1}$ is universal for any n. This property is only valid within the pure ϕ^4 model.

(ii) As long as we limit ourselves to this model, it is clear that the adjustable parameter $c(u,\rho)$, if nonzero, may be eliminated through a redefinition of ρ, X^*, and Y^*. Of course $c(u,\rho)$ could vanish, but it is unlikely because of its dependence on all of the couplings, and not merely on $|u-u^*|$ as usually thought.[28] Hence, the number of adjustable parameters is reduced to three.

One thus has the confirmation of the validity of the description of $\mathscr{D}_{\text{preas}}$ in terms of the three free parameters proposed in Refs. 1 and 2, related, respectively, to the following:

(1) a length scale (noted g_0 in Refs. 1 and 2), which fixes the dimensional factor $\rho^{D_{LN}}$;

(2) a temperature scale (noted θ) related to X^*;

(3) a field scale (noted Ψ) related to Y^*.

From now on, we can limit our study of $\mathscr{D}_{\text{preas}}$ to the pure ϕ^4 model at infinite cutoff, since the higher transients and the cutoff induce only a redefinition of the free parameters which, by their nature, must be determined from experiment. Let us stress the fact that this infinite-cutoff limit corresponds to $\rho=0$ and thus concerns the bare theory at $\Lambda=\infty$ (we recall that $\rho=\mu/\Lambda$). Nevertheless, the cutoff dependence of the bare theory has two fundamentally different origins (as recalled in Appendix A): first, the lattice structure and second, the physical effect of the long-range correlations between the fluctuations at the CP which is expressed in the dimensionalization of all the bare (physical) quantities in terms of Λ [in particular in the $\rho^{D_{LN}}$ factor of Eq. (2.24), see Appendix A]. The infinite-cutoff limit of the bare theory, in which we are interested, corresponds to the continuous limit of a zero lattice spacing and thus to $\Lambda\to\infty$ with all bare quantities (r_0,g_0) fixed. In the next section we show how this can be realized together with the presentation of the advantages of working within the massive FT directly at $d=3$.

III. PARISI APPROACH

In this section we present the fundamental technical arguments which led us to work within the massive FT directly for $d=3$ as proposed by Parisi.[15] We want first to discuss the infinite-cutoff limit of the bare theory which is alluded to at the end of Sec. III.

A. Infinite-cutoff limit of the bare theory below four dimensions

In order to avoid confusion we must emphasize that this limit taken for the bare theory (at g_0 and r_0 fixed) is not contrary to custom. In practice, the usual infinite-cutoff limit (for the renormalized theory) is equivalent to an implicit definition of a bare theory at infinite Λ. It amounts to setting ρ $(=\mu/\Lambda)$ equal to zero and not only $\lambda=0$ (i.e., not only $\bar{\rho}=0$) in the renormalized quantities of Sec. II, while μ is kept nonzero. From the preceding section it follows that the two bare theories (Λ finite or infinite) are equivalent in $\mathscr{D}_{\text{preas}}$ up to a change of the three adjustable parameters.

Of course a finite cutoff is essential in the RG approach[5,7] (see Appendix A). The infinite-cutoff limit, which facilitates the calculations, is only justified after the renormalization procedure has been set up (see Sec. II).

For $d=3$ (but also for other rational values of d below four[14]) the bare theory presents ultraviolet (uv) divergences reminiscent of those of the two-point correlation function (at zero momentum) for $d=4$. A mass shift is sufficient to eliminate them and must be differentiated from the true renormalization process related to the introduction of the Z_i's of Sec. II. Indeed, this latter renormalization expresses the fundamental scale-invariance hypothesis of the RG approach (see Sec. II C) while the mass shift is useful for defining the critical parameter (scaling field):

$$\bar{r}_0=r_0-r_{0c}\ . \tag{3.1}$$

r_{0c} is the critical bare mass defined in Eq. (2.10) which corresponds precisely to the subtraction condition of all the ultraviolet (uv) divergences of the bare theory below four dimensions.

At $d=4$, the mass shift in the renormalization process of field theories cannot be distinguished from the other renormalizations (Z_i's) since only uv divergences are concerned. In the critical limit one is only interested in the resummation of the ir divergences which occur when $\bar{r}_0$ goes to zero (at g_0 and Λ fixed). For dimensional reasons the quadratic uv divergences summed within r_{0c} are completely disconnected from the ir divergences. On the other hand, the uv divergences summed within the Z_i's are linked to the ir ones (and vice versa) through logarithms of $\Lambda/\bar{r}_0$ (Λ and $\bar{r}_0$ have the same dimension while g_0 is dimensionless, hence $\Lambda/\bar{r}_0$ is the single dimensionless quantity that can be constructed with Λ and $\bar{r}_0$).

Consequently, since the mass shift is disconnected from the critical limit, r_{0c} may be used, as counterterm, to define the bare theory for $d=3$ for infinite Λ. It does not matter that r_{0c} is infinite since it will never appear afterwards, $\bar{r}_0$ being the single bare mass useful in the RG approach (scaling field).

A coherent theoretical scheme for the definition of a bare theory for $d=3$ and Λ infinite would follow these steps:

(1) In order to avoid the uv singularities which occur at $d=4-2/k$ $(k=1,2,3,\dots)$ (in particular at $d=3$), one makes an analytic continuation in d (Refs. 14 and 29) (dimensional regularization). One may thus drop Λ and deal with $\Gamma_0(\{q,p\};r_0,g_0,\epsilon)$ in which $\epsilon=4-d$ is different from the above rational values.

(2) One then eliminates the pole at $\epsilon=1$ by mean of a mass shift, introducing a new bare mass r_0' defined by

$$r_0=r_0'+\delta r_0(\epsilon) \tag{3.2}$$

in which $\delta r_0(\epsilon)$ subtracts the pole at $\epsilon=1$. The new bare correlation functions $\Gamma_0(\{q,p\};r_0',g_0,\epsilon)$ are then well defined at $d=3$ $(\epsilon=1)$. They are formally equal to the correlation functions expressed in terms of r_0 as long as ϵ does not take the above rational values. They will now be considered as the physical correlation functions. This essentially means that the bare quantities have dimensions expressed in terms of the microscopic length $a\sim 1/\Lambda$: $r_0'\sim\Lambda^2$ and $g_0\sim\Lambda^{\epsilon}$. This distinction in the role of Λ is explained in Appendix A.

(3) One introduces a critical parameter (scaling field):

$$\bar{r}_0'=r_0'-r_{0c}' \tag{3.3}$$

with r_{0c}' defined by

$$\Gamma_0^{(0,2)}(\{0\};r_{0c}',g_0,\epsilon)=0\ , \tag{3.4}$$

at $d=3$, r_{0c}' is finite and in principle calculable. One also has

$$\Gamma_0(\{q,p\};\bar{r}_0',g_0,\epsilon)=\Gamma_0(\{q,p\};r_0',g_0,\epsilon)\ . \tag{3.5}$$

(4) One then considers the renormalization transformations of the field ϕ_0, the coupling constant g_0, and the critical parameters $\bar{r}_0'$ which introduce the Z_i's.

(5) After obtaining the critical behavior of the renormalized correlation functions $\Gamma_R(\{q,p\};t,u,\mu,\epsilon)$, one returns to the bare correlation functions $\Gamma_0(\{q,p\};\bar{r}_0',g_0,\epsilon)$ by means of the inverse renormalization transformations.

(6) The cutoff dependence is then reintroduced through the three adjustable parameters of Refs. 1 and 2 (see Sec. II) to obtain the realistic bare correlation functions within $\mathscr{D}_{\text{preas}}$.

However this program, although correct in principle, is not adapted to practical calculations, owing to the perturbative nature of the FT approach. The reason is that, as we shall see below (Sec. III B) the critical bare mass r_{0c}' cannot be determined perturbatively. This is one of the difficulties, already noted in the Introduction, associated with the definition of the critical theory (zero-mass theory). In the following subsections we shall recall the origins of these problems and show how the massive FT framework circumvents them.

B. Spurious infrared singularities and the critical bare mass for $d<4$

From now on we shall deal with the bare correlation functions $\Gamma_0(\{q,p\};r_0,g_0,\epsilon)$, dimensionally regularized. Within the ϵ-expansion scheme it is usual to consider the critical bare mass r_{0c} as being identically zero. The perturbative equation which defines r_{0c} has the following form:

$$r_{0c}=\sum_{k=1}^{\infty} r_{0c}^{(1-k\epsilon/2)}a_k(\epsilon)g_0^k\ , \tag{3.6}$$

where $a_k(\epsilon)$ represents the Feynman graph contributions of order k and has poles[14] at $\epsilon=2/k$ $(k=1,2,3,\ldots)$.

One easily sees in Eq. (3.6) that the series which defines r_{0c} will be identically zero provided $k\epsilon<2$ for any k. Within the ϵ expansion it is formally the case and one has $r_{0c}\equiv 0$ at each order. However, this supposes infinitesimally small ϵ and prevents the possibility of setting $\epsilon=1$ in the results without having prescribed the way the singularities of the correlation functions located at the rationals[14] $\epsilon=2q/p$ and generated by the massless bare theory (ir singularities) have been circumvented. As Parisi claimed,[15] the ϵ-expansion scheme could never give any quantitative information on the critical behavior without an additional hypothesis on the resummation of these ir singularities. This limitation, however, concerns only the critical amplitudes and not the determination of either the critical exponents, which requires only consideration of the poles at $\epsilon=0$ $(d=4)$ through the Z_i's, or the universal amplitude combinations. This is well illustrated by the recent work of Le Guillou and Zinn-Justin.[18]

As mentioned in the preceding subsection, the dimensional regularization provides an analytic continuation which allows an approach to $d=3$ without the difficulty of the ir singularities of the massless theory. Only the uv singularities located at $\epsilon=1$ $(d=3)$ are of importance. These singularities can then be absorbed in a redefinition of the bare mass [Eq. (3.2)]. Let us give explicitly the mass shift at $d=3$ already introduced in a previous paper.[30]

The coefficients $a_k(\epsilon)$ in Eq. (3.6) have, in particular, poles at $\epsilon=1$ generated by the simple divergent graph (or subgraph) drawn in Fig. 1. The value of this graph, $G(\epsilon)$, contributes linearly to Eq. (3.6) through $a_2(\epsilon)$ which has a simple pole at $\epsilon=1$. Let us perform the change of mass given by Eq. (3.2) such that $\delta r_0(\epsilon)$ subtracts only the poles at $\epsilon=1$. Owing to the superrenormalizable character of the ϕ^4 theory at $d=3$, $\delta r_0(\epsilon)$ will have only one term, determined by requiring that the combination

$$D(\epsilon)=g_0^2(r_0')^{1-\epsilon}a_2(\epsilon)+\delta r_0(\epsilon) \tag{3.7}$$

no longer have a pole at $\epsilon=1$. For this, we perform an expansion around $\epsilon=1$ of $D(\epsilon)$ knowing that

$$a_2(\epsilon)=f_2(\epsilon)/(\epsilon-1) \tag{3.8}$$

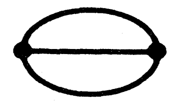

FIG. 1. Graph which gives a pole at $\epsilon=1$ and generates all the ultraviolet divergences of the ϕ^4 field theory at $d=3$.

with $f_2(1)$ a well-defined number. We find

$$D(\epsilon)=g_0^{2/\epsilon}[f_2(1)/(\epsilon-1)-f_2(1)\ln(r_0'/g_0^{2/\epsilon}) + C_1+O(\epsilon-1)]+\delta r_0(\epsilon)\ , \tag{3.9}$$

in which C_1 is a constant.

The elimination of the pole at $\epsilon=1$, gives

$$\delta r_0(\epsilon)=-g_0^{2/\epsilon}f_2(1)/(\epsilon-1) \tag{3.10}$$

and a finite $D(1)$,

$$D(1)=-g_0^2f_2(1)\ln(r_0'/g_0^2)+C_1\ . \tag{3.11}$$

The bare correlation functions expressed in terms of r_0' and g_0 at $d=3$ are then finite and obtained from the relation

$$\Gamma_0(\{q,p\};r_0',g_0)=\lim_{\epsilon\to 1}\Gamma_0(\{q,p\};r_0,g_0,\epsilon) \tag{3.12}$$

with r_0' defined by Eqs. (3.2) and (3.10).

The drawback is the nonanalyticity in the coupling constant which appears through logarithms of g_0 in Eq. (3.11). This is a consequence of g_0 being the single dimensioned parameter (at the CP) once Λ is eliminated. For the same reason the bare critical mass r_{0c}' at $d=3$ has the form

$$r_{0c}'=g_0^2\mathscr{M}\ , \tag{3.13}$$

in which $\mathscr{M}$ is a number which cannot be determined perturbatively.[14] In the next subsection we show that the massive FT framework circumvents these difficulties.[31]

C. Massive field-theory framework

To study the critical behavior, it is not necessary to define the theory at the CP and especially the critical bare mass from which arise the problems indicated above. It seems incompatible with the perturbative FT approach, to use a linear (analytic in temperature) scaling field as critical parameter. It is preferable to introduce, as critical parameter, a quantity which already contains a nonanalyticity (nonlinear scaling field) such as the inverse correlation length (or the inverse susceptibility).

The massive FT corresponds to this choice. Usually the nonlinear scaling field (the mass, see below) is introduced at the level of the renormalized theory. From Sec. III B we must differentiate the mass shift which subtracts the poles at $\epsilon=1$, from the other renormalization steps. The procedure for defining a massive theory for $d=3$ can be seen as follows.

Starting from the bare correlation functions $\Gamma_0(\{q,p\};r_0,g_0,\epsilon)$, we perform the mass shift

$$r_0=m^2+\delta m^2(\epsilon)\ , \tag{3.14}$$

in which δm^2 is defined by the condition

$$\frac{\Gamma_0^{(0,2)}(\{0\};m^2+\delta m^2,g_0,\epsilon)}{\frac{\partial}{\partial p^2}\Gamma_0^{(0,2)}(\{p\};m^2+\delta m^2,g_0,\epsilon)}=m^2 . \quad (3.15)$$

In this case,[32] m is the inverse of the correlation length. It is easy to realize that δm^2 plays a role similar to that of $\delta r_0(\epsilon)$ previously introduced in Eq. (3.2). In other words, it eliminates all the poles at $\epsilon=1$. We thus obtain a bare theory finite at $d=3$ in terms of the nonlinear scaling field m, the correlation functions of which will be noted $\Gamma_0(\{q,p\};m,g_0)$. We emphasize that the two bare theories at $d=3$ introduced up to now (with $\bar{r}'_0$ and m) are identical through a finite shift of the mass at $d=3$:

$$\Gamma_0(\{q,p\};\bar{r}'_0,g)=\Gamma(\{q,p\};m,g) \quad (3.16)$$

with

$$\bar{r}'_0=m^2+\lim_{\epsilon\to 1}[\delta m^2(\epsilon)-\delta r_0(\epsilon)] . \quad (3.17)$$

Equation (3.16), which relates the formulations in terms of linear ($\bar{r}'_0$) and nonlinear (m) scaling fields will be important in the following to return to the description in terms of τ through Eqs. (2.5), (3.1), and (3.2). The advantage of dealing with $\Gamma_0(\{q,p\};m,g_0)$ is that they are free of logarithms of g_0 and may be calculated perturbatively.

Up to now we have realized the three first steps of section III A. The fourth step is the introduction of the renormalization process which relates the bare quantities to the renormalized ones and reads for any d:

$$\phi_0(x)=[Z_3(g)]^{1/2}\phi_R(x) , \quad (3.18a)$$

$$g_0=m^\epsilon g Z_1(g)/[Z_3(g)]^2 , \quad (3.18b)$$

$$[\phi_0(x)]^2=[Z_3(g)/Z_2(g)][\phi_R(x)]_R^2 , \quad (3.18c)$$

in which the subscript R refers to renormalized quantities, while the renormalized and bare correlation functions are related by

$$\Gamma_R(\{q,p\};m,g,\epsilon)=[Z_3(g)]^{N/2-L}[Z_2(g)]^L \times\Gamma_0(\{q,p\};m,g_0,\epsilon) . \quad (3.19)$$

The renormalization functions $Z_i(g)$, $i=1$ to 3, are usually defined by the following equations:[13]

$$\frac{\partial}{\partial p^2}\Gamma_R^{(0,2)}(p;m,g,\epsilon)\,|_{p^2=0}=1 , \quad (3.20a)$$

$$\Gamma_R^{(0,4)}(\{0\};m,g,\epsilon)=m^\epsilon g , \quad (3.20b)$$

$$\Gamma_R^{(1,2)}(\{0\};m,g,\epsilon)=1 . \quad (3.20c)$$

These equations are called subtraction conditions in the language of field theorists who are interested in the situation at $d=4$ and in the renormalized theory. It seems to us simpler to write the explicit definitions of the Z_i's which, from Eqs. (3.18)–(3.20), read

$$[Z_3(g)]^{-1}=\frac{\partial}{\partial p^2}\Gamma_0^{(0,2)}(p;m,g_0,\epsilon)\,|_{p^2=0} , \quad (3.21a)$$

$$[Z_1(g)]^{-1}=\Gamma_0^{(0,4)}(\{0\};m,g_0,\epsilon)/g_0 , \quad (3.21b)$$

$$[Z_2(g)]^{-1}=\Gamma_0^{(1,2)}(\{0\},m,g_0,\epsilon) . \quad (3.21c)$$

From this we see clearly that the Z_i's are finite dimensionless functions of g_0/m^ϵ [or of g through Eq. (3.18b)] at $d=3$. It is precisely this way that the high-order series have been calculated[19] at $d=3$ to obtain the estimates[16,17] of the critical exponents. Before recalling the steps which led, from knowledge of the Z_i's to these estimates, let us make some comments on the significance of the renormalization process introduced by Eqs. (3.18).

The Wilson RG approach is based on the hypothesis of scale invariance at the CP. This means that the description of the system in terms of the physical parameters ($T-T_c$, H, and couplings) at a given length scale (here the microscopic scale $1/\Lambda$ for the bare theory) is equivalent to that made at a macroscopic length scale (the correlation length $\xi=m^{-1}$) through a change of normalization of the physical parameters.

Equation (3.18b) relates the two length scales Λ^{-1} (g_0 is order Λ^ϵ) and $\xi=m^{-1}$, while the renormalization of the coupling $u_0=g_0/\Lambda^\epsilon$ is performed through Z_1, and the new coupling is designated by g. Equation (3.18a) corresponds to a redefinition of the magnetic field h of the bare theory (see Appendix A on the introduction of the field h) through Z_3. This renormalization is customarily written as a renormalization of $\phi_0(x)$ coupled to h in the Hamiltonian. It is indeed equivalent since ϕ_0 is an integration variable within the partition function so that a change in the normalization of ϕ_0 then corresponds to a change of h. For the same reason, Eq. (3.18c) which introduces Z_2 corresponds to the renormalization of the *linear* scaling field $\bar{r}'_0$ (or $\bar{r}_0$) expressed here as a renormalization of the square of the renormalized $\phi_R(x)$ to which it is coupled. Equations (3.18) define the renormalized physical parameters (attached to the length scale ξ) in relation to the bare ones ($h,u_0,\bar{r}'_0$) (attached to the length scale Λ^{-1}). The definitions of the Z_i's [Eqs. (3.21)] are then suggested by the similarity of the problems solved by field theorists for $d=4$ and those raised by the critical limit at any d. We stress the fact that this similarity could not be understood without having first expressed the scale-invariance hypothesis through Eqs. (3.18) or similar ones. The reference to $d=4$ is only a guide, and Eqs. (3.21) may be used for $d=3$.

In Sec. II we used a different renormalization scheme: the so called μ renormalization.[12] In that scheme one tries to describe the situation *at* the CP. In that case no relevant length scale other than Λ^{-1} is available (ξ is infinite); the arbitrariness of the length scale μ^{-1} expresses completely the scale-invariance hypothesis at the CP. This contrasts with the massive framework, in which it is only approximately formulated (the CP is not yet reached). This explains why the RG equations take a simpler form in the μ scheme than in the massive scheme. However, as already mentioned, the CP is a singular point whose description is an idealization. The μ scheme allows the complete expression of the consequences of the fundamental-theoretical ideas, but some difficulties arise when explicit calculations are made within it (see Secs. III A and III B). The massive scheme is more realistic since, as in real systems, the CP is only asymptotically approached. If the fundamental formal ideas of the RG approach may be useful in understanding the critical behavior in real systems, then we assume that the conclusions of Sec. II may be correct for the massive theory when m goes to zero.[33] Consequently, consideration of the massive theory at $\Lambda=\infty$ directly at $d=3$, as introduced in this section, is justified to obtain the critical behavior.

Let us now recall briefly how the critical-exponent estimates have been obtained within the massive theory. In the massive theory, the critical limit corresponds to $m\to 0$ at g_0 fixed. Hence, Eq. (3.18b) defines the renormalized coupling g as a function of temperature such that the combination $gZ_1(g)/[Z_3(g)]^2$ goes to infinity as $m^{-\epsilon}$ ($\epsilon>0$). The essential hypothesis usually made is that g reaches a finite value (g^*) when $m\to 0$. Following the standard presentation, g^* is the single nontrivial zero of the function $W(g)$ defined by[13]

$$W(g)=-\epsilon(d\ln\{gZ_1(g)/[Z_3(g)]^2\}/dg)^{-1}\ . \tag{3.22}$$

Near the fixed point g^*, $W(g)$ is supposed to behave as

$$W(g)=\omega(g-g^*)+O((g-g^*)^2) \tag{3.23}$$

with positive ω.

The critical exponents are obtained from the functions $\eta_3(g)$ and $\eta_2(g)$ defined as[13]

$$\eta_3(g)=W(g)\frac{d}{dg}\ln[Z_3(g)]\ , \tag{3.24}$$

$$\eta_2(g)=W(g)\frac{d}{dg}\ln[Z_2(g)]\ , \tag{3.25}$$

whose values are estimated at the fixed point g^*:

$$\eta=\eta_3(g^*)\ , \tag{3.26a}$$

$$\eta_2=\eta_2(g^*)\ . \tag{3.26b}$$

The other critical exponents are then obtained via the scaling laws which are automatically verified in FT. For example, the exponent ν of the correlation length is

$$\nu=(2-\eta+\eta_2)^{-1}\ . \tag{3.27}$$

The computation[19] at $d=3$, of the series $Z_i(g)$ ($i=1,2$, and 3) to sixth order in powers of g, provides the series for $W(g)$ and for the critical exponents at the same order. The divergent nature of these series has been characterized by the determination of the large-order behavior[34] of their terms. This has suggested resummation methods of the series whose details may be found in Ref. 17. The zero of the Wilson function $W(g)$ is then evaluated, which gives a numerical verification of the hypothesis of the existence of g^*. Finally, the series for ω, η, and η_2 are resummed for $g=g^*$. The study of various resummation methods in several cases has suggested criteria for estimating error bars.[17]

As recalled in Sec. IV, the resummation method and the principle of error-bar determination are independent of value of g. This allows a systematic numerical study of the approach of the CP via the approach of g^* by g. It is the subject of the next section to show that this can be done, with a similar accuracy as for the exponents,[17] for the three measurable quantities ξ, χ, and C using the series computed in Ref. 19.

IV. NONASYMPTOTIC CRITICAL BEHAVIOR FROM MASSIVE FIELD THEORY AT $d=3$

In this section we show in detail how the nonasymptotic critical behaviors for the physical quantities ξ, χ, and C of the disordered phase presented[1] and used[2] in previous papers, have been obtained. We first present the principle of the method which is based on a nonperturbative treatment of the RG equation. This was made possible by the high-order perturbation series calculated by Nickel, Meiron, and Baker[19] and the method of resummation of the divergent series introduced by Le Guillou and Zinn-Justin[17] (Sec. IV B). The precision of our numerical treatment together with the direct calculation in the bare theory (dimensionally regularized as in Sec. III) allows a comparison with experiments and, in our opinion, will improve the experimental tests of theoretical predictions (see Sec. V).

A. Principle of the method

Our starting point is the massive ϕ^4 FT at infinite cutoff presented in Sec. III C which is considered directly at $d=3$. The renormalization functions $Z_i(g)$ ($i=1-3$) calculated by Nickel *et al.*[19] correspond to the bare correlation functions at infinite cutoff after the mass renormalization at $d=3$ [Eqs. (3.21)]. From these definitions and the considerations of Appendix A, one easily verifies that the physical ξ and χ are respectively given by

$$\xi^{-1}=m\ , \tag{4.1a}$$

$$\chi^{-1}=m^2/Z_3(g)\ . \tag{4.1b}$$

We drop the factor $1/c_0^2$ in the definition [Eq. (A9) of Appendix A] of χ^{-1} for the sake of clarity. It will be restored in Sec. V when comparison with real systems will be envisaged.

Equation (3.18b) defines m as a function of g at g_0 fixed. At $d=3$ it reads

$$g_0=mgZ_1(g)/[Z_3(g)]^2\ . \tag{4.2}$$

g_0 is a constant proportional (at $d=3$) to the inverse of the microscopic length a or $1/\Lambda$ (see Appendix A). It is a characteristic of the real system considered and will be an adjustable parameter (see Sec. V). From Eqs. (4.1) and (4.2) one obtains the dimensionless (starred) expressions for ξ and χ, in terms of g:

$$\xi^*=gZ_1(g)/[Z_3(g)]^2\ , \tag{4.3a}$$

$$\chi^*=g^2[Z_1(g)]^2/[Z_3(g)]^3\ , \tag{4.3b}$$

in which

$$\xi^*=\xi g_0\ , \tag{4.4a}$$

$$\chi^*=\chi g_0^2\ . \tag{4.4b}$$

When $m\to 0$ at g_0 fixed (as one approaches the CP), g goes to the fixed point g^* defined as the first nontrivial zero of $W(g)$, which, at $d=3$ reads

$$W(g)=-(d\ln\{gZ_1(g)/[Z_3(g)]^2\}/dg)^{-1}\ . \tag{4.5}$$

Hence the renormalized coupling constant g plays the same role as $\bar{u}(\lambda)$ in Sec. II C. Consequently Eqs. (4.3) give implicitly the critical behavior of ξ and of χ when g varies. It remains to relate these variations to that of the linear scaling field $\bar{r}'_0$ of Eq. (3.3). In order to maintain the notations used in previous papers, we shall use the symbol t instead of $\bar{r}'_0$, with[35]

$$t = r'_0 - r'_{0c} \; . \tag{4.6}$$

This linear scaling field no longer appears explicitly in the massive theory. It is, however, accessible from Eq. (3.21c) which, from the definition of the ϕ^2 insertions, and Eqs. (3.14)–(3.17) and (4.6), may be written as

$$[\partial\Gamma_0^{(0,2)}(\{0\};t,g_0)/\partial t]_{g_0} = +[Z_2(g)]^{-1} \; . \tag{4.7}$$

Using the Eq. (3.15) which defines m as ξ^{-1} and Eqs. (3.18b) and (3.21a), one may write equivalently

$$\frac{dt^*}{dg} = +Z_2(g)(d\{[Z_3(g)]^3/[gZ_1(g)]^2\}/dg) \; , \tag{4.8}$$

in which $t^* = t/g_0^2$ is dimensionless.

The integration of Eq. (4.8), with the initial condition $t^*(g^*)=0$, gives the expression of t^* in terms of g which we sought:

$$t^*(g) = -\int_g^{g^*} dx \left\{ Z_2(x)\left[d\left\{ \frac{[Z_3(x)]^3}{[xZ_1(x)]^2} \right\} \Big/ dx \right] \right\} . \tag{4.9}$$

Let us make some comments about this expression in relation to the discussion of Sec. III C.

The functions $Z_i(g)$ are given as power series in g. If one expands them in powers of x in Eq. (4.9) and integrates term by term, one will obtain singular terms for small g, and in particular a logarithm of g. These singularities are related to the definition of the critical bare theory at $d=3$ (see Sec. III C) and occur only far from the CP (small values of g). If, instead of working perturbatively, one integrates after having resummed the power series of the integrand, one will have no singularity if the integral converges.[15] The only possible divergence could come from x near g^* where the Z_i's are singular (g is kept strictly greater than zero). From the definitions of $W(g)$ [Eq. (4.5)], $\eta_3(g)$ and $\eta_2(g)$ [Eqs. (3.24) and (3.25)], and the expansion (3.23), one easily checks that as $x \to g^*$ from below one has

$$Z_3(x) \sim (g^* - x)^{\eta/\omega} \; , \tag{4.10a}$$

$$Z_2(x) \sim (g^* - x)^{\eta_2/\omega} \; , \tag{4.10b}$$

$$Z_1(x) \sim (g^* - x)^{2\eta - 1} \; . \tag{4.10c}$$

The integrand of Eq. (4.9) thus behaves as $(g^*-x)^{(\eta_2-\eta+1)/\omega}$, which leads for g close to g^*, to

$$t^*(g) \sim (g^* - g)^{1/\Delta} \; . \tag{4.11}$$

This result is valid if Δ ($=\omega\nu$) is positive. This exponent controls the corrections to the scaling behavior and its positivity assures the convergence of the integral (4.9).

It is on similar nonperturbative considerations and the hypothesis of negligibility of the right-hand side of the Callan-Symanzik equation that Parisi[15] based his proposal for calculating the critical exponents from the ϕ^4 model at integer dimensions $d=2$ or 3.

We are now in a position to obtain nonasymptotic critical behavior. The strategy is the following.

(1) Owing to the nonperturbative character of Eq. (4.9) the function $t^*(g)$ is obtained numerically at discrete values g_p of g ($0 \lesssim g_p \lesssim g^*$). This gives $t_p^*(g_p)$ such that $10^{-15} \lesssim t^* \lesssim 10^{15}$. In the next section we show how to perform the resummation of the series defining the integrand.

(2) Likewise, we resum the series of $\xi^*(g)$ and $\chi^*(g)$ from Eqs. (4.3) for the same set of values g_p of g yielding $\xi_p^*(g_p)$ and $\chi_p^*(g_p)$.

(3) The discretized numerical variations so obtained are continuously interpolated by phenomenological functions which reproduce with high accuracy the variation of ξ^* and χ^* in terms of t^* in the reduced range $10^{-15} \lesssim t \lesssim 10^{-2}$. These functions are presented in Sec. IV D for various values of the number of components n of the order parameter.

Until now we have not introduced the series which gives, as for ξ and χ in Eqs. (4.3), the specific heat. This quantity was not explicitly considered in Ref. 19, but was extracted from it in another article.[36] According to the notation used in Refs. 19 and 36, the bare correlation function $\Gamma_0^{(2,0)}(\{0\};\bar{r}_0,g_0)$, which is related to the specific heat (see Appendix A), is given by

$$\Gamma_0^{(2,0)}(\{0\};\bar{r}_0,g_0) = 6Z_5^{-1}(g)/g_0 \; , \tag{4.12}$$

in which the series $Z_5(g)$ (as Z_1, Z_2, and Z_3) are known up to the sixth-loop order (see Table I). The principle for obtaining $C^*(t^*)$ is the same as for ξ^* and χ^* presented just above. Let us now look at the numerical realization of this program.

B. Numerical treatment

As mentioned in the preceding subsection, we need to sum the series $Z_i(g)$ ($i=1,2,3,5$) presented in Table I with great accuracy for any value of g between 0 and g^*. The method of resummation used is that introduced by Le Guillou and Zinn-Justin,[17] which is based on the Borel transform of the series and conformal mappings. These transformations and further refinements of the method is suggested by the large order behavior of the series.[34] Let us denote by $z_i^{(k)}$ the kth order of the series $Z_i(g)$. Then for large k one can show[34] that

$$z_i^{(k)} \sim k!(-a)^k k^{b_i} c_i[1+O(1/k)] \; , \tag{4.13}$$

in which a, b_i, and c_i are known numbers.[34] We shall not recall the details of the resummation method here, as it has already been presented in great detail.[17] We simply indicate that we have used the simplified version which is also presented in detail in Ref. 27. The point we want to discuss now is different.

In principle, the resummation method gives rules for estimating the values of the Z_i's for any value of g (the method is independent of the value of g, and only converges less well as g increases). It also gives means of appreciating the sensitivity of the results with respect to the parameters introduced[17,27] (estimates of error bars). However, this is only true for value of g at which the Z_i's are

TABLE I. Series $[Z_i(\lambda)]^{-1}=\sum_{k=0}^{6}a_k\lambda^k$ for the symmetry $O(n)$ $(n=1,2,3)$ considered in this work. The numbers come from Ref. 19; see also Ref. 36 for $Z_5^{-1}(\lambda)/\lambda$ (a typing error in this reference is corrected here). The coupling λ is related to g through $g=-48\pi\lambda[Z_3(\lambda)]^{3/2}/Z_1(\lambda)$.

n	Z_1^{-1}	Z_2^{-1}	Z_3^{-1}	Z_5^{-1}/λ
1	1.0	1.0	1.0	0.0
	9.0	3.0	0.0	0.5
	98.999 999 999 60	21.0	0.444 444 548 15	1.5
	1245.198 933 648	199.545 169 287 4	5.111 177 322 60	8.773 907 565 00
	17 370.757 048 10	2282.813 025 630	60.348 829 923 0	72.519 023 056 0
	264 016.846 831 9	29 833.960 643 97	778.402 645 637	749.984 306 250
	4 321 786.169 513	432 776.546 616 2	10 950.295 614 4	9107.071 872 07
2	1.0	1.0	1.0	0.0
	10.0	4.0	0.0	1.0
	121.333 333 332 6	32.0	0.592 592 696 30	4.0
	1672.215 308 646	337.887 197 579 0	7.572 114 552 00	27.397 086 840 0
	25 416.380 571 06	4241.085 197 333	98.542 107 632 0	254.067 295 208
	418 762.338 105 0	60 332.111 959 85	1392.099 662 33	2890.111 619 20
	7 396 132.322 372	946 988.921 775 7	21 334.474 045 2	38 188.138 063 0
3	1.0	1.0	1.0	0.0
	11.0	5.0	0.0	1.5
	145.666 666 665 6	45.0	0.740 740 844 44	7.5
	2177.826 461 878	522.142 711 8018	10.411 657 509 0	58.869 537 825 0
	35 732.226 496 41	7122.121 801 299	147.800 231 681	605.157 241 752
	632 832.987 281 4	109 339.694 933 7	2264.117 691 16	7503.828 885 75
	11 968 416.691 48	1 842 682.709 973	37 449.768 403 6	107 060.358 495

not singular. As shown by Eqs. (4.10), they are singular near g^*, in whose vicinity we are essentially interested. The functions $W(g)$, $\eta_3(g)$, and $\eta_2(g)$, whose estimates at g^* are needed for obtaining the critical exponents, are nonsingular functions of g [$W(g)$ has only a simple zero at g^*]. This suggests expressing the Z_i's in terms of these nonsingular functions by using Eqs. (3.24), (3.25), and (4.5). After integration one has

$$gZ_1(g)=y_0Z_1(y_0)\times\exp\left[\int_{y_0}^{g}\{[2\eta_3(x)-1]/W(x)\}dx\right], \tag{4.14a}$$

$$Z_2(g)=Z_2(y_0)\exp\left[\int_{y_0}^{g}[\eta_2(x)/W(x)]dx\right], \tag{4.14b}$$

$$Z_3(g)=Z_3(y_0)\exp\left[\int_{y_0}^{g}[\eta_3(x)/W(x)]dx\right], \tag{4.14c}$$

in which $W(g)$, $\eta_3(g)$, and $\eta_2(g)$ are series obtained from the series $Z_i(g)$ of Table I by Eqs. (3.24), (3.25), and (4.5). The constant of integration y_0 is chosen small enough to estimate, from the series Z_i themselves, the factor $Z_i(y_0)$.

The functions $W(x)$, $\eta_3(x)$, and $\eta_2(x)$ are then evaluated for any value of x. Since the principle of the resummation method is independent of x, it is easy to include the estimates of the series inside a computer routine for integral evaluation. We thus obtain estimates of the Z_i's at fixed g. The integration is repeated for a large set of values g_p of g. The difference $g_{p+1}-g_p$ is chosen smaller and smaller as g approaches g^* in order to estimate well the strong contribution to the Z_i's which comes from $g\sim g^*$. This numerical analysis, presented for the functions Z_i, has been performed for ξ^*, χ^*, and t^*, whose definitions [Eqs. (4.3), (4.4), and (4.9)] have been written by using Eq. (4.14):

$$\xi^*(g)=\xi^*(y_0)\exp\left[-\int_{y_0}^{g}dx/W(x)\right], \tag{4.15a}$$

$$\chi^*(g)=\chi^*(y_0)\exp\left[-\int_{y_0}^{g}dx\,\gamma(x)/[\nu(x)W(x)]\right], \tag{4.15b}$$

$$t^*(g)=-t_0\int_{g}^{g^*}dx\,\gamma(x)/[\nu(x)W(x)]\times\exp\left[\int_{y_0}^{x}dz/[\nu(z)W(z)]\right], \tag{4.15c}$$

in which $t_0=Z_2(y_0)/\chi^*(y_0)$, $\chi^*(y_0)$, and $\xi^*(y_0)$ are given by Eqs. (4.3) for $g=y_0$.

The expression for $t^*(g)$ introduces a double integration, but it is only a question of computer time to estimate it within a similar accuracy as for a simple integration. The series $\nu(x)$ and $\gamma(x)$ which appear in the integrands of Eqs. (4.15) are obtained from Table I by using Eqs. (3.24), (3.25), (3.27), and (4.5) and the definition

$$\gamma(x)=\nu(x)[2-\eta_3(x)]\ . \tag{4.16}$$

In order to be consistent with the error correlations we have reestimated the fixed-point value g^* instead of using that of Ref. 17. The reason is that the error estimate of g^* accounted for the sensitivity in the change of the resummation method. In our work we could not use more than one method in the routine. The value we use for g^*

is compared to that of Ref. 17 in Sec. IV D.

We perform our numerical integrations two times according to the cases g^*_{max} and g^*_{min} which correspond to the upper and lower values of the fixed point. For each of these two cases there exist also the upper and lower estimates for ν, γ, and W denoted also by subscripts max and min. We have chosen the combination of these bounds which corresponds to the envelope of the possible functions $t^*(g)$ [and also for $\xi^*(g)$ and $\chi^*(g)$].

Our primary result is thus the discretized variations of t^*, ξ^*, and χ^* with respect to the renormalized coupling-constant values g_p given for the maximum and minimum. We present in Table II the 10 first points we have obtained for the maximum case. One can easily verify from this table the asymptotic behaviors:

$$t^*(g) \sim (g^* - g)^{1/\Delta} , \quad (4.17a)$$

$$\xi^*(g) \sim (g^* - g)^{-\nu/\Delta} , \quad (4.17b)$$

$$\chi^*(g) \sim (g^* - g)^{-\gamma/\Delta} , \quad (4.17c)$$

with the values for Δ, ν, and γ given in Table III.

One may realize that such a numerical presentation of the results is not well adapted for a comparison with experimental data. In Sec. IV D we present our final results under a form of continuous functions determined phenomenologically from these primary discretized variations. This has the advantages of eliminating the explicit dependence on the renormalized (unphysical) coupling g and of allowing a simpler comparison with the previous estimates of universal quantities calculated at the fixed point.[17,27,36] Before discussing this part of the work, we shall now present the specific-heat case which needs a slightly more complex numerical treatment.

C. Specific-heat case

The numerical treatment, as presented in the preceding section, was possible because of the asymptotic pure scaling behavior of the quantities considered (as g^* is approached). Consequently, the series $Z_i(g)$ $(i=1-3)$ calculated by Nickel *et al.*[19] gave simultaneously the functions studied and their singularities at g^*. For the specific heat the situation is different. First, it is defined from the series $Z_5(g)$ unrelated to the $Z_i(g)$ $(i=1-3)$ for which Eqs. (4.14) allow the numerical treatment described above. Second, the singular behavior of C cannot be easily factorized owing to its particular critical behavior:

$$C(t) \sim (A_0^+/\alpha) t^{-\alpha}(1 + a_0 t^{\Delta} + \cdots) + B_0 , \quad (4.18)$$

which contains a constant term B_0. The subscript 0 is used to distinguish the physical from the renormalized amplitudes (see Sec. V).

In order to have a pure scaling function we thus perform a derivative of $C(t)$ with respect to t at g_0 fixed. From the definition of C given in Appendix A and Eq. (4.12) one easily obtains

$$(\partial C/\partial t)_{g_0} = -(6/g_0^3) \frac{dZ_5^{-1}(g)/dg}{Z_2 d[Z_3^3/(gZ_1)^2]/dg} \quad (4.19)$$

as a power series in g. Near g^* the function $(\partial C/\partial t)_{g_0}$ behaves as $(g^*-g)^{-3(1-\nu)/\Delta}$, since $\alpha = 2-3\nu$ (at $d=3$). This means that it has the same singularity as the combination:

$$\{[Z_2(g)Z_3(g)]/[gZ_1(g)]\}^{-3} .$$

Factorizing out this combination of series in Eq. (4.19) one can write it in the form

$$\left[\frac{\partial C}{\partial t}\right]_{g_0} = \frac{6}{g_0^3} F(g) \left[\frac{y_0 Z_1(y_0)}{Z_2(y_0) Z_3(y_0)}\right]^3 \times \exp\left[-3\int_{y_0}^{g} dx \frac{1-\eta_3(x)+\eta_2(x)}{W(x)}\right] , \quad (4.20a)$$

TABLE II. Ten first primary results (for $n=1$) of the discretized evolutions of t^*, ξ^*, χ^*, and C^* as the renormalized coupling constant g goes far from g^*. The coupling variation is represented through $v = g(n+8)/48\pi$. Only the upper bound called max in the text is displayed. The fixed-point value in that case is $v^*_{max} = 1.420\,214\,705$ ($v^*_{min} = 1.410\,942\,26$) to be compared to the estimate $v^* = 1.416 \pm 0.005$ of Ref. 17.

v	t^*	$(\xi^*)^{-1}$	$(\chi^*)^{-1}$	C^*
1.420 214 000	2.4808×10^{-16}	3.0769×10^{-10}	1.5545×10^{-19}	84.1785
1.420 200 625	1.1004×10^{-13}	1.4355×10^{-8}	3.026×10^{-16}	41.2814
1.420 187 25	4.2846×10^{-13}	3.3825×10^{-8}	1.6386×10^{-15}	35.0348
1.420 1605	1.7111×10^{-12}	8.0986×10^{-8}	9.1576×10^{-15}	29.5605
1.420 107	6.9235×10^{-12}	1.955×10^{-7}	5.2016×10^{-14}	24.8165
1.42	2.8202×10^{-11}	4.7393×10^{-7}	2.9791×10^{-13}	20.7289
1.418 75	1.4080×10^{-9}	5.5772×10^{-6}	3.8404×10^{-11}	12.1712
1.4175	4.9531×10^{-9}	1.2324×10^{-5}	1.8327×10^{-10}	10.1144
1.415	1.8775×10^{-8}	2.854×10^{-5}	9.5927×10^{-10}	8.2266
1.41	7.4331×10^{-8}	6.790×10^{-5}	5.2963×10^{-9}	6.5512
⋮	⋮	⋮	⋮	⋮

Table III. Numerical values of the parameters found by adjustment of the function $F^*(t^*)$ [Eq. (4.22)] to the primary discretized evolutions of ξ^*, χ^*, and C^* for the symmetry $O(n)$ ($n=1,2,3$). In each case two sets of values are given (two successive lines) according to the bounds max (upper line) and min defined in the text. These bounds give an indication of the accuracy of the work. The functions so defined reproduce the ϕ^4 model at $d=3$ for $t \lesssim 10^{-2}$ (see text). The values for e and Δ can be compared directly (see Table IV) to the respective estimates obtained in Ref. 17. For the other universal characteristics of the asymptotic critical behavior of these functions see also Table IV.

F^*	n	e	Δ	X_1	X_2	X_3	X_4	X_5	X_6
ξ^*	1	0.6305	0.491 25	0.4708	1.5166	0.1984	25.446	0.1957	
		0.6291	0.500 31	0.4779	2.6876	0.1451	27.714	0.1804	
	2	0.6698	0.5201	0.3878	0.8277	0.4536	29.070	0.2369	
		0.6679	0.5274	0.3956	0.5203	0.7215	30.365	0.2279	
	3	0.7056	0.5498	0.3317	4.648	0.1611	33.427	0.2509	
		0.7038	0.5504	0.3367	3.3105	0.1968	33.152	0.2511	
χ^*	1	1.241 94	0.491 25	0.2629	2.0445	0.2021	20.069	0.3912	
		1.239 49	0.500 31	0.2698	1.4206	0.2831	21.443	0.3740	
	2	1.3179	0.5201	0.1813	0.4001	1.2694	24.100	0.4693	
		1.3140	0.5274	0.1884	0.3413	1.5102	24.934	0.4521	
	3	1.3892	0.5498	0.1317	4.5895	0.2910	29.749	0.4790	
		1.3836	0.5504	0.1383	2.5935	0.3372	27.546	0.5057	
C^*	1	0.108 50	0.491 25	1.7493	32.101	0.2623	16.962	-0.3976×10^{-1}	-3.8799
		0.112 64	0.500 31	1.5882	27.184	-3.4164	27.677	3.6424	-3.5757
	2	-9.545×10^{-3}	0.5201	-85.468	36.927	-1.856×10^{-2}	5.113×10^{-4}	1.5253	79.964
		-3.604×10^{-3}	0.5274	-207.003	34.175	-7.374×10^{-3}	8.943	6.988×10^{-4}	201.975
	3	-0.1169	0.5498	-19.461	39.706	-0.2087	13.078	-2.7×10^{-3}	8.971
		-0.1115	0.5504	-19.094	37.283	-0.2064	12.086	6.11×10^{-3}	9.186

$$F(g)=-\left[\frac{dZ_5^{-1}(g)}{dg}\right]\left[\frac{Z_3(g)Z_2(g)}{gZ_1(g)}\right]^3 \times\left[\frac{Z_2 d}{dg}\left[\frac{Z_3^3(g)}{g^2Z_1^2(g)}\right]\right]^{-1}, \quad (4.20b)$$

in which the singularity near g^* is treated as in the preceding section and the series $F(g)$ no longer has a singularity, it can thus be evaluated in the same way as $W(g)$, $\nu(g)$, and $\gamma(g)$. Hence, it remains to integrate with respect to t to find again the specific heat $C(t)$. Finally, the expression we have to evaluate for $C(g)$ is the following:

$$C^*(g)=-\frac{6y_0Z_1(y_0)}{[Z_2(y_0)]^2} \times\int_{y_0}^{g}dx\frac{\gamma(x)F(x)}{\nu(x)W(x)} \times\exp\left[\int_{y_0}^{x}dz\frac{-2+3\nu(z)}{W(z)\nu(z)}\right]+C^*(y_0)\,, \quad (4.21)$$

in which $C^*(g)$ stands for $C(g)g_0$ for dimensional reasons.

This expression can easily be found by using the procedures given in the preceding section. The constant of integration $C^*(y_0)$ is numerically determined from the original series [Eq. (4.12)] of C^* [y_0 is chosen sufficiently small to estimate $C^*(y_0)$ from the perturbative expansion itself]. We stress the fact, which will have importance in what follows, that the additive critical part B_0 of $C(g)$ in Eq. (4.18) is implicitly contained in Eq. (4.21). In other words, the evolution in g of $C(g)$ obtained in this way contains the complete critical behavior of C (the singular part and the constant B_0).

The numerical treatment of $C^*(g)$ follows the same method as that for $t^*(g)$ described in the preceding section. The 10 first points of $C^*_{\max}(g_p)$ are displayed also in Table II.

D. Final form of the results

We present in this section the final form of the nonasymptotic critical behavior of $\xi^*(t^*)$, $\chi^*(t^*)$, and $C^*(t^*)$ which comes from the ϕ^4 model at $d=3$. We have performed a smoothing of the discretized results whose derivation is presented in Secs. IV B and IV C. The useful information is displayed in Table III. In Table IV we compare the universal characteristics of our phenomenological theoretical functions with the previous estimates.[17,27,36] We also give in this table the estimates of the universal combination $R_{B_{\rm cr}}$ derived in Ref. 20.

We have looked for phenomenological functions which could reproduce with a relative error of less than 10^{-4} the variation of ξ^*, χ^*, and C^* in terms of t^*. This eliminates the explicit reference to the renormalized (unphysical) coupling constant g. We have chosen the form of the functions in such a way that the universal exponents (critical and subcritical) appear explicitly and that the usual universal combinations of amplitudes may be easily reconstituted. The other characteristics of the functions are arbitrary. We have found the general following form to be convenient:

$$F^*(t^*)=X_1t^{*^{-e}}(1+X_2t^{*^{\Delta}})^{X_3} \times(1+X_4t^{*^{\Delta}})^{X_5}+X_6\,, \quad (4.22)$$

in which F^* stands for any of the functions ξ^*, χ^*, and C^*, while e is the critical exponent characteristic of the function considered, and Δ has its usual significance. The quantities X_i ($i=1-6$) are adjusted to the points we have obtained from the numerical study at discretized values g_p of g (see Sec. IV B). This particular form is, thus, not universal. The form specified in Eq. (4.22) for $F^*(t^*)$ cannot reproduce the entire crossover, of the ϕ^4 model, to mean-field behavior. We limit the fit of the discretized evolution to values of t^* smaller than 10^{-2}. In a previous paper[1] we gave a complete reproduction of this crossover by using a more complex form for $F^*(t^*)$. In particular, an effective exponent $D(t^*)$ reproduced the crossover which also affects the correction exponent. It reproduced an evolution from the value Δ (≈ 0.5 for $n=1$) near the CP to exactly $\frac{1}{2}$ far from the CP. Since the ϕ^4 model cannot be thought to be quantitatively valid for small g (large t^*), we choose Eq. (4.22) (limited to $t^*\lesssim 10^{-2}$), which allows an easier and more correct expansion around $t^*=0$, the vicinity we are interested in.

As can be seen from Table IV, our error bars on the universal characteristics of the critical behavior are smaller than in previous work.[17,27,36] As already mentioned, we could not, without making the numerical calculation untractable, carry out a complete study of the sensitivity on the resummation method used. Hence, only one criterion is chosen to determine the error for each resummed series. However, the very small error bars which appear, in particular for quantities associated with the first corrections, are also a consequence of the fact that in the present work the correlation between the errors are taken into account better. When calculating separately a universal quantity, one must add the errors on all the terms independently estimated which compose the desired quantity without looking at the correlations between the errors. In our treatment we resum a small number of series [$W(x)$, $\nu(x)$, $\gamma(x)$, $F(x)$] to obtain the complete critical behavior, and we combine the error estimate in order to obtain the envelope (max,min). The correlations between errors are thus taken into account. We stress the fact that our numerical treatment is entirely independent of the previous work[17,27,36] and the agreement displayed in Table IV must be seen as a check of the correctness of the numerical study. Figure 2 shows, on the effective exponent $\gamma_{\rm eff}(t^*)$ defined by Eq. (5.10) below, the evolution with t^* of the error estimates. The error decreases when t^* increases; this is due to the fact that the renormalized coupling constant g also decreases, and consequently the series are better resummed. In the next section we discuss the interest of knowing the precise evolution of $\xi^*(t^*)$, $\chi^*(t^*)$, and $C^*(t^*)$ from the ϕ^4 model for a comparison between theory and experiment.

TABLE IV. Comparison between estimates of universal combinations of amplitudes from this work (Table III) and previous ones (Refs. 17, 27, and 36). The agreement is good despite the various numerical integrations performed in the present work. The tendency is to underestimate the error in particular for Δ and the correction amplitude ratios. Let us specify, however, that in Refs. 17, 27, and 36 the correlations between errors were not taken into account as it is in the present treatment. This explains, in part, the minimization of the errors. The last column gives estimates of the universal combination $R_{B_{cr}}$ [see Eq. (7.7) and Ref. 20].

n	ν	γ	α	Δ	R_ξ^+	a_ξ^+/a_χ^+	a_c/a_χ^+	$R_{B_{cr}}^+$
1	0.6298(7)	1.2407(12)	0.1106(21)	0.496(5)	0.2700(7)	0.64	8.6(2)	−0.7081(5)
	0.6300(15)	1.2410(20)	0.1100(45)	0.498(20)	0.2699(8)	0.65(5)	8.5(9)	
2	0.6689(10)	1.3160(20)	−0.0066(30)	0.524(4)	0.3606(20)	0.615(5)	5.95(15)	−1.057(22)
	0.6690(20)	1.3160(25)	−0.0070(60)	0.522(18)	0.3597(10)	0.600(40)	5.9(5)	
3	0.7047(9)	1.3864(28)	−0.1142(27)	0.5501(3)	0.4347(20)	0.60(1)	4.6	−1.3785(41)
	0.7050(30)	1.3860(40)	−0.1150(90)	0.550(16)	0.4319(17)	0.59(6)	4.6(5)	

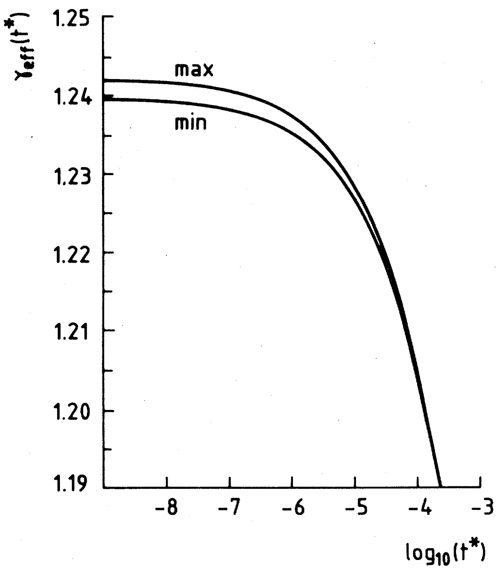

FIG. 2. Evolution of the error of the numerical analysis as it is estimated in this work for $\gamma_{\rm eff}(t^*)$ [see Eq. (5.10)]. The error decreases as t^* increases owing to the fact that the coupling g also decreases. The significance of the notation max and min is given in the text.

V. USE OF THE RESULTS FOR A COMPARISON WITH EXPERIMENT

A. Adjustable parameters and two-scale-factor universality

The functions given in the preceding section are theoretical results whose validity is limited to the preasymptotic critical domain $\mathscr{D}_{\rm preas}$. This means that they cannot be used in a range of t^* in which the expansion around $t^*=0$ reveals a non-negligible contribution of the second correction to scaling. In the following subsection this limitation will be used as a criterion to determine the size of $\mathscr{D}_{\rm preas}$. At present we want to discuss the introduction of the adjustable parameters of Refs. 1 and 2 in more detail.

At the end of Sec. II we showed that, provided that the first correction never vanishes, there are only three adjustable parameters. The origin of these three adjustable parameters may be easily seen, since in the ϕ^4 bare theory at infinite cutoff, only three parameters characterize a system. They are the coupling g_0, the scaling field t^*, and the magnetic scaling field $h^*=h/g_0^{5/2}$ (h is introduced in Appendix A). From the hypothesis of analyticity, they are, within $\mathscr{D}_{\rm preas}$, related to the physical parameters as follows:

$$t^*=\theta\tau\ , \tag{5.1a}$$

$$h^*=\psi H\ , \tag{5.1b}$$

$$g_0=u_0/a\ . \tag{5.1c}$$

These relations come from Eqs. (A4) of Appendix A with $\theta=b_0/g_0^2$ and $\psi=c_0/g_0^{5/2}$, while the coupling constant g_0, the single-dimensioned parameter, is expressed in terms of a microscopic length a ($g_0=u_0/a$) which may be chosen *a priori* according to the physical system studied. The three adjustable parameters are thus θ, ψ, and u_0.

From the definitions of the physical correlation length $\xi_{\text{expt}}(\tau)$ and susceptibility $\chi_{\text{expt}}(\tau)$ given in Appendix A and those of the ϕ^4 model given in Table III, we have the following correspondences within $\mathscr{D}_{\text{preas}}$:

$$\xi_{\text{expt}}(\tau)=(a/u_0)\xi^*(\theta\tau) \;, \tag{5.2a}$$

$$\chi_{\text{expt}}(\tau)=(u_0/a)^3\psi^2\chi^*(\theta\tau) \;, \tag{5.2b}$$

in which only u_0, θ, and ψ are adjustable parameters. Let us note that expression (5.2a) for the correlation length has the right dimension [a length as indicated by a which may be chosen in, for example, a liquid-gas system as[2] $a=(k_B T_c/P_c)^{1/3}$, where P_c is the critical pressure]. Equation (5.2b) gives to χ_{expt} the dimension of an inverse volume; the reduction of the measured susceptibility to this dimension must be envisaged first. The precise choice of the reduction process does not matter as long as only the inhomogeneous phase is concerned, since the scale ψ does not enter either in ξ or in C (see below).

Owing to the presence of this nonuniversal scale ψ in the asymptotic critical amplitude of χ, there does not exist a universal relation between the two amplitudes of ξ and χ. However, the two confluent amplitudes depend only on θ and their ratio is a universal number. In Table IV we give the numerical value of this ratio as derived from an expansion around τ (or t^*) equal to zero. To be explicit, these expansions performed on Eqs. (5.2) give

$$\xi_{\text{expt}}(\tau)\sim (a/u_0)X_1^{\xi}(\theta\tau)^{-\nu} \times[1+(X_2^{\xi}X_3^{\xi}+X_4^{\xi}X_5^{\xi})(\theta\tau)^{\Delta}+\cdots] \;, \tag{5.3a}$$

$$\chi_{\text{expt}}(\tau)\sim (u_0/a)^3\psi^2X_1^{\chi}(\theta\tau)^{-\gamma} \times[1+(X_2^{\chi}X_3^{\chi}+X_4^{\chi}X_5^{\chi})(\theta\tau)^{\Delta}+\cdots] \;. \tag{5.3b}$$

The universal ratio is thus equal to $(X_2^{\xi}X_3^{\xi}+X_4^{\xi}X_5^{\xi})/(X_2^{\chi}X_3^{\chi}+X_4^{\chi}X_5^{\chi})$. This result is a consequence of the fact that θ must have the same value in the two functions adjusted to the experimental data. It is thus the first possible test of the theoretical predictions provided that the experimental data are accurate enough to observe correction to scaling.

The third physical quantity, the specific heat, is much more interesting. It also has a more complicated form. As explained in Appendix B, in addition to the calculated function C^*, we must add a regular part which does not belong to the critical one. The comparison with the measured specific heat per unit volume divided by k_B will thus be made from our function C^* (see Table III) using the following relation:

$$C_{\text{expt}}(\tau)=(u_0/a)^3\theta^2C^*(\theta\tau)+B_{\text{bg}}(\tau) \;, \tag{5.4}$$

in which $B_{\text{bg}}(\tau)$ must be determined from the experimental data far from the CP. In Fig. 3 we illustrate what could be a possible estimate of $B_{\text{bg}}(\tau)$ in the case of the superfluid transition of ^{4}He near the λ point (at the saturated vapor pressure). Hence $B_{\text{bg}}(\tau)$ is not an adjustable parameter in the analysis of the data near the CP.

The experimental determination that we propose for $B_{\text{bg}}(\tau)$ is simply an interpolation in the critical domain of

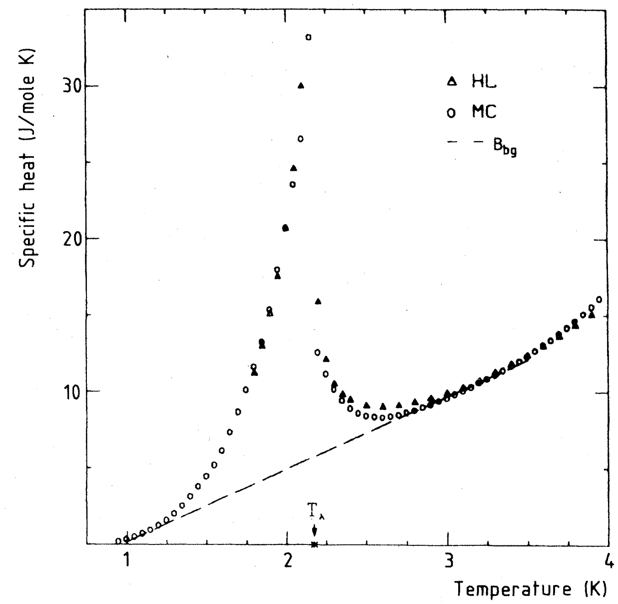

FIG. 3. Determination of the background term $B_{\text{bg}}(\tau)$ (dashed line) of the specific heat of ^{4}He near the λ transition at saturated vapor pressure from Hill and Lounasmaa (HL) (Ref. 38) and Mc Carty (MC) (Ref. 39) $[\tau=(T-T_c)/T_c]$.

the regular behavior of C_{expt} far from the CP. In any case, the asymptotic critical behavior of C_{expt}, as noted by Singsaas and Ahlers,[37] is not very sensitive to the exact value of B_{bg} determined from the experimental data far from the CP. The specific-heat behavior is the sum of this regular behavior and a singular one. The latter, generated by the long-range correlations between the fluctuations, also contains a regular part: the critical constant B_{cr}.[20] This constant, in the case where α is negative (the ^{4}He superfluid transition, for example) is equal to the difference, at the CP, between the value of C and that of the regular part $B_{\text{bg}}(0)$. We have shown[20] that B_{cr} is, apart from factors that we will introduce below, calculable within the pure ϕ^4 model. This means that no additive contribution to B_{cr} comes from the higher-transient or finite-cutoff effects. In order to calculate B_{cr} it is necessary that the pure ϕ^4 model generates only the singular behavior of C and does not contain a part of $B_{\text{bg}}(\tau)$. This means that it must give $C^*\sim(X_1t^{*-\alpha}+X_6)$ when $t^*\to 0$ and $C^*\to 0$ when $t^*\to\infty$. The form presented in Table III does not have this property because it reproduces the critical behavior of the ϕ^4 model only from $t^*\approx 0$ up to $t^*\approx 10^{-2}$. However, we have performed for $n=1$ a complete reproduction up to $t^*\approx 10^{15}$, and the form proposed for C^* in the Table I of Ref. 1 has the correct behavior in the two limits (as can be easily checked). This result does not depend on the value of n since the limit $t^*\to\infty$ corresponds to the limit $g\to 0$ and the series $Z_5^{-1}(g)$, which gives C, has no constant term (see Table I of the present paper).

If one expands Eq. (5.4) near $\tau=0$, we obtain from Eq. (4.22) and the X_i given in Table III:

$$C_{\text{expt}}(\tau)=(u_0/a)^3\theta^2\{X_1^c(\theta\tau)^{-\alpha}[1+(X_2^cX_3^c+X_4^cX_5^c)(\theta\tau)^{\Delta}+\cdots]+X_6\}+B_{\text{bg}}(\tau)\ . \quad (5.5)$$

By comparing this with Eq. (5.3), we see clearly that the combination

$$R_\xi^+=\alpha\tau^2C_{\text{expt}}^{\text{sing}}(\tau)[\xi_{\text{expt}}(\tau)]^3$$

is universal with

$$R_\xi^+=\alpha X_1^c[X_1^\xi]^3\ , \quad (5.6)$$

whose values, given in Table IV, are compared to the previous (and independent) estimates.[36] Similarly, any ratio of the confluent amplitudes of ξ, χ, and C is universal.[26,27] In addition, there is a new universal combination[20] of amplitudes which concerns only the specific heat in a given phase (here the inhomogeneous phase). In Ref. 20 this universal quantity is called $R_{B_{\text{cr}}}^+$, and from Eq. (5.5) its value is given by

$$R_{B_{\text{cr}}}^+=X_1^c\,|\,X_2^cX_3^c+X_4^cX_5^c\,|^{\alpha/\Delta}/X_6\ . \quad (5.7)$$

From Table III one easily obtains estimates for $B_{B_{\text{cr}}}$ which are displayed in Table IV. We claimed that we could not understand the case $\alpha=0$ without the universality of $R_{B_{\text{cr}}}$. In that case one expects a logarithmic behavior, $C_{\text{sing}}^{(0)}\sim -A_0\ln\tau$, which occurs at some particular value of n, say n_0. Hence the singular behavior of C, whose general asymptotic form is $C_{\text{sing}}\sim(A/\alpha)\tau^{-\alpha}+B_{\text{cr}}$, must pass continuously through $C_{\text{sing}}^{(0)}$ as n varies. This is possible if, for $n\sim n_0$, $B_{\text{cr}}\sim-(A_0/\alpha)$ such that

$$C_{\text{sing}}^{(0)}=\lim_{\alpha\to 0}(A_0/\alpha)(\tau^{-\alpha}-1)\ .$$

We do not know the value n_0, but at $n=2$, α is very small and negative while for $n=1$ it is positive. In Table III X_1 and X_6, for C^*, play the roles of A_0/α and B_{cr}. If one compares their values for the three values $n=1,2$, and 3, one sees that at $n=2$, X_1 and X_6 have much greater values than for $n=1$ or 3. They are also of the same order (for $n=1$ or 3 there is almost a factor 2) and of the opposite sign. Clearly, n_0 is not far from $n=2$. This indicates that X_6 (given in Table III) has the property required for B_{cr} and has been correctly determined.

All these considerations show clearly that our functions with the minimal set of adjustable parameters θ, ψ, and u_0 quantitatively contain *all the universal characteristics* of the critical behavior within $\mathscr{D}_{\text{preas}}$ as predicted by the theory. In addition, it appears clear that the two-scale-factor universality, whose consequence is usually restricted to the universality of R_ξ^+,[40] has a larger importance and is valid in the whole $\mathscr{D}_{\text{preas}}$.

The two-scale-factor universality expresses the fact that apart from a length-scale dependence u_0 which is factorized (from a simple dimensional analysis), the critical behavior within $\mathscr{D}_{\text{preas}}$ is universal provided that the temperature θ and the magnetic field (ψ) scales are properly chosen according to the physical system. This is true only if the first confluent correction is not identically zero. We have shown in Sec. II that this is not the case at the fixed point which corresponds to some particular value u_0^* of u_0 and to $a=0$ (infinite cutoff). Then, not only is the first correction zero, but also all the other corrections (by definition of the fixed point). This is not a realistic situation since the length a is never zero in a real system. However, although very unlikely, it is possible that the corrections induced by the ϕ^4 model (and so the first one in the real behavior) might be identically zero for Ising-like system ($n=1$) (systems with a positive α). Because of the constraint $R_{B_{\text{cr}}}$, this could not be possible for systems with $n\geq 2$ for which the critical constant B_{cr} would be infinite[20] (α is negative). In Ising-like systems with a zero first correction, the critical constant B_{cr} would also be zero and the asymptotic critical behavior of C_{expt} would have the form

$$C_{\text{expt}}(\tau)=(A^+/\alpha)\tau^{-\alpha}[1+O(\tau,\tau^{\Delta_2})]+B_{\text{bg}}(\tau)\ , \quad (5.8)$$

and similarly for ξ and χ. Our nonasymptotic functions should, then, not be very useful in analyzing the experimental data in such systems.

In order to be clear on the question of the number of adjustable parameters needed within $\mathscr{D}_{\text{preas}}$ and the possibility of a vanishing first confluent correction, we must restore the presence of a fourth adjustable parameter [related to $c(u,\rho)$ in Sec. II] by a change of the scales θ, ψ, and u_0 which become

$$\theta'=c^{-1/\Delta}\theta\ , \quad (5.9a)$$

$$u'=c^{\nu/\Delta}u_0\ , \quad (5.9b)$$

$$\psi'=c^{-(\gamma+3\nu)/\Delta}\psi\ . \quad (5.9c)$$

Hence, the new parameter c may vanish and this would allow the corrections in powers of τ^{Δ} to be zero according to the Sec. II. Nevertheless, in the case of $c\neq 0$, only the three adjustable parameters θ, ψ, and u_0 remain relevant in $\mathscr{D}_{\text{preas}}$. The adjustable parameters being introduced, let us now look at the practical use and interest of our nonasymptotic critical behavior.

B. Determination of the size of $\mathscr{D}_{\text{preas}}$

One of the great advantages of our nonasymptotic functions for ξ, χ, and C is that they provide the possibility of estimating the size of $\mathscr{D}_{\text{preas}}$. The expansions around $t^*=0$ of the functions generate the WE of the pure ϕ^4 model (i.e., neglecting analytic corrections and higher-transient effects). Hence, we obtain an illustration of how a WE converges by comparing the expansions and the complete functions as t^* varies. This can be based not only on pure theoretical grounds but also on the true critical behavior. Let us illustrate this by considering the results of the analysis of the data[41] on xenon made in Ref. 2. This system corresponds to an Ising-like system ($n=1$). Let us focus our attention on the susceptibility for which the measurements are most accurate.[41] They are presented relative to a reference temperature $\tau sibr$ as the ratio $\chi_{\text{expt}}(\tau)/\chi_{\text{expt}}(\tau_r)$. Hence the scale factor ψ was eliminated and the comparison could be made directly with our results by using the function $\mathcal{X}^*(\theta\tau)/\mathcal{X}^*(\theta\tau_r)$. Of course, only θ, related to the size of $\mathcal{D}_{\text{preas}}$, can be determined in this way. We are not able to appreciate the true influence, beyond this region, of the other corrections neglected in the ϕ^4 model.

However, we indicate that the work of Newman and Riedel[21] suggests that for $n=1$ the exponents of the second and third nonanalytic confluent corrections Δ_2 and Δ_3 are very close to 2Δ and 3Δ as in the pure ϕ^4 model. In addition, the first analytic correction is very much like a $\tau^{2\Delta}$ term while the second one should be relevant farther from T_c than the $\tau^{3\Delta}$ term. Consequently, if not perfectly correct, our model may give a rather good illustration of the influence of the corrections to scaling in real systems even out of $\mathcal{D}_{\text{preas}}$.

Within the available temperature range $10^{-1} \lesssim \tau \lesssim 10^{-4}$ of the measurements, Güttinger and Cannell observed[41] the possibility of three correction terms controlled, respectively, by Δ, 2Δ, and 3Δ. This observation was only possible because of the great accuracy of the measurements, and was corroborated by the fact that we could reproduce the data with a rather well-defined value of θ:[2] $\theta=(1.91\pm0.85)\times10^{-2}$. Furthermore, we have shown[2] from the other thermodynamical measurements that this value was coherent on the global set of experimental data (ξ, $\mathcal{X}$, and C).

The best way to illustrate the effects of the corrections to scaling is to consider the effective critical exponent[10] defined, in the case of $\mathcal{X}$, by

$$\gamma_{\text{eff}}(\tau)=-d\ln[\mathcal{X}(\tau)]/d\ln\tau\ . \tag{5.10}$$

The deviation from a constant critical value γ indicates the effect of the correction terms. From their experimental data on xenon, Cannell and Güttinger[42] extracted a dependence on τ for γ_{eff}. In Fig. 4 we present the successive influences, on $\gamma_{\text{eff}}(t^*)$, of the three first corrections to scaling compared to the complete variation given by the ϕ^4 model (Table III). From a simple shift of the logarithm of t^* we indicate the relative position of the measurements in xenon.[41] We are thus in a position to determine the range of τ for xenon, where asymptotic, preasymptotic, and other regimes respectively dominate. One observes in Fig. 4 that the overlap with $\mathcal{D}_{\text{preas}}$ is too small to obtain very accurate experimental determination of the asymptotic critical behavior of xenon. If, as claimed in Ref. 41, theorists must reduce the number of free parameters, reducing the gravitational effects is also an important experimental task. In this spirit, we have made[43] a comparison between $C^*(t^*)$ with experimental data in ^{4}He ($n=2$) obtained recently by Lipa and Chui.[23] The gravitational effects in this system are much smaller and the data usually analyzed with only one correction to scaling. The universality of $R_{B_{\text{cr}}}$ automatically satisfied by Eq. (5.5) and the control of the size of $\mathcal{D}_{\text{preas}}$ that we obtain by using the complete function C^* of Table III are very constraining. They provide an opportunity to appreciate the reality of the slight deviation observed[23,37] between the theoretical and experimental estimates of α. We show that analytic confluent corrections have much more importance than is usually thought.

In summary, we think that we have partly replied to the hope expressed by Fisher[44] in 1974. Our calculations may be used to "reliably indicate the size of the asymptotic critical region" within experimental accuracy. They are sufficiently precise to make the theory "fully—in the sense, globally—testable" from experiments. There remain, however, many other problems to be considered such as the equation of state including corrections to the scaling form. The consideration of the homogeneous phase (below T_c) along single critical lines (isochore or isotherm) is first needed. In a subsequent paper,[45] we will present the way to derive nonasymptotic critical behavior in this phase within the massive FT at $d=3$. Preliminary results have already been published[46] which concern estimates of universal combinations of critical amplitudes.

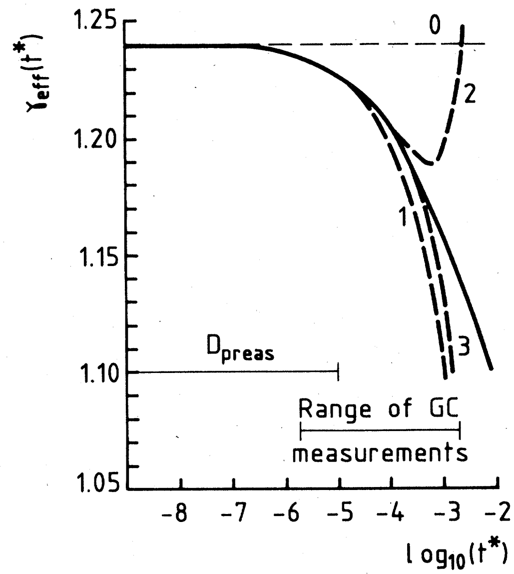

FIG. 4. Effective exponent $\gamma_{\text{eff}}(t^*)$ [see Eq. (5.10)] provides a good illustration of the influence of the different corrections to scaling. A pure scaling law would give a straight horizontal line at $\gamma_{\text{eff}}=1.2419$ (dashed curve, 0). The solid line indicates $\gamma_{\text{eff}}(t^*)$ as it follows from the nonasymptotic function $\mathcal{X}(t^*)$ given in Table III. The other dashed curves give the evolution of the expansion around $t^*=0$ of $\gamma_{\text{eff}}(t^*)$ truncated at the first (1), second (2), and third (3) confluent corrections. The scale of t^* indicated is purely theoretical. A comparison with the actual range of τ in the measurements of Güttinger and Cannell (GC) in xenon is made through a change of scale: $t^*=\theta\tau$ with $\theta\sim1.9\times10^{-2}$ (see Ref. 2). One clearly sees that the xenon data do not reach the domain $\mathcal{D}_{\text{preas}}$. An efficient comparison with theory is thus difficult in such a system where the gravitational effects are too important to allow a correct determination of the asymptotic critical behavior.

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APPENDIX A: BARE THEORY AND PHYSICS OF CRITICAL PHENOMENA

In this appendix we recall some details (many of them are well known) about the relation between the FT approach and statistical physics. The relevance of the following Hamiltonian:

$$\mathscr{H}\{\phi_0\}=\int d^dx\{\tfrac{1}{2}[(\nabla\phi_0)^2(x)+r_0\phi_0^2(x)] +(g_0/4!)\phi_0^4(x)-h\phi_0(x)\} \tag{A1}$$

is well known.[5,13] We shall not recall the arguments here, but simply limit ourselves to a review of the main features useful to the understanding of this paper.

The field $\phi_0(x)$ represents a local fluctuating parameter whose statistical mean value $\langle\phi_0(x)\rangle=M_0(x)$ is the physical (local) order parameter of a given system. In ferromagnets it is the magnetization; in liquid-gas systems it stands for the deviation of the density from its critical value. In the case where the system is homogeneous (as is generally the case) $M_0(x)$ is a constant M_0. For Ising-like systems, ϕ_0 has only one component; in other systems it could have n components ($n=1,2,3,\ldots$), and then stands for a set $(\phi_0^{(1)},\phi_0^{(2)},\ldots,\phi_0^{n})$. The implicit notations in Eq. (A1) had to be understood as $\phi_0^2=\sum_{i=1}^{n}(\phi_0^{(i)})^2$ and $\phi_0^4=(\phi_0^2)^2$. We shall not consider the possible breaking of the $O(n)$ symmetry (this will be emphasized in a subsequent paper[45]).

The statistical average is performed with the weight

$$\mathscr{P}\{\phi_0\}=\mathscr{N}^{-1}\exp(-\mathscr{H}\{\phi_0\}), \tag{A2}$$

where $\mathscr{N}$ is a normalization constant and the curly brackets indicate the functional character of $\mathscr{P}$. The mean value of any functional $\mathscr{A}\{\phi\}$ is given by the functional integral:

$$\langle\mathscr{A}\rangle=\int\mathscr{D}\phi_0\mathscr{A}\{\phi_0\}\mathscr{P}\{\phi_0\}. \tag{A3}$$

The dependence on the physical temperature T and on the physical magnetic field H (the field thermodynamically conjugated to the order parameter of the system considered) is implicitly contained in the parameters r_0, g_0, and h via the hypothesis of analyticity[5] around the CP ($H=0$, $\tau=0$):

$$r_0=r_{0c}+b_0\tau+O(H^2\tau,\tau^2), \tag{A4a}$$

$$g_0=\text{const}+O(H^2,\tau), \tag{A4b}$$

$$h=c_0H+O(H^2,H\tau), \tag{A4c}$$

where r_0, b_0, c_0, and const are constants which depend on the physical system considered and τ is $(T-T_c)/T_c$.

In the expansions (A4) the limitation to the first relevant terms already supposes that the system is close to the CP. In this paper we neglect the higher terms of the Taylor expansions (A4). They must be restored when the validity domain of the ϕ^4 model is discussed (see Secs. II and V).

The Hamiltonian (A1) is related to the thermodynamic potential $\Phi(h,t)$ (the Gibbs free energy) through the relations

$$\Phi(h,\tau)=\Phi_{\text{reg}}(\tau)-k_BT\ln[Z(h,\tau)], \tag{A5a}$$

$$Z(h,\tau)=\int\mathscr{D}\phi_0\mathscr{P}\{\phi_0\}, \tag{A5b}$$

in which k_B is the Boltzmann constant and $\Phi_{\text{reg}}(\tau)$ corresponds to a regular part of the Gibbs free energy. Another way to account for this regular part is to add a "constant" (independent of ϕ_0) term in $\mathscr{H}$. Like r_0, g_0, and h, this constant would have an analytic dependence on τ. The net separation we make between that regular part and the critical part of the Gibbs free energy allows a clear distinction between the critical regular part and the "background" term of the specific heat (see Appendix B).

The Helmholtz free energy $F(M_0,\tau)$ is, as usual, obtained from $\Phi(H,\tau)$ by the Legendre transform:

$$F(M_0,\tau)=\Phi(H,\tau)+k_BTHM_0, \tag{A6a}$$

$$M_0=-(\partial\Phi/\partial H)_\tau/k_BT. \tag{A6b}$$

One easily relates these free energies to the generating functionals for the connected ($W\{J\}$) and the one-particle irreducible ($\Gamma\{M_0\}$) correlation functions familiar to field theorists.[13] Let us suppose that the field h depends on x, and let us call it $J(x)$. In Eq. (A5b) $Z(h,\tau)$ becomes $Z\{J\}$, a functional of J, but remains a simple function of τ (through r_0).

$W\{J\}$ and $\Gamma\{M\}$ are defined as follows:

$$W\{J\}=\ln(Z\{J\}), \tag{A7}$$

$$\Gamma\{M\}=-W\{J\}+\int d^dx[J(x)M_0(x)], \tag{A8a}$$

$$M_0(x)=\delta W\{J\}/\delta J(x). \tag{A8b}$$

If one compares Eqs. (A5a) and (A6a) to Eqs. (A7) and (A8), one sees that they differ only by regular terms in τ and H. Notice that the k_BT term is nothing but $k_BT_c(1+\tau)$ and generates only analytic correction terms in τ. From the same reasons which lead us to neglect the higher terms in Eqs. (A4), we shall take $k_BT=k_BT_c$ in Eqs. (A5) and (A6).

The physical quantities in which we are interested (in the disordered phase) are the susceptibility χ, the correlation length ξ, and the specific heat C. The first two quantities are strictly related to the fluctuations [the regular part $\Phi_{\text{reg}}(\tau)$ will not contribute to them]. A functional manipulation leads, from the definition $\chi(\tau,M_0)=(\partial M_0/\partial H)_\tau$ and Eqs. (A4)–(A8) to

$$[\chi(\tau,M_0)]^{-1}=\Gamma_0^{(0,2)}(\{0\};r_0,g_0,M_0)/c_0^2, \tag{A9}$$

in which $\Gamma_0^{(L,N)}(\{q,p\};r_0,g_0,M_0)$ are the Fourier transforms of the one-particle irreducible correlation functions corresponding to

$$\langle\tfrac{1}{2}\phi_0^2(y_1)\cdots\tfrac{1}{2}\phi_0^2(y_L)\phi_0(x_1)\cdots\phi_0(x_N)\rangle.$$

The notation $\{q,p\}$ in Eq. (A9) stands for

$$\{q_1,q_2,\ldots,q_L,p_1,p_2,\ldots,p_N\}$$

the set of wave vectors Fourier conjugated to the set

$$\{y_1,y_2,\ldots,y_L,x_1,x_2,\ldots,x_N\}.$$

Similarly, the second moment of the correlation function $\langle \phi_0(x_1)\phi_0(x_2)\rangle$ defines ξ as

$$[\xi(\tau,M_0)]^{-2}=\frac{\Gamma_0^{(0,2)}(\{0\};r_0,g_0,M_0)}{\frac{\partial}{\partial p^2}\Gamma_0^{(0,2)}(\{p\};r_0,g_0,M_0)\,|_{p^2=0}} . \quad (A10)$$

This time there is no extra factor.

As for the specific heat, it is slightly more complicated since it requires the second derivative of F with respect to T:

$$C(\tau,M_0)=-Td^2F(\tau,M_0)/dT^2 . \quad (A11)$$

The regular part $\Phi_{\rm reg}$ will thus enter in the definition of C. From Eqs. (A4) and the remark that the derivatives of $\Gamma\{M\}$ and $W\{J\}$ with respect to r_0 (at g_0 and h fixed) generate the insertion of $-\frac{1}{2}\int d^dx\,\phi_0^2(x)$ in the correlation functions, it is easy to check that

$$C(\tau,M_0)=-k_B b_0^2\Gamma_0^{(2,0)}(\{0\};r_0,g_0,M_0)+C_{\rm reg}(\tau) \quad (A12)$$

with

$$C_{\rm reg}(\tau)=-k_B d^2\Phi_{\rm reg}(\tau)/d\tau^2 , \quad (A13)$$

apart from analytic corrections in τ and H which, once more, are neglected, while they must be restored when more than one confluent correction to scaling is needed in three-dimensional systems.[43]

As explained in Sec. II, neglecting these regular terms is consistent with the limitation of the description of the preasymptotic critical domain $\mathscr{D}_{\rm preas}$ in which the ϕ^4 Hamiltonian is supposed to be valid. Indeed the Hamiltonian (A1) is not general enough to describe the whole critical domain. Many (an infinite number) of other terms (like ϕ_0^6, for example) would be needed.

We show in Sec. II that the effects of these higher transients may be qualitatively reproduced by the consideration of an explicit finite cutoff Λ, which has not yet been introduced in this appendix. This cutoff should enter in the Hamiltonian (A1) from two different physical sources.

(1) It recalls the discretized structure of any real systems: for example, the range of the molecular forces or the lattice spacing of the Ising model. We shall denote this microscopic length by a. In the continuous limit, the cutoff Λ is proportional to $1/a$ and prevents the Feynman integrals (in the perturbative expansion in powers of g_0) from uv divergence. This dependence on the cutoff appears explicitly as an additional argument within the correlation functions (i.e., independently of the bare quantities r_0, g_0, and M_0) in Secs. II and III. The infinite-cutoff limit considered at $d=3$ in this paper concerns this Λ dependence. It is distinguished from the following source.

(2) The long-range correlations between the fluctuations in the vicinity of the CP make any wavelength L of fluctuation between a and ξ (intermediate fluctuations) relevant to the critical phenomena. Owing to the strong correlations, these intermediate fluctuations have small amplitudes and their effects are essentially limited to the transmission of the small scale structure to the large scale. This cascade picture is increasingly correct as ξ grows. The final effect of the strong correlations is the transmission, over any length scale, of the microscopic structure (over the length scale a) of the system. This is expressed in the FT approach by a strong dependence on a (or Λ^{-1}) of any bare parameter. In the system of units used here, where only the lengths are considered, one thus has

$$\phi\sim\Lambda^{(d-2)/2} , \quad (A14a)$$

$$r\sim\Lambda^2 , \quad (A14b)$$

$$g\sim\Lambda^{(4-d)} , \quad (A14c)$$

$$h\sim\Lambda^{(d+2)/2} . \quad (A14d)$$

This cutoff dependence is formally separated in Sec. III from that of source (1) above. This allows a clear distinction to be kept between the bare theory at $\Lambda=\infty$ (the bare quantities being kept finite and dimensioned by Λ) and the unphysical renormalized theory.

Let us now look at the relation between the infinite-cutoff limit (at fixed bare parameter) and the critical limit. In Sec. III we show that one must first introduce a critical parameter (scaling field) whose zero value defines the CP. For convenience we shall choose here the inverse correlation length m as defined in Eq. (3.15). This definition does not require considering the infinite-cutoff limit, and we shall thus suppose Λ finite in the following. The physical (bare) correlation functions at zero momenta, along the critical isochore above T_c are given by $\Gamma_0^{(L,N)}(\{0\};m,g_0,\Lambda)$. From dimensional analysis, the perturbative expansion has the following general form:

$$\Gamma_0^{(L,N)}(\{0\};m,g_0,\Lambda)=m^{D_{LN}}\sum_k(g_0m^{-\epsilon})^k f_{k,d}^{(L,N)}(\Lambda/m) , \quad (A15)$$

in which D_{LN} is the (classical) dimension of $\Gamma_0^{(L,N)}$:

$$D_{LN}=d-2L-N(d-2)/2 . \quad (A16)$$

The critical limit corresponds to $m\to 0$ at g_0 and Λ fixed. The problem raised by this limit is the following: If the infinite sum in Eq. (A15) has a finite limit (i.e., nonsingular at $m=0$) then the critical behavior will be controlled by the classical exponent, (the dependence on m will be controlled by D_{LN}. Conversely a singular behavior of the sum, as m^σ, will change the exponent into $D_{LN}+\sigma$. The study of the critical limit is thus that of the limit of the infinite sum in Eq. (A15).

The hypothesis of scale invariance suggests the following change in the coupling (see Sec. III):

$$g_0=m^{-\epsilon}S(g,m/\Lambda) . \quad (A17)$$

For simplicity we do not write the renormalization of the field ϕ_0 but it should be considered in a complete discussion.

From Eqs. (A15) and (A16) one may write

$$\Gamma_0^{(L,N)}(\{0\};m,g_0,\Lambda)/m^{D_{LN}}=\sum_k[S(g,m/\Lambda)]^k f_k^{(L,N)}(\Lambda/m) . \quad (A18)$$

One sees that any explicit reference to a particular value of the dimension of the space has disappeared in the right-hand side and the limit $m \to 0$ corresponds to that of $\Lambda \to \infty$. The reference to the situation at $d=4$ comes simply from the fact that the problem of how to define $S(g,m/\Lambda)$ (and also the renormalization of ϕ_0) in order to factor the singularities when $\Lambda \to \infty$ has already been solved. The structure of the renormalization functions (S) being determined, it remains to calculate them at any d.

APPENDIX B: SUBTRACTIONS OF FIELD THEORY AT $d=4$ AND THE CRITICAL LIMIT

In FT one is interested in the construction of a finite theory at $d=4$ and Λ infinite. The bare theory presents uv divergences and the renormalization procedure corresponds to a change of the bare parameters in order to subtract, at all orders in perturbation theory, these uv divergences. The number and the kinds of changes are determined by looking at the primitive uv divergences of the correlation functions. From simple dimensional analysis, the primitively divergent correlation functions are $\Gamma_0^{(0,0)}$, $\Gamma_0^{(1,0)}$, $\Gamma_0^{(0,2)}$, $\Gamma_0^{(1,2)}$, $\Gamma_0^{(2,0)}$, and $\Gamma_0^{(0,4)}$, whose superficial degrees of divergence D_{LN} at $d=4$ are, respectively, $D_{00}=4$, $D_{10}=2$, $D_{02}=2$, $D_{12}=0$, $D_{20}=0$, and $D_{04}=0$. According to the value of D_{LN}, these divergences are of the following type: quartic (4), quadratic (2), and logarithmic (0). Usually, in the FT approach to critical phenomena[13] only the last four correlation functions, in the list given above, are considered. The reason is that they contain logarithmic uv divergences while the first two do not. The correlation functions $\Gamma_0^{(0,0)}$ and $\Gamma_0^{(1,0)}$ have no spatial dependence (their Fourier transforms do not depend on momentum) and the divergence displayed are strictly quartic or quadratic. For dimensional reasons, these divergences are not related to the ir divergences responsible for the nonclassical values of the critical exponents. Factoring out Λ instead of m [as in Eq. (A15)], one obtains at $d=4$

$$\Gamma_0^{(L,N)}(\{0\};m,g_0,\Lambda)=\Lambda^{D_{LN}}\sum_k g_0^k \tilde{f}_{k,4}^{(L,N)}(m/\Lambda) \; . \qquad \text{(B1)}$$

By comparing this with Eq. (A15), one sees that the interesting structure of uv divergences for the critical limit is that displayed within the infinite sum

$$\sum_k g_0^k \tilde{f}_{k,4}^{(L,N)}(m/\Lambda) \; .$$

It is clear that it contains only logarithmic primitive uv divergences, the two limits $\Lambda \to \infty$ and $m \to 0$ being connected through logarithms of m/Λ. Consequently, only the renormalization functions introduced to subtract these logarithmic primitive uv divergences are relevant for the study of the critical limit. Although the two-point correlation function $\Gamma_0^{(0,2)}$ is superficially quadratically divergent, its derivative with respect to the momentum will generate logarithmic uv divergences, resummed within a renormalization of the field ϕ_0 through the function Z_3 (see Secs. II and III). The subtraction of the quadratic singularities is made by a shift of the mass which simply corresponds to the definition of the distance to T_c. The logarithmic uv divergences related to $\Gamma_0^{(0,4)}$ and $\Gamma_0^{(1,2)}$ introduce a change of the coupling (through Z_1) and of the linear scaling field (through Z_2). That related to $\Gamma_0^{(2,0)}$ (the specific heat C) is particular and introduces an additive renormalization function (called A in Ref. 20) which is responsible in the nonhomogeneity of the RG equation satisfied by $\Gamma_0^{(2,0)}$. This nonhomogeneity generates[20] a critical constant $B_{\rm cr}$ to be added to the singular part of the specific heat in $\tau^{-\alpha}$. This critical constant $B_{\rm cr}$ belongs to the critical behavior and must be distinguished from the regular part of the specific heat by writing asymptotically

$$C \sim A^+\tau^{-\alpha}+B_{\rm cr}+B_{\rm bg}(\tau) \; , \qquad \text{(B2)}$$

in which $B_{\rm bg}(\tau)$ is a noncritical (regular) part (a purely background term). $B_{\rm bg}(\tau)$ cannot be determined within the FT approach; it is related to the subtraction of the quartic and quadratic uv divergences of $\Gamma_0^{(0,0)}$ (the Helmholtz free energy) and $\Gamma_0^{(1,0)}$. As the quadratic divergence of the two-point correlation function, these subtractions are not relevant to the critical limit and correspond to an adjustment of the regular part $F_{\rm reg}$ coming from $\Phi_{\rm reg}$ (see Appendix A). $B_{\rm bg}(\tau)$ must be determined (as must T_c) from experimental data (far from T_c, see Fig. 3). Indeed, $F_{\rm reg}$ is the free energy that the system should display in the absence of the long-range correlations between fluctuations.

The distinction between $B_{\rm cr}$ and $B_{\rm bg}(\tau)$ in Eq. (B2) is very important since, as shown in Ref. 20, there exists a universal combination between the asymptotic critical amplitude, the amplitude of the first confluent correction of the singular part of C and $B_{\rm cr}$ (see also Table IV in which we give estimates of this universal combination).

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