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Incompressible states of the fractionally quantized Hall effect in the presence of impurities: A finite-size study

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We have studied the effect of impurities on the incompressible states of two-dimensional electrons in a strong magnetic field by finite-size numerical calculations in the spherical geometry. For short-ranged impurity potentials the Laughlin ground state shows no screening and is found to be stable regardless of the potential strength. An analogy with acceptor and donor states of a semiconductor is drawn and ionization energies are estimated.

Laughlin's Jastrow-function picture[1] of the incompressible ground state of interacting electrons bound to a clean two-dimensional substrate at $\frac{1}{3}$ Landau-level filling is now well confirmed[2,3] and explains the observed fractionally quantized Hall effect with Hall constant $R_H = h/\nu e^2$, $\nu = \frac{1}{3}$. The elementary excitations of the clean system have also been characterized:[3] $q = \pm\frac{1}{3}e$ quasiparticle and quasihole defects of the incompressible state and a neutral collective excitation[3-5] that becomes a well separated "excitonic" pair of opposite charge defects at large wave numbers.

The spherical geometry introduced by Haldane[6] has proved to be a powerful device for numerical finite-size studies of the clean system and in an obvious extension we have applied it to study the effect of isolated point impurities on the incompressible Laughlin-Jastrow (LJ) state.

It is important to understand the effects of impurities, as they are likely to dominate dissipative transport properties such as the finite-temperature Ohmic resistance, unless extremely clean samples can be prepared. The activation energies reported in Ref. 7 are significantly lower than those found in Refs. 3 and 4 for a clean system with pure Coulomb interactions. However, since various effects such as finite inversion-layer thickness, Landau-level mixing, and impurities sensitively affect these energies, our study is not aimed at quantitative fits to experiment; instead we study model impurity potentials in a system with purely Coulomb interactions and vanishing inversion-layer thickness.

It is very instructive to study the effect of a short-range or delta-function impurity potential. In fact, any potential with a range much less than the magnetic length $l = (\hbar/eB)^{1/2} = 66$ Å at $B = 15$ T is effectively a delta function with binding energy

$$g = \frac{1}{\pi l^2}\int d^2\mathbf{r}\, V(\mathbf{r}) e - (r/l)^2$$

for particles in the lowest Landau level. More importantly, the charge-density response to a weak delta-function impurity potential directly defines the real-space form of the ground-state linear response function.

In the case of the $\nu = 1$ incompressible fluid (the filled Landau level) where electron interactions can be neglected $(e^2/4\pi\epsilon_0 l << \hbar\omega_c)$, the effect of a delta-function impurity has been studied by Prange.[8] A single impurity level is pushed above or below each Landau level depending on whether the potential is repulsive or attractive. If one neglects spin, and for simplicity treats a spinless Fermi gas, there is no change in the ground-state quantum numbers due to the impurity potential even if its strength becomes infinite. (In fact, if g is significantly larger than the gap $\hbar\omega_c$ the potential strength is effectively infinite.)

In an obvious analogy with a semiconductor, a "donor" or "acceptor" level is pushed into the gap between the filled lowest Landau level and the empty second Landau level. Particles in the second Landau level and holes in the lowest Landau level are the "defects" of the incompressible state: The donor and acceptor states can each exist in both a neutral state, and a "singly ionized" state left after the release of an oppositely charged defect.

In our numerical study of the effect of the delta-function impurity in the $\nu = \frac{1}{3}$ state, we find a rather similar behavior, except that now the donor (repulsive impurity) or acceptor (attractive impurity) can exist in exotic *fractional-charge* states.

We chose to study the six-electron system at $\nu = \frac{1}{3}$ filling using the spherical geometry. This is a convenient size, with matrix dimensions less than 500. The incompressible ground state occurs[3,6] at net magnetic flux $2S = 15$ flux quanta, corresponding to a sphere radius $R = \sqrt{S}\, l = 2.73l$. In the absence of an impurity, states fall into multiplets characterized by the rotational quantum number L; with an impurity at the north pole, only the azimuthal rotational quantum number M remains to classify states. The quantum number M measures the first moment of the charge-density distribution on the surface of the sphere:

$$M = R^2 \int d^2\Omega\, (\cos\Theta)\rho(\hat{\Omega})\ ;$$

changes in M thus indicate charge redistribution.

Figure 1 shows how the uniform charge-density profile of the pure incompressible state is modified in the presence of various strength delta-function impurity potentials. The ground-state quantum number M remains at the pure system value $M = 0$ even in the limit of an infinitely strong impurity potential, just as in the filled Landau-level case $\nu = 1$. This reflects the incompressible nature of the LJ state: *No*

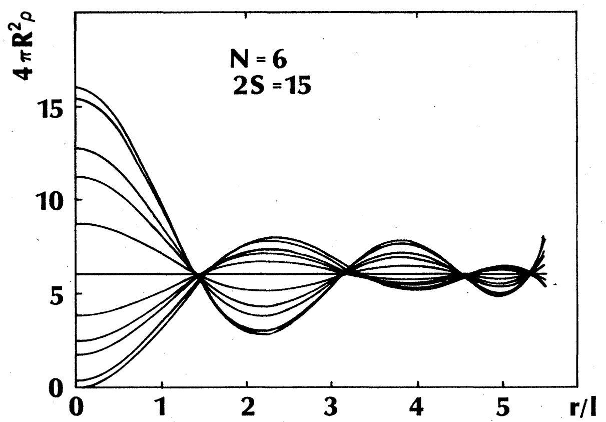

FIG. 1. Charge-density profile of the six-electron system at $\nu=\frac{1}{3}$ ($2S=15$ flux quanta on the sphere) (Refs. 3 and 6) in the presence of various strengths of the delta-function impurity potential. r is the (chord) distance from the impurity on the spherical surface of radius $R=2.73l$. In the $\nu=\frac{1}{3}$ pure state the surface density ρ is given by $4\pi R^2\rho=6$, while in the $\nu=1$ state it would take the maximum value $4\pi R^2\rho=16$. The strengths of impurity potential shown are $g=+\infty$, $+0.3$, $+0.1$, $+0.07$, $+0.035$, 0, -0.035, -0.07, -0.1, -0.3, $-\infty$, for which the charge density rises from $4\pi R^2\rho=0$ to 16.

net screening charge accumulates on the impurity, in marked contrast to a metallic system. Instead there is a local oscillatory polarization of the charge density in the neighborhood of the impurity. The oscillation of the charge density about the *background* value at large distances supports this conclusion. At the position of the impurity, the charge density increases from zero to the maximum value for the lowest Landau-level states as the impurity strength rises from $-\infty$ to $+\infty$. The period of this oscillation is governed by the *linear* response of the LJ state, and remains essentially unchanged even well outside of the linear response regime. From Fig. 1, the first node of the polarization charge density occurs at a distance of about $1.45l^{-1}$ from the impurity. One can readily estimate the linear-response profile: If the response function $\chi(q)$ was entirely concentrated at $q=q_0$ then the response $\delta\rho(\mathbf{r})$ would be proportional to the Bessel function $J_0(q_0r)$, with the first node at $q_0r=2.4$, i.e., $q_0\simeq 1.65l^{-1}$. This is in line with the recent observation by Girvin, MacDonald, and Platzman[4] that the numerical results of Ref. 3 for the collective excitation dispersion can be understood if the linear response is dominated by the collective mode in the region of the rotonlike minimum of the excitation energy at $q\simeq 1.4l^{-1}$. In fact, a recent numerical study by one of us[9] has confirmed that the linear response is well described by the single-mode approximation suggested by Girvin, MacDonald, and Platzman.

Figure 1 also shows that the characteristic coupling energy at which the impurity becomes strong is $|g|\simeq 0.1$ (in units of $e^2/4\pi\epsilon_0 l$) which is comparable to the energy gap of the incompressible state.[3] If the potential is significantly stronger than this, it will have effectively infinite strength. This motivates the special study of the infinite strength delta-function potentials as these are likely to model all sufficiently strong short-ranged potentials.

The overlap of the exact pure state with the neutral ground-state wave functions in the presence of short-ranged impurity potentials interpolates from unity to 0.6023 and 0.7820 as the impurity charge is varied from zero to $+\infty$ and $-\infty$, respectively. These values are strikingly close to $\sqrt{(6/16)}=0.6123$ and $\sqrt{(10/16)}=0.7906$, which are the overlaps of the pure ground-state wave function $\Psi_{\rm pure}$ for six electrons at $\nu=\frac{1}{3}$, with $\hat{P}\Psi_{\rm pure}$, $\hat{P}$ projects out the component of the wave function where the occupation of the lowest Landau-level orbital centered on the impurity is either zero (repulsive) or one (attractive). In fact, the overlap of the exact ground-state wave function with $\hat{P}\Psi_{\rm pure}$ was found to be 98.9% and 98.3% for infinite strength repulsive and attractive cases, respectively.

The excitation spectrum and charge profiles of low-energy states of the neutral system with an infinite strength repulsive delta-function impurity are shown in Fig. 2. In the series of low-lying excitation with $M=-1,-2,\ldots,$ the neutral impurity releases a quasiparticle defect, ending up in a state of charge $-\frac{1}{3}e$. By analogy to semiconductor physics, we describe the impurity as a donor (of quasiparticle defects) and the ionization process as $D(0)\rightarrow D(-\frac{1}{3}e)+P$, where $D(q)$ are the charge states of the donor and P is the released charge $+\frac{1}{3}e$ defect. The quantum number ΔM is a measure of the radius at which the defect orbits around the impurity. The outward progress of the defect as $|\Delta M|$ increases is clearly seen in the charge-density profiles of Fig. 2: The quasiparticle defect has a minimum in the charge density at its center,[3] which can be identified in Fig. 2.

Figure 3 shows analogous results for the fractionally charged ($q=-\frac{1}{3}e$) system obtained by adding one flux quantum to the $\nu=\frac{1}{3}$ state. In the absence of the impurity, the ground state would be a degenerate multiplet with $L=\frac{1}{2}N(=3)$ describing the possible states of the free

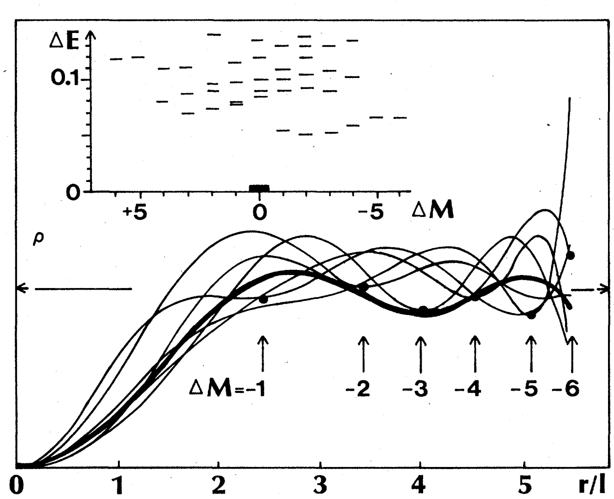

FIG. 2. Charge-density profiles of ground state (thick line) and the low-lying excitations of the neutral $\nu=\frac{1}{3}$ system with $N=6$, $2S=15$, and an infinite strength repulsive delta-function impurity. Filled points show minima in the charge density identified with the center of a quasiparticle defect emitted by the impurity. The neutralizing background charge density (total charge $6e$) is indicated by the horizontal arrows. Inset shows excitation energies (in units $e^2/4\pi\epsilon_0 l$) vs ΔM, the change in azimuthal quantum number.

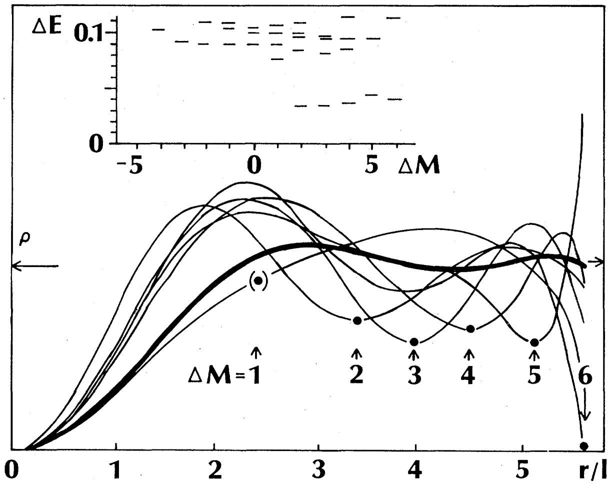

FIG. 3. As Fig. 2, but for the charged ($q=-\frac{1}{3}e$) system with $N=6$, $2S=17$ flux quanta. The background charge density neutralizes the LJ condensate (total charge $6\frac{1}{3}e$), filled points indicate the center of a quasihole defect emitted in the ionization process $D(-\frac{1}{3}e)\rightarrow D(0)+h$ (in the case $\Delta M=1$, only the suggestion of an inflection point indicates the center of the defect, and this state has a much higher energy than the other low-lying states).

"quasihole" defect. The infinitely strong repulsive delta-function impurity potential binds the hole: Since the charge density at the center of the defect is almost[3] zero in the pure case, there is very little modification of the ground-state charge-density profile, which describes the $D(-\frac{1}{3}e)$ state of the donor. The low-lying excitations again have a simple interpretation: the process $D(-\frac{1}{3}e)\rightarrow D(0)+h$, where a quasihole defect "h" is released is clearly seen (Fig. 3). ΔM again indicates the radius at which the released defect orbits around the impurity, and its center can be identified as a minimum in the charge-density profile. The ground-state quantum number $M_0=-3$ indicates binding of the hole to the impurity.

If $\Delta^+ + \Delta^-$ is the intrinsic energy gap for creating a pair of widely separated quasihole and quasiparticle defects, then the energy gap for the donor ionization process $D(0)\rightarrow D(-\frac{1}{3}e)+P$ can be estimated as

$$\Delta^+ + \Delta^- - [E_0(\infty)-E_0(0)+E_-(0)-E_-(\infty)]\ ,$$

where $E_0(g)$ and $E_-(g)$ are, respectively, the ground-state energies of the neutral and $q=-\frac{1}{3}e$ systems in the presence of an impurity of strength g. Taking the raw 6-electron data, we find the impurity reduces the gap by $0.039e^2/4\pi\epsilon_0 l$: For a more refined estimate, the extrapolation to large N should be performed; we leave this for a more detailed study.

This estimate of the ionization energy ($\simeq 0.081e^2/4\pi\epsilon_0 l$) agrees reasonably well with that obtained directly from the neutral excitation spectra (e.g., Fig. 2): It should coincide with the limit as $|\Delta M|\rightarrow\infty$ of the low-lying branch of the excitation spectrum.

We characterize the infinitely attractive short-range impurity as an acceptor level for quasiparticle defects. For the 6-electron system we find the energy for the ionization process $A(0)\rightarrow A(+\frac{1}{3}e)+h$ to be $0.080e^2/4\pi\epsilon_0 l$. This is essentially identical to the ionization energy of the donor level. Details of the excitation spectrum and charge-density profiles of the acceptor level will be given elsewhere.

We studied the *doubly* ionized state of the acceptor impurity $A(+\frac{2}{3}e)$ at flux $2S=13$ flux quanta. For the pure system this flux change corresponds to adding two quasiparticle defects to the neutral $\nu=\frac{1}{3}$ system. The infinitely strong attractive impurity potential binds both quasiparticles. However, we have found that the second quasiparticle is only marginally bound: In the 6-electron system its ionization energy is $0.0012e^2/4\pi\epsilon_0 l$. This may be a consequence of the strong short-ranged repulsion of the two quasiparticles.[3]

We have also studied the effects of longer-range impurity potentials. Figure 4 shows the density response of the LJ incompressible fluid ground state ($L=0$) for six electrons at $\nu=\frac{1}{3}$ to a *Coulomb* impurity potential. We have varied the charge of the impurity from $+0.5e$ to $-0.5e$. In contrast to short-ranged impurities the LJ incompressible state is found to become unstable if the charge of the impurity is made sufficiently large. The azimuthal quantum number of the new ground state is no longer zero signifying a sudden rearrangement of charge around the impurity by local nucleation of an exciton. This feature is quite evident in Fig. 4 (thick lines). Prior to the transition the linear response regime and the *unscreened* behavior is also clearly seen. The inset in Fig. 4 shows the rather strong dependence of the gap (separation of the ground state from the first-excited state) on the charge of the impurity. The gap vanishes at the transition point and remains relatively small above it. For six electrons the critical charges are $+0.38e$ and $-0.30e$

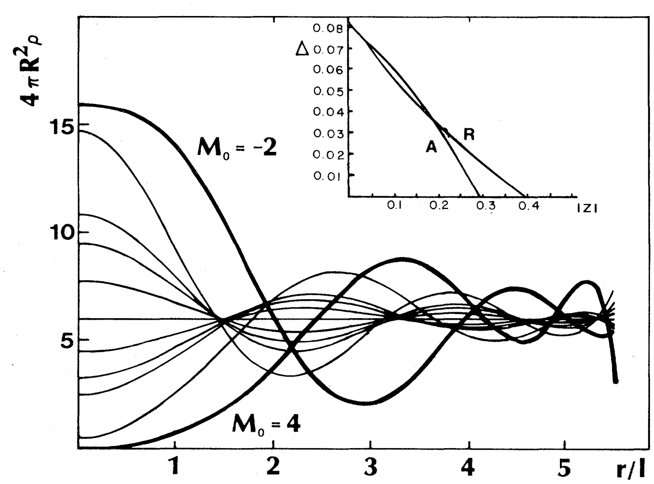

FIG. 4. Ground-state charge-density profiles (as in Fig. 1; $N=6$, $2S=15$ flux quanta $-\nu=\frac{1}{3}$) for a number of long-range *Coulomb* impurity potentials. The strength of the charges shown are $Z/e=+0.5$, $+0.3$, $+0.1$, $+0.07$, $+0.035$, 0, -0.035, -0.07, -0.1, -0.3, -0.5. The density $4\pi R^2\rho$ again rises from 0 to 16. The thick curves are the charge density above the point of instability, where a large redistribution of the charge density has taken place. The new ground-state quantum numbers are indicated for each case. The inset is the gap (energy of the first-excited state relative to the ground state in units of $e^2/4\pi\epsilon_0 l$) as a function of the impurity charge for the attractive (A) and the repulsive (R) cases.

with corresponding azimuthal ground-state quantum numbers $M_0 = -2$ and $+4$ for the repulsive and attractive impurity potentials, respectively. We believe the asymmetry in the quantum numbers to be a consequence of the characteristic density profile of the quasiparticle defect:[3] The maximum of the charge density does not occur at the center of the defect.

In summary, the incompressibility of the Laughlin condensate prevents screening of the impurity. We expect this and other features reported in this paper to persist in the thermodynamic limit. We have made a detailed study of short-ranged impurity potentials with infinite strength. We conclude by noting that the conditions for this model to be applicable to physical systems are

$$\hbar\omega_c >> V_0 >> \Delta >> V_m, \quad m \geq 1 ,$$

where Δ is the gap in the spectrum of the fluid and V_m is a set of pseudopotential coefficients[10] characterizing the interactions. In general, these conditions require that the range $r_0 << l$.

We remark that the picture presented here is only applicable to *dilute* systems to the extent that impurities can be considered as independent.

While this work was in its final stage of completion we received a copy of the preceding paper[11] reporting features similar to those presented in Fig. 4, but for attractive Coulomb impurity potentials and for 4-, 5-, and 6-electron size systems in various geometries.

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[1]R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2]D. Yoshioka, Phys. Rev. B **29**, 6833 (1984); J. Phys. Soc. Jpn. **53**, 3740 (1984); and (unpublished).
[3]F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985).
[4]S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. Lett. **54**, 581 (1985).
[5]R. B. Laughlin, in *Proceedings of the Seventeenth International Conference on Low-Temperature Physics, Karlsruhe, Germany, 1984*, edited by U. Eckern, A. Schmid, W. Weber, and H. Wühl (North-Holland, Amsterdam, 1984).
[6]F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[7]S. Kawaji, J. Wakabayashi, J. Yoshino, and H. Sakaki, J. Phys. Soc. Jpn. **53**, 1915 (1984); A. M. Chang, P. Berglund, D. C. Tsui, H. L. Störmer, and J. C. M. Hwang, Phys. Rev. Lett. **53**, 997 (1984).
[8]R. E. Prange, Phys. Rev. B **23**, 4802 (1981).
[9]F. D. M. Haldane, Bull. Am. Phys. Soc. **30**, 381 (1985); and (unpublished).
[10]When projected onto the lowest Landau level the original interaction $V(r)$ is described by a set of pseudopotential coefficients V_m, which are the interaction energies of two particles in pair state of relative angular momentum m.
[11]F. C. Zhang, V. Z. Vulovic, Y. Guo, and S. Das Sarma, preceding paper, Phys. Rev. B **32**, 6920 (1985).