Neutron refractive index: A Fermi-Huygens theory

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A multiple-scattering calculation of the neutron refractive index is performed by an extension of the Fermi-Huygens technique. The extension involves projecting the problem into a one-dimensional walk by integrating out the transverse coordinate in a semi-infinite medium and then partially summing parts of the walk to infinite order. The square of the refractive index is given by $n^2-1=-(4\pi\rho b/k_0^2)/[1+(4\pi\rho b^2/nk_0)\int_0^\infty da\, e^{ik_0a}\sin(nk_0a)h(a)]$, where k_0 is the incident wave propagation vector, b the nuclear scattering length, ρ the number density of nuclei ($\rho\equiv 1/a_0^3$, say), and $h(a)=g(a)-1$, where $g(a)$ is the pair distribution function. The results parallel those obtained by constitutive equation methods, and offer a physical picture of local-field effects. When the mean scattering length vanishes (total incoherence), correlated multiple scattering yields $n^2-1\sim(b/a_0)^4(k_0a_0)^{-2}\ln[(k_0a_0)^{-1}]$. Thus, the refractive index is exceedingly close to unity unless b is large (a resonance) or $k_0\to 0$ (ultracold neutrons). The presence of the logarithmic term indicates that randomness in the scattering field apparently reduces the effective dimension.

I. INTRODUCTION

Neutron-optical techniques now yield measurements of very high accuracy[1] for the neutron refractive index n. The accuracy of calculations of the neutron scattering length b using these measurements is, however, limited by the accuracy of available approximations relating n and b. The best known of these approximations is the Fermi[2] thin-slab formula

$$n=1-2\pi\rho b/k_0^2\,, \tag{1.1}$$

where ρ is the number of scattering nuclei per unit volume and k_0 the propagation constant in free space. Relation (1.1) depends on many approximations including the neglect of multiple scattering. A more rigorous analysis[3] finds

$$n^2-1=-4\pi\rho b/k_0^2\,. \tag{1.2}$$

To go beyond this relation, correlations between nuclei must be taken into account.

There are essentially two ways to proceed. The first, reviewed by Sears,[3] is what Lloyd and Berry[4] classify as the "hierarchy" method where the ensemble averaged (coherent) wave function with N atoms fixed is calculated in terms of that with $N+1$, and so on. The names of Foldy[5] and Lax[6] are most often associated with these methods. The second method, called by Lloyd and Berry[4] the "resummation" or Green's function method, was first applied to particles or waves in random potentials by Edwards.[7] Here, the scattering series is averaged term by term and resummed in a Dyson-type of analysis. A similar Dyson equation with the same diagrammatic structure was exploited in an n-component field theory with random potentials by Edwards and Warner.[8] In the calculation of the refractive index presented below we follow this second method. Its advantages are a clear interpretation of the multiple-scattering events and a transparency of approximation level.

In Sec. II we outline how n is defined in an infinite medium and then proceed to the case of a plane wave incident on a semi-infinite slab. We treat this special geometry for several reasons: The errors inherent in (1.1), namely excluding multiple scattering in the reverse direction, are also those which prevent a calculation of reflectivity of a neutron plane wave from a slab. Our treatment of multiple scattering naturally yields reflection (Appendix B). Additionally, the explicit resummation of the multiple scattering series illustrates how to handle the problem of the last scattering center in slab geometry when ensemble averaging the multiple scattering. (Lloyd and Berry[4] discuss this point in comparing different theories of multiple scattering.) Our result in Sec. III for the refractive index is an explicit example of a resummation and reduces to the result of Sears[3] obtained by the hierarchy method when $n-1$ is very small and when the neutron wavelength is very long. It is

$$n^2-1=+\frac{4\pi\rho b/k_0^2}{1+(4\pi\rho b/nk_0^2)\int_0^\infty dy\, e^{iy}\sin(ny)[g(y/k_0)-1]} \tag{1.3}$$

and shows the effect of correlations through the pair distribution function g. Besides being more accurate, our result is also of methodological interest, as it makes contact with multiple-scattering treatments of other subjects, for instance electrons in random potentials, and has the advantage of giving a clear picture of the level of approximation involved.

Our Huygens technique of following definite sequences of wave scattering shows its greatest power when considering situations where the mean scattering length $\bar{b}$ is zero. In Sec. IV we show that more complicated trajectories and correlations can yield a phase shifted coherent wave, i.e., an $n \neq 1$, even when $\bar{b}=0$, a result difficult to obtain by other methods. Repeated scatterings under these conditions yield a dimensional reduction characteristic of many random field problems. In Sec. IV we also treat the related problem of evanescence.

II. PRELIMINARIES

In this section we review the neutron-nucleus scattering properties, reproduce the refractive indices (1.1) and (1.2), and discuss the Huygens method used in subsequent sections. When a neutron wave $\psi_0(\mathbf{r})$ is incident at a scattering center at $\mathbf{r}=0$ the resultant wave $\psi(\mathbf{r})$ is

$$\psi(\mathbf{r})=\psi_0(\mathbf{r})+\psi_0(0)e^{ik_0 r}f(\theta)/r \tag{2.1}$$

with[9]

$$f(\theta)=\frac{1}{2ik_0}\sum_n (2n+1)(e^{2i\eta(n)}-1)P_n(\cos\theta) , \tag{2.2}$$

where θ is the angle between the incident and the final directions, P_n is the nth-order Legendre polynomial, and $\eta(n)$ is the phase shift of the nth partial wave. In thermal neutron scattering, because the neutron wavelength is very large compared to nuclear distances, the scattering is isotropic, that is, S wave. Accordingly, only one partial wave ($n=0$) enters into (2.2), and the scattering is described by a single (nuclear) length b defined by

$$b=\lim_{k_0\to 0} f(\theta)=\frac{\eta(0)}{k_0}+\frac{i\eta^2(0)}{k_0} \tag{2.3}$$

$$\equiv -\hat{b}+ik_0\hat{b}^{\,2} , \tag{2.4}$$

where the phase shift $\eta(0)$ is small and in the absence of absorption is also real. The imaginary part $ik_0\hat{b}^{\,2}$ of (2.4) ensures the satisfaction of the optical theorem.[9,10] The minus sign in the definition of $\hat{b}$ is the Fermi convention. For most materials n^2-1 is less than zero.

Thus the scattered wave at $\mathbf{r}'$ originating from $\mathbf{r}$ is

$$\psi_f(r')\sim -bG^0(\mathbf{r}',\mathbf{r})\psi_i(\mathbf{r}) , \tag{2.5}$$

where $G^0(\mathbf{r}',\mathbf{r})$ is the propagator or Green's function satisfying outgoing wave boundary conditions and

$$(\nabla^2_{r'}+k_0^2)G^0(\mathbf{r}',\mathbf{r})=4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{2.6}$$

or

$$G^0(\mathbf{r}',\mathbf{r})=\frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|r-r'|} .$$

We also have the Fourier transform of (2.6)

$$G^0_{\mathbf{k}}=4\pi/(k^2-k_0^2) , \tag{2.7}$$

which shows the simple pole character of $G^0(k)$. With many scatterers the wave ψ_i incident at $\mathbf{r}$ may propagate to $\mathbf{r}'$ by undergoing l scatterings to become

$$\psi^{(l)}(\mathbf{r}')=G^0(\mathbf{r}',\mathbf{r}_1)(-b_1)G^0(\mathbf{r}_1,\mathbf{r}_2)(-b)\times\cdots\times(-b_l)\times G^0(\mathbf{r}_l,\mathbf{r})(-\mathrm{b})\psi_{\mathrm{i}}(\mathbf{r}) . \tag{2.8}$$

This situation is represented diagrammatically in Fig. 1. There are $l+1$ G^0 factors and $l+1$ factors of $-b$ in (2.8). The total amplitude for going from $\mathbf{r}$ to $\mathbf{r}'$ is

$$\psi(\mathbf{r}')=\sum_l \psi^{(l)}(\mathbf{r}') \tag{2.9}$$

with a sum over all possible l scattering paths in $\psi^{(l)}$. Ignoring for the moment spin and isotopic variations, we replace the summation in (2.9) by

$$\rho^l\int d\mathbf{r}_1\int d\mathbf{r}_2\cdots\int d\mathbf{r}_l\, g(\mathbf{r}_1,\mathbf{r}_2)\times g(\mathbf{r}_2,\mathbf{r}_3)\times\cdots\times g(\mathbf{r}_l,\mathbf{r}) ,$$

where ρ is the number density of nuclei and $g(r_i,r_{i+1})$ is the pair distribution function describing the probability of finding a nucleus at $\mathbf{r}_i$ given that there is one at $\mathbf{r}_{i+1}$. In replacing the set of actual sites by probabilities of finding them, we are ensemble averaging, discussed more fully in Sec. III. By taking pair correlations we are breaking the hierarchy alluded to before. Averaging restores translational invariance. We appeal to the convolution theorem and get for l scatterings

$$\psi^{(l)}(\mathbf{r}')=-b\psi_i(\mathbf{r})(-\rho b)^l\times\int\frac{d\mathbf{k}}{(2\pi)^{3/2}}e^{-i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}[G^0_{\mathbf{k}}(H_{\mathbf{k}})^l] , \tag{2.10}$$

with $H_{\mathbf{k}}$ arising from the propagation between intermediate scatterings

$$H_{\mathbf{k}}=\int\frac{d\mathbf{r}}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{r}}G^0(r)[g(r)-1+1]$$

$$=G^0_{\mathbf{k}}+J(\mathbf{k}) \tag{2.11}$$

with

$$J(\mathbf{k})=\int\frac{d\mathbf{r}}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{r}}G^0(r)[g(r)-1] . \tag{2.12}$$

Inserting (2.10) into (2.9), we find

$$\psi(\mathbf{r}')=-b\psi_i(\mathbf{r})\int\frac{G^0_k e^{-i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}}{(1+\rho b H_{\mathbf{k}})(2\pi)^{3/2}}d\mathbf{k} \tag{2.13}$$

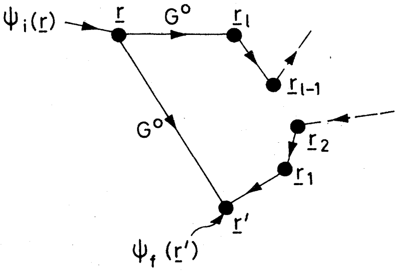

FIG. 1. A schematic of paths between $\mathbf{r}$ and $\mathbf{r}'$ showing no scatterings and a path involving l scatterings.

and after identifying this result with (2.5), we find that the effective propagator G_k for the medium is

$$G_k = G_k^0/(1-\rho b H_k) = \frac{4\pi}{k^2-k_0^2+4\pi\rho b+\rho b J(k)(k^2-k_0^2)} . \quad (2.14)$$

When $g(r)=1$ (random distribution of scattering centers), $J=0$. Hence, the poles of G_k are displaced from k_0^2 to $k_0^2-4\pi\rho b$ which implies the propagation constant k_0^2 has been altered to $k_0^2(1-4\pi\rho b/k_0^2)$. This is the refractive index result (1.2) expected from a change in the *mean* potential experienced by the neutron. Correlations are measured by J, motivating the separation in (2.11), and indicate a further shift in the pole, that is, further changes in the propagation constant.

We emphasize that the above has not been for plane waves for which a refractive index is appropriate and that we have exploited translational symmetry, which is absent in the case of a plane wave incident on a slab. We have also ignored the complex character of b whereupon the wave intensity is not decreasing, in violation of the optical theorem. Additionally, correlations and fluctuations are dealt with at the level of the *pair* distribution $g(\mathbf{r})$.

We now will contrast this derivation of n with the Fermi-Huygens method.[2] Here a plane wave incident on a sufficiently thin slab emerges as a plane wave with a phase shift proportional to the slab thickness. If we take the z axis perpendicular to the slab and let $\boldsymbol{\sigma}$ be a vector in the plane (see Fig. 2), then in the presence of multiple scattering the total wave amplitude at $(0,dz)$ becomes

$$\psi(dz)=\psi_0(dz)-\sum_i \exp\left[\frac{ik_0[\sigma_i^2+(dz)^2]^{1/2}}{|\sigma_i^2+(dz)^2|^{1/2}}\right]\psi_0(\boldsymbol{\sigma}_i)b . \quad (2.15)$$

Again we convert the sum over scattering sites i into an integral over the slab,

$$\psi(dz)=\psi_0(dz)-b\rho\, dz\int d\sigma \frac{e^{ik_0[\sigma^2+(dz)^2]^{1/2}}}{[\sigma^2+(dz)^2]^{1/2}} \quad (2.16)$$

and by changing the integration variable $\sigma^2+(dz)^2$ to y^2, we obtain to order (dz)

$$\psi(dz)=\left[1-\frac{2\pi i\rho b\, dz}{k_0}\right]e^{ik_0 dz} \equiv \exp[ik_0(1-2\pi\rho b/k_0^2)dz] . \quad (2.17)$$

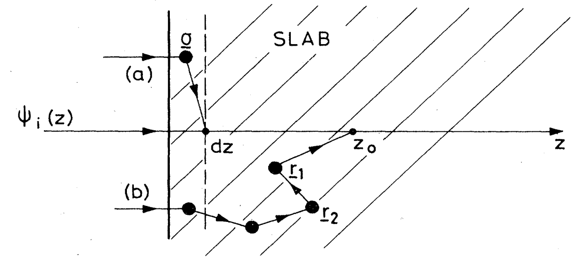

FIG. 2. Two configurations are shown: (a) A differential slab of thickness dz with one scattering event at coordinate $(\boldsymbol{\sigma},0)$ and $\psi_f(dz)$ at the final point $(0,dz)$. The two-dimensional vector $\boldsymbol{\sigma}$ denotes position in the plane. (b) A semi-infinite slab with, in general, a path involving l scatterings, and $\psi_f(z_0)$ evaluated at the final point $(0,z_0)$.

Thus expression (1.1) is recovered. By repeating such slabs and allowing scattering in each slab only to be forward into the next, we can reduce Eq. (2.17) to the differential equation

$$d\psi(z)/dz=(1-2\pi\rho b/k_0^2)ik_0\psi(z) , \quad (2.18)$$

which yields $\psi \sim e^{ik_0 nz}$ with (1.1) for n. This pasting together of the thin slab results is quite approximate. The essence of the method though is already clear; for s-wave scattering one can convert the problem [(2.17) and (2.18)] into an effectively one-dimensional calculation. Under similar conditions other quasi-s-wave problems can also be reduced to a one-dimensional calculation. We now proceed to perform such a reduction for the multiple scattering treatment of the slab including correlations.

III. RESUMMATION FOR THE SEMI-INFINITE SLAB

A. Uncorrelated sites

We now consider case (b) of Fig. 2 where a plane wave $\psi_0(z)$ is incident at $z=0$ and a path of l scatterings is needed to get to the point of observation at $z=z_0$. The $l=0$ result is the straight-through beam that interferes with the scattered waves to yield a plane wave with a modified propagation constant $k_0 n$. Considering the path $r_l \to r_{l-1} \to \cdots \to r_1 \to r_0$ with l intermediate sites, we have

$$\psi^{(l)}(z_0)=(-b)^l\sum_{\mathbf{r}_1}\frac{e^{ik_0|\mathbf{r}_{01}|}}{|\mathbf{r}_{01}|}\times\cdots\times\sum_{\mathbf{r}_{l-1}}\frac{e^{ik_0|\mathbf{r}_{l-2,l-1}|}}{|\mathbf{r}_{l-2,l-1}|}\sum_{\mathbf{r}_l}\frac{e^{ik_0|\mathbf{r}_{l-1,l}|}}{|\mathbf{r}_{l-1,l}|}\psi_0(z_l) , \quad (3.1)$$

where the internuclear distances are denoted by $\mathbf{r}_{i,i-1}=\mathbf{r}_i-\mathbf{r}_{i-1}$ and the plane-wave amplitude incident on the first scattering site, r_l, is $\psi_0(z_l)=e^{ik_0 z_l}$.

At this point we pause to establish exactly what is calculated when we reconstruct a phase-shifted plane wave from all the scattered waves. By averaging over scattering sites we are in effect averaging the wave amplitude. This wave, ψ, is known as the coherent wave. A test for coherent contributions is whether in forming $|\psi|^2$ the contribution survives averaging (quantum mechanical or statistical). Examples of incoherent contributions including neutron spin flip and are given in Sec. IV. For the moment we restrict discussion to the coherent wave.

To evaluate (3.1), we start by turning the summations into integrals over the slab. The average of $\psi^l(z_0)$ is thereby effected:

$$\sum_i \to \rho \int dz \int \left[\prod_i d\sigma_i\right] e^{ik_0 z_i} \frac{\exp\{ik_0[\sigma_{i,i-1}^2+(z_i-z_{i-1})^2]^{1/2}\}}{[\sigma_{i,i-1}^2+(z_i-z_{i-1})^2]^{1/2}} . \tag{3.2}$$

In doing this we neglect correlations between the scattering centers. However, multiple scattering from correlated sites can produce important effects, and in Appendix A we present the treatment of pairwise correlations of scattering sites, and at the end of this section we discuss their consequences.

Changing to the variable y as discussed in Appendix A, we find that the integral in the plane of the slab at constant z_i gives

$$\sum_i \to 2\pi\rho \int_0^\infty \left[\prod_i dz_i\right] \int_{|z_{i-1}-z_i|}^\infty dy \exp(ik_0 y) = \frac{2\pi i \rho}{k_0} \int_0^\infty dz_i \exp(ik_0 |z_{i-1}-z_i|) . \tag{3.3}$$

The wave scattered l times is then

$$\psi^l(z_0) = \left[-\frac{2\pi i \rho b}{k_0}\right]^l \int_0^\infty \left[\prod_{i=1}^l dz_i\right] \exp\left[ik_0\left[\sum_{j=1}^l |z_{j-1}-z_j|\right]\right] \psi_0(z_l) . \tag{3.4}$$

It is illustrative to recover previous results of pasting thin slabs together. To do this we simply restrict successive scattering events to be to the right, that is, $z_j \le z_{j-1}$, whereupon the exponent in (3.4) simplifies to $\exp(ik_0 z_0)$. The integrals over z_i with the restricted limits yield $z_0^l/l!$ and the total wave becomes

$$\psi(z_0) = \psi_0(z) + \sum_{l=1}^\infty \left[\frac{-2\pi i \rho b}{k_0}\right]^l \frac{z_0^l}{l!} e^{ik_0 z_0} \tag{3.5a}$$

$$= \exp\left[ik_0 z_0 - \frac{2\pi i \rho b}{k_0} z_0\right]$$

$$\equiv \exp(ik_0 n z_0) , \tag{3.5b}$$

whereupon the refractive index, n, is $1-2\pi\rho b/k_0^2$. Clearly, in order to get the correct answer for the refractive index under multiple scatterings we must allow for trajectories with backscattered portions like those in Fig. 2(b). Equally, such possibilities are needed to calculate reflection and transmission coefficients for the beam impinging on the slab and underline the need to go beyond the multiple thin-slab treatment.

Returning to (3.4) we make another change of variables

$$a_i = z_{i-1} - z_i, \quad i = 1,2,\ldots,l$$
$$a_{l+1} = z_l \tag{3.6}$$

to convert the z integrals to

$$\int da_1 \int da_2 \cdots \int da_{l+1} \delta\left[\sum_{i=1}^{l+1} a_i - z_0\right] ,$$

where the δ function restricts the trajectory of scattering events to end at the point of observation z_0. Using $\delta(x) = \int (d\lambda/2\pi) e^{i\lambda x}$, we next write $\psi^{(l)}(z_0)$ as

$$\psi^l(z_0) = -\left[\frac{2\pi i \rho b}{k_0}\right]^l \int_0^\infty da_{l+1} \int_{-\infty}^{z_0} da_1 \int_{-\infty}^\infty \left[\prod_{i=2}^l da_i\right] \int \frac{d\lambda}{2\pi} \exp\left[ik_0 \sum_{i=1}^{l+1} |a_i| + i\lambda \sum_{i=1}^{l+1} a_i - i\lambda z_0\right] . \tag{3.7}$$

In this expression the a_i integrals yield

$$f(\lambda) = \int_{-\infty}^\infty da \exp(ik_0|a| + i\lambda a)$$
$$= 2ik_0/(k_0+\lambda)(k_0-\lambda) . \tag{3.8}$$

For the case of transmission ($z_0 > 0$) the coherent l-scattered field (3.7) reduces to

$$\psi^l(z_0) = \left[\frac{-2\pi i \rho b}{k_0}\right]^l \int_{-\infty}^\infty \frac{d\lambda}{2\pi} [f(\lambda)]^{l-1} \frac{i^2 e^{-i\lambda z_0}}{k_0+\lambda} \left[\frac{1}{k_0-\lambda} + \frac{1}{k_0+\lambda}(1-e^{iz_0(\lambda+k_0)})\right] \tag{3.9}$$

with the additional factors arising from the a_{l+1} and a_1 integrals, respectively. We now perform the integral over λ by first noting that k_0 was assumed to have a small positive imaginary part ϵ so the poles in the complex λ plane are at $\lambda = k_0 + i\epsilon$ and $\lambda = -k_0 - i\epsilon$. In view of the $e^{-i\lambda z_0}$ factor in (3.11) we take the pole in the lower half plane and find that the $l = 1$ (the Born term) is

$$\psi^1(z_0)=\frac{2\pi i\rho b}{k_0}e^{ik_0z_0}\left[z_0-\frac{1}{2ik_0}\right] . \tag{3.10}$$

The first term recovers the thin-slab result. Higher order processes, $l\geq 2$, give lth order poles at $\lambda=-k_0-i\epsilon$, and have as their leading contribution each of the terms in (3.5a). Since we argue that by allowing for the full range of scattering at each step takes us beyond (3.5a) and its implications, we must take the extra backscattering terms [the $-1/2ik_0$ in the $l=1$ expression (3.10) for instance]. To do this, we insert (3.9) into the geometrical series (2.9) and discover that

$$\psi(z_0)=\psi_0(z_0)-\frac{2\pi i\rho b}{k_0}\int\frac{d\lambda}{2\pi}\frac{i^2e^{-i\lambda z_0}}{(k_0+\lambda)}\left[\frac{1}{k_0-\lambda}+\frac{1}{k_0+\lambda}(1-e^{iz_0(\lambda+k_0)})\right]\left[\frac{1}{1+\frac{2\pi i\rho b}{k_0}f(\lambda)}\right] . \tag{3.11}$$

Thus the effect of summing the infinity of terms ψ^l is to move the pole at $\lambda=-k_0-i\epsilon$ to the zero of $1+2\pi i\rho bf(\lambda)/k_0$, that is, to a λ such that

$$\lambda=\pm k_0(1-2\beta/k_0)^{1/2} \tag{3.12}$$

with β defined as $2\pi\rho b/k_0$. Next doing the integral in (3.11) by the method of residues, we get two terms, namely

$$\frac{\beta/k_0}{[1-(1-2\beta/k_0)^{1/2}][(1-2\beta/k_0)^{1/2}]}e^{ik_0(1-2\beta/k_0)^{1/2}z_0} \tag{3.13}$$

and

$$-\frac{\beta e^{ik_0z_0}}{2k_0(1-2\beta/k_0)^{1/2}}\frac{[1+(1-2\beta/k_0)^{1/2}]}{[1-(1-2\beta/k_0)^{1/2}]} . \tag{3.14}$$

The first term is a plane wave, with a phase evolution modified from that of the incident wave, from which we deduce a refractive index

$$n=(1-2\beta/k_0)^{1/2} \text{ or } n^2-1=-4\pi\rho b/k_0^2 . \tag{3.15}$$

The modified wave is then

$$\frac{1+n}{2n}e^{ik_0nz_0} \tag{3.16}$$

while the second wave has the same phase evolution and opposite amplitude to the incident wave:

$$-\frac{(1+n)^2}{4n}e^{ik_0z_0} . \tag{3.17}$$

From (3.11) we see that the unscattered wave $\psi_0(z_0)$ is cancelled to order β/k_0, that is $n-1$, by (3.17).

In summary, the result of summing all trajectories shown in Fig. 2(b) is to generate a phase-shifted plane wave and a plane wave to cancel the incident wave in the usual Huygens manner. We have considered subsequent scatterings to be chosen successively at random, except that repeated scatterings between the same two sites have been omitted. The trajectory in Fig. 3(a) is the simplest with this complication—more than two passes between pairs of sites could be imagined as well as more complicated connections between various sites on the trajectory. The current method of reducing the problem to a one-dimensional walk can also handle this complexity; an example is given in the next section. We would not expect these additional scatterings to be important for liquids. They are more important for perfect crystals where dynamical (multiple scattering) effects can be subtle.

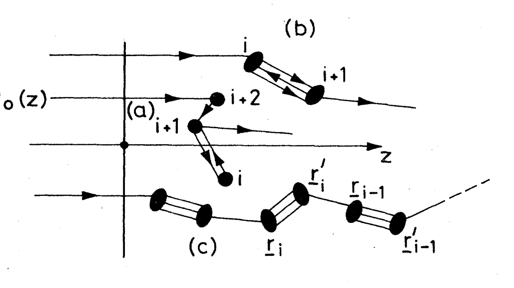

FIG. 3. A plane wave ψ_0 incident on a semi-infinite medium $z\geq 0$. Three trajectories are shown: (a) A double scattering off one site $i+1$. (b) Four scatterings from sites i and $i+1$ allowing the restoration of nuclear spin wave functions when spin flip is present. The possibilities are tabulated in Table 1 and are the simplest contributing scattering events when $\bar{b}=0$. (c) The simplest linking together of diagrams of the type (b) to give a new restricted multiple-scattering summation.

B. Correlated sites

We now consider the effect of correlations between pairs of sites. Here we replace the probability of finding site $i+1$ being independent of site i, expressed by the density ρ, by pair correlation functions

$$g(\mathbf{r}_{i+1},\mathbf{r}_i)\equiv g(|\mathbf{r}_{i+1}-\mathbf{r}_i|) .$$

The analysis follows the above and is presented in Appendix A. The principal result is obtaining an expression similar to (3.11), but with a new denominator

$$1+\frac{2\pi i\rho b}{k_0}f(\lambda)\rightarrow 1+\frac{2\pi i\rho b}{k_0}\hat{G}(\lambda) , \tag{3.18}$$

where from (A10)

$$\hat{G}(\lambda)=f(\lambda)-\frac{2ik_0}{\lambda}\int_0^\infty e^{ik_0a}\sin(\lambda a)h(a)da \tag{3.19}$$

and $h(a)=g(a)-1$ measures the deviations away from a uniform system, the most appropriate scheme for correlations.[4] The poles in (3.11) with f replaced by $\hat{G}$ give the new propagation vector $\lambda=k_0n$ where λ is the root of $1+2\pi i\rho b\hat{G}(\lambda)/k_0$ which, with (3.19), yields

$$1-\frac{2\beta k_0}{k_0^2-\lambda^2}+\frac{2\beta k_0}{\lambda}\int_0^\infty e^{ik_0a}\sin(\lambda a)h(a)da=0 . \tag{3.20}$$

Writing the root λ as nk_0 and with a variable change one obtains for n

$$n^2-1=-\frac{4\pi\rho b/k_0^2}{1+(4\pi\rho b/nk_0^2)\int_0^\infty dy\, e^{iy}\sin(ny)[g(y/k_0)-1]} . \tag{3.21}$$

We can approximate the integral by assuming, as for a liquid, that $h(a)$ is significant only around $a \sim a_0$, the first neighbor shell. Then, if $nk_0a_0 \ll 1$, the sine can be expanded in the region where $h(a)$ contributes. The n dependence of the integral vanishes, and we get

$$n^2-1=-\frac{2\beta/k_0}{1+2\beta k_0\int_0^\infty ah(a)da} , \tag{3.22}$$

where the exponential has been expanded to the same accuracy and the fact that $\int da\, a^2h(a)=0$ exploited.

It is useful to make contact with the local-field approach[5] used by Sears.[3] He defines the local field such that the expression for n^2-1 is

$$n^2-1=(2\beta/k_0)C , \tag{3.23}$$

where C is the local-field enhancement. Our long wavelength expression (3.22) is the same as his Eqs. (6.16) and (6.23).

The full expression (3.20) for λ (and hence n) differs from the result of Sears.[3] His follows from ours by setting $\lambda=k_0$ (or equivalently $n=1$) in the last term of (3.21) which yields

$$n^2-1=-\frac{2\beta/k_0}{1+2\beta\int_0^\infty e^{ik_0a}\sin(k_0a)h(a)da} , \tag{3.24}$$

that is, our full formula gives terms higher order in b which Sears explicitly ignores (his Eq. 6.18 and comments following).

Sears gives details on the magnitude of the correlation effect on the refractive index. One sees immediately that it is small by making the integrand in (3.24) dimensionless so that

$$n^2-1=-\frac{2\beta/k_0}{1+2(\beta/k_0)\int dx\, e^{ix}\sin(x)h'(x/a_0k_0)} , \tag{3.25}$$

where $h'(a/a_0)$ is the pair distribution of a dimensionless variable. The length a_0 characterizes packing and correlation in the fluid. For $a_0k_0 \sim 1$ the integral will give some characteristic number of order unity. In contrast the prefactor β/k_0 of the integral in the denominator is small. It is $\sim 2\pi b/a_0^3k_0^2$ since $\rho \sim 1/a_0^3$ for the liquid and so within our assumptions the prefactor reduces to $\sim b/a_0$, that is, a nuclear length divided by an atomic length. For the same reason the numerator $2\beta/k_0$ is also small. However, because of the extreme sensitivity of optical methods, local-field effects (the corrections represented by the denominator) should be observable. An interesting case is near a resonance of a nucleus since there b depends sharply on energy and can become very large. Absorption also becomes large, but questions of total external reflection remain interesting since they may represent an efficient way of monochromating epithermal neutrons.

IV. BEAM ATTENUATION AND INCOHERENT SCATTERING

The refractive indices calculated thus far have not led to attenuating waves. To develop attenuation we include the effects of scattering out of the beam and of incoherence. The former is expressed by the optical theorem[10] and follows naturally from a complex scattering length b, even in the absence of absorption, which in turn yields Sec. IV A, a phase shifted regenerated wave and, by interference, evanescence. This is the physical mechanism by which an imaginary part appearing in the partial wave analysis, (2.3), ensures conservation of scattered and unscattered flux. The evanescence generally arises from a randomness in scattering lengths because of an isotopic or elemental variation in the scattering centers or because the scattering nuclei and neutron can both have spin. In Sec. IV B we find that certain scattering events generate spherical waves which, because of their incoherence with the primary beam or with other scattered waves, are not susceptible to being synthesized into a coherent plane wave with a modified propagation constant k_0n. This too leads to evanescence. In Sec. IV C we also find, even when there is complete incoherence, that correlation in the scattering can again lead to a phase-shifted plane wave, $n\neq 1$, in contrast with the results of Sec. IV B and with a dimensional reduction characteristic of many random field problems.

A. Complex b

The effect of complex b is most easily seen from the results of Sec. III. Where b appears it should, according to (2.4), be replaced by $b-ik_0b^2$. Since b only occurs in the combination defined as $\beta=2\pi\rho b/k_0$ it is clearly β in (3.12)—(3.15) that we must consider to be complex and we must take care to ensure that the roots of $1+i\beta f(\lambda)$ remain in the appropriate part of the complex plane. We first look at the problem without correlations and see from (3.12) that replacing β by $\beta'+i\beta''$ where

$$\beta'=2\pi\rho b/k_0 \quad\text{and}\quad \beta''=-2\pi\rho b^2 \tag{4.1}$$

generates an imaginary component to λ. Simple analysis shows that the root is

$$\lambda=-k_0(1-2\beta'/k_0)^{1/2} \tag{4.2}$$

where we ignore terms like $(\beta'')^2/k_0^2$ and $\beta'\beta''/k_0^2$. The real part of the refractive index n' remains as before, but now there is an imaginary part $n''=\beta''/n'k_0$ signaling decay in amplitude

$$|\psi| \sim \exp(-k_0n''z)=\exp(-2\pi\rho b^2z/n') . \tag{4.3}$$

If $n'\simeq 1$, the decay length of wave intensity is the conventional result $4\pi\rho b^2$. When correlations between scattering centers are present, we find from (3.21) that the above result for n'' simply gets altered to

$$n''=\beta'' \Big/ \left[n'k_0 \left[1+2\beta'k_0 \int_0^\infty da\, ah(a) \right] \right] , \qquad (4.4)$$

where the real part n' is given by (3.22). Hence as before the relevant part of the refractive index, here n'', shows the effect of correlations only at higher order in b. In this case $n''\sim b^2$ with corrections at $O(b^3)$ due to $h(a)$. A more important influence on the beam evanescence than correlation is the existence of incoherent scattering which we now discuss.

B. Spin incoherence

We limit our discussion to spin incoherent scattering because it is particular to neutrons. Until now we have not specified the spin state of the neutron n and have treated the nuclei p as spinless. In reality, since neutrons are spin $\frac{1}{2}$, we need a spinor part in their wave function. For concreteness we assume that the scattering nuclei are also spin $\frac{1}{2}$ in which case their spin wave functions also have a spinor part. The scattering length associated with the pseudopotential between neutron and nucleus now depends on whether the total spin $(\mathbf{I}=\mathbf{I}_n+\mathbf{I}_p)$ is 1 (triplet) or 0 (singlet). We denote these lengths by b_t and b_s, respectively. Following Fermi,[2] we then identify the fundamental scattering processes in Table I, in which the correspondence between the notation $(\uparrow)$ and the spinor χ is

$$(\uparrow)\equiv \begin{bmatrix}1\\0\end{bmatrix},\quad (\downarrow)\equiv \begin{bmatrix}0\\1\end{bmatrix} . \qquad (4.5)$$

From the Table we see that processes a and d represent pure triplet states of total spin and spin flip is impossible. Processes b and c involve the decomposition of the mixed initial state into triplet and singlet states with the associated weights b_t and b_s in the final state. Reconstitution of the final state shows flipped and nonflipped final states with weights $(b_t+b_s)/2$ and $(b_t-b_s)/2$, respectively.

Since an unpolarized neutron beam has up and down components with equal weight and arbitrary relative phase we could have assumed in Sec. III, without loss of generality, that the incident neutron was spin up. Then the b that enters is the mean value $\bar{b}=(3b_t+b_s)/4$ because from an initial state $(\uparrow_n)$ a final state $(\uparrow_n)$ can be attained either via scattering from $(\uparrow_p)$, process a, or from $(\downarrow_p)$ without spin flip, the relevant part of process b, each nuclear spin state occurring with probability $\frac{1}{2}$.

For an lth order process, (3.1) has a prefactor

$$\{\tfrac{1}{2}b_t+\tfrac{1}{2}[\tfrac{1}{2}(b_t+b_s)]\}^l$$

which when expanded gives all the possible intermediate processes, appropriately weighted by the probabilities of that succession of nuclear spins encountered and the appropriate scattering lengths being used. Rewriting this as $\bar{b}^{\,l}$ yields the appropriate definition of the mean scattering length $\bar{b}$. A totally incoherent scatterer is defined to have $\bar{b}=0$, that is $b_s=-3b_t$. Evidently the spin flipped component of process b does not contribute to the growth of a down-spin coherent wave since when multiple scattering is neglected

$$\psi_\downarrow^1\equiv\psi^1\begin{bmatrix}0\\1\end{bmatrix}=-\frac{1}{2}\frac{b_t-b_s}{2}\sum_{j=1}^{N}\frac{e^{ik_0r_{10}(j)}}{r_{10}(j)}e^{ik_0z_1(j)}\begin{bmatrix}0\\1\end{bmatrix} . \qquad (4.6)$$

The spatial part of the wave function (4.6) arises from once-scattered waves originating from the N scattering centers at $r_1(j)$ that are initially spin down with probability $\frac{1}{2}$. This is the same as before. The spinor results from the spin-flipped process at site j, hence the amplitude $(b_t-b_s)/2$. However, a spin-down coherent wave is not really generated since when taking $\psi_\downarrow^{1*}\psi_\downarrow^1$ to get the total intensity the nuclear spinors must be taken into account, that is, for a particular j in the sum (4.6), the final target spinor is

$$\left[\prod_{\substack{i,j=1\\ i\neq j}}^{N}\chi_i\right]\chi_j(\uparrow) , \qquad (4.7)$$

where the χ_i for $i\neq j$ are unchanged and the spinor for the site j, having been found down, has flipped up. The spin part of typical element of $\psi^*\psi$ is then

$$\left[\prod_{\substack{i',j'=1\\ i'\neq j'}}^{N}\chi_{i'}\right]\chi_{j'}(\uparrow)\left[\prod_{\substack{i,j=1\\ i\neq j}}^{N}\chi_i\right]\chi_j(\uparrow) \qquad (4.8)$$

and these N-particle wave functions are orthogonal unless $j'=j$ so only N and not N^2 elements in $\psi^*\psi$ are nonzero and we are unable to construct a coherent wave from any spin-flipped components of ψ. In turn, we therefore deduce that the scattering length entering the coherent wave is indeed $\bar{b}$ defined above according to the weights of the two available processes. This is an explicit example of a process not contributing to the mean wave amplitude. A more subtle process overcoming this incoherence is treated in Sec. IV C.

We also note the decay of the wave is described as before by the imaginary components of the b's, and performing the same analysis as before for ψ^l where the imaginary parts enter as $\frac{1}{2}(b_t^2+b_s^2)$, we find that the second term enters as a result of the allowable part of process b. It then follows that

$$n''=4\pi\rho\left[\frac{3b_t^2+b_s^2}{4}\right] , \qquad (4.9)$$

which is nonzero even when $\bar{b}=0$.

TABLE I. Amplitudes of the incident and scattered waves for different scattering processes. b_s is the singlet state scattering length; b_t is the triplet state scattering length. $(\uparrow_n\uparrow_p)$ is the neutron and nucleus both in spin-up states, etc.

Process	Incident amplitude	Scattered amplitude
a	$(\uparrow_n\uparrow_p)$	$-b_t(\uparrow_n\uparrow_p)$
b	$(\uparrow_n\downarrow_p)$	$-\frac{b_s+b_t}{2}(\uparrow_n\downarrow_p)-\frac{-b_s+b_t}{2}(\downarrow_n\uparrow_p)$
c	$(\downarrow_n\uparrow_p)$	$-\frac{-b_s+b_t}{2}(\uparrow_n\downarrow_p)-\frac{b_s+b_t}{2}(\downarrow_n\uparrow_p)$
d	$(\downarrow_n\downarrow_p)$	$-b_t(\downarrow_n\downarrow_p)$

C. Total incoherence

It remains to ask whether there are processes that contribute to the coherent wave when the mean scattering length $\bar{b}$ is zero. Since we are considering multiple scattering, it is possible that repeated scatterings could restore the nuclear spinors. The orthogonality argument for the final states in (4.8) and the consequent incoherence is thereby circumvented.

The argument against the creation of a down-spin coherent wave is also valid when multiple scattering is present, since a final neutron down-spin implies that a minimum of one nucleus, and certainly an odd number of nuclei, must be flipped up. Hence restrictions on the contributions to $\psi_{\downarrow}^{l*}\psi_{\downarrow}^{l}$ are again involved when evaluating the intensity of the resultant wave. We thus restrict ourselves to up-spin resultant waves where, in the multiple scatterings leading to the final wave, spin flips, if any, are such that the nuclear states are restored to their initial configuration. The simplest example is that of Fig. 3(b) where, if spin flip takes place, there is the possibility for another to restore the nuclei. Nonflip processes are also significant since they do not yield $\bar{b}^4 \equiv 0$ as in the analogous case of four independent sites used in our previous arguments about $\psi^{l=4}$ and in defining $\bar{b}$.

Instead, if we think of these processes involving four sites, then there is a most definite correlation between sites 1+3 and 2+4. In Table II are the possible sequences corresponding to the four equiprobable initial spin states of the two nuclei.

In Table II we have written out the factors in such a way so the sequence of scattering events is apparent. We call $\overline{b^4}$ the total scattering length factor, with the appropriate factor of $\frac{1}{4}$ from equal *a priori* probability of the nuclear spin states of the pair of centers,

$$\overline{b^4}=5b_t^4 \quad \text{when } \bar{b}=0\ . \tag{4.10}$$

The spatial part of the wave function can also be evaluated. The wave function incident on the first site has to be multiplied by a phase and amplitude in respect of the four scattering events and the three internal propagations. This multiplicative factor is then

TABLE II. Scattering weights for initial spin states of two nuclei when the incident neutron has spin-up.

Nuclear configuration	Weight
$(\uparrow_{p1}\uparrow_{p2})$	b_t^4
$(\downarrow_{p1}\uparrow_{p2})$	$\frac{b_t+b_s}{2}b_t\frac{b_t+b_s}{2}b_t+\frac{b_t-b_s}{2}\frac{b_t+b_s}{2}\frac{b_t-b_s}{2}b_t$
$(\uparrow_{p1}\downarrow_{p2})$	$b_t\frac{b_t+b_s}{2}b_t\frac{b_t+b_s}{2}+b_t\frac{b_t-b_s}{2}\frac{b_t+b_s}{2}\frac{b_t-b_s}{2}$
$(\downarrow_{p1}\downarrow_{p2})$	$\left[\frac{b_t+b_s}{2}\right]^4+\frac{b_t-b_s}{2}b_t\frac{b_t-b_s}{2}\frac{b_t+b_s}{2}+\frac{b_t+b_s}{2}\frac{b_t-b_s}{2}b_t\frac{b_t-b_s}{2}$

$$\overline{b^4}\frac{e^{3ik_0|\mathbf{r}_{i,i+1}|}}{|\mathbf{r}_{i,i+1}|^3}\ . \tag{4.11}$$

We can now extend the analysis of Sec. III to the case where $\bar{b}=0$. For $l\geq 1$, where l now counts the number of *pairs* of points $\mathbf{r}_i,\mathbf{r}_i'$, we can connect together diagrams of the type Fig. 3(b) to form the equivalent of Fig. 2(b), shown in Fig. 3(c). Of course, more complicated events are possible while still fulfilling the conditions on the nuclear wave functions adhered to above. We shall neglect these complications. What is not allowed are diagrams of the form of Fig. 3(b) connected by simple scattered events such as in Sec. III since each of these intermediate points contribute a weight $\bar{b}=0$.

Following the analysis of Sec. III for ψ^l we convert the sums over (4.11) into integrals. The trick, as before, is to do $\int d\sigma$ in the transverse plane, that is

$$\rho\overline{b^4}\int dz_i\int d\sigma \exp\{3ik_0[(z_i-z_i')^2+(\sigma_i-\sigma_i')^2]^{1/2}\}/\{[(z_i-z_i')^2+(\sigma_i-\sigma_i')^2]^{1/2}\}^3 \tag{4.12}$$

which on change of variables reduces to

$$\rho\overline{b^4}\int dz_i\, 2\pi\int_{|z_i-z_i'|}^{\infty} dy\, e^{3ik_0 y}\frac{g(y)}{y^2} \equiv 2\pi\rho\overline{b^4}\int dz_i\, I(|z_i-z_i'|)\ . \tag{4.13}$$

The other part of each element of the l-scattered wave is the propagation from $\mathbf{r}_i'$ to $\mathbf{r}_{i-1}$ and is precisely as before, yielding as in (3.3)

$$\frac{2\pi i\rho}{k_0}\int_0^\infty dz_i' \exp(ik_0|z_i'-z_{i-1}|) \tag{4.14}$$

an expression leading to the $f(\lambda)$ term. Following the previous derivation, we do the integral over z_i and z_i' by changing to the a variables with the λ integration expressing the constraint that the trajectory must end at the point of observation z_0. The important part of this calculation is the term equivalent $f(\lambda)$ of Sec. III which determines where the poles in the λ integration are and therefore what the new propagation constant and refractive index are. We call this term $F(\lambda)$, and hence the geometric term in the $\sum_l \psi^l$ is

$$F(\lambda)=\overline{b^4}\left[\frac{2\pi i\rho}{k_0}f(\lambda)\right]2\pi\rho\int da\, e^{i\lambda a}I(|a|)\ . \tag{4.15}$$

The part is the large parentheses results from (4.14) as in Sec. III; the last part results from (4.13). The last integral we denote by f' and is

$$f'(\lambda)=\int_{-\infty}^{\infty} da\, e^{i\lambda a}\int_{|a|}^{\infty}\frac{dy}{y^2}e^{3ik_0 y}\Theta(y-a_0)\ , \quad (4.16)$$

where we neglect any correlations from $g(y)$ except that from a hard sphere repulsion with radius a_0 expressed by the unit step function Θ. This is necessary since with three, rather than one, internal propagations the y integral over separations of sites would become divergent. Performing the leading integral by parts we get

$$f'(\lambda)=\frac{1}{i\lambda}\int_{a_0}^{\infty} da\frac{1}{a^2}(e^{i\lambda a}-e^{-i\lambda a})e^{3ik_0 a} \quad (4.17)$$

whereupon $F(\lambda)$ becomes

$$F(\lambda)=\frac{-(4\pi\rho)^2\overline{b^4}}{k_0^2-\lambda^2}\frac{1}{\lambda}\int_{a_0}^{\infty} da\frac{1}{a^2}\sin(\lambda a)e^{3ik_0 a}\ . \quad (4.18)$$

The geometrical series yields $1/[1-F(\lambda)]$ which has a pole located by

$$k_0^2-\lambda^2+(4\pi\rho)^2\overline{b^4}\frac{1}{\lambda}\int_{a_0}^{\infty} da\frac{1}{a^2}\sin(\lambda a)e^{3ik_0 a}=0\ . \quad (4.19)$$

Putting $\lambda=k_0$ in the last term as Sears[3] does, writing $\lambda=nk_0$, and taking only the real part, we find that

$$n^2=1+\frac{(4\pi\rho)^2\overline{b^4}}{2k_0^2}\int_{k_0a_0}^{\infty} dx\frac{1}{x^2}[\sin(4x)-\sin(2x)]\ . \quad (4.20)$$

The second term on the right is small. The refractive index increment $n-1\sim b^4$ must be very small since one can write

$$n-1\sim\left[\frac{b}{a_0}\right]^4\left[\frac{\lambda_n}{a_0}\right]^2 H\left[\frac{a_0}{\lambda_n}\right] \quad (4.21)$$
$$\sim\left[\frac{b}{a_0}\right]^4\left[\frac{\lambda_n}{a_0}\right]^2 \ln\left[\frac{\lambda_n}{a_0}\right]\ ,$$

where $\rho\sim 1/a_0^3$ depends on the atomic length a_0 and $k_0\equiv 2\pi/\lambda_n$ has been expressed in terms of the neutron's wavelength, λ_n. The ratio b/a_0 of a nuclear to atomic length is small ($\sim 10^{-4}$). The ratio λ_n/a_0 can be large at the long wavelengths of neutron optical experiments. The integral, $H(a_0/\lambda_n)$, is evidently like $\ln(\lambda_n/a_0)+\text{const}$ as $\lambda_n/a_0\to\infty$. The appearance of a logarithm in the refractive index, a result perhaps expected in two dimensions, arises because we have a random field b with $\bar{b}=0$. Hence, the repeated propagations needed to get a finite result yield a dimensional reduction, characteristic of many random field problems. In practice, of course, if the system is not entirely incoherent, the conventional term will probably dominate. When $\bar{b}=0$ the criterion for whether (4.20) is significant in practice is whether in a thickness L there is significant phase *shift* relative to a wave in free space, $(n-1)k_0L$, compared with the loss of intensity due to simple spin-flip scattering; that is, compared with an exponent like $4\pi\rho b^2 L$. We thus look at

$$4\pi\rho b^2:\ \frac{1}{4}\frac{(4\pi\rho)^2b^4}{k_0}\ln\left[\frac{1}{k_0a_0}\right]$$

or

$$1:\ \frac{\pi\rho b^2}{k_0}\ln\left[\frac{1}{k_0a_0}\right]\sim 1\Big/\left[\left[\frac{b}{a_0}\right]^2\frac{\lambda_n}{a_0}\ln\left[\frac{\lambda_n}{a_0}\right]\right]\ . \quad (4.22)$$

This ratio is only ~ 1 for long, probably unattainable, neutron wavelengths. Although this result appears to have little physical application, unless we are near a resonance so that b is large, it illustrates how multiple scattering, combined with variations of scattering length, can lead to subtle effects.

V. CONCLUSIONS

We used a multiple-scattering approach to calculate the neutron refractive index, with and without the effect of correlations in the position of scattering centers. This approach is an alternative to the constitutive relation approach reviewed and extended by Sears.[3] The results for the coherent wave depend on the level of complexity in the diagrams summed. At the simplest level of successive sites being distributed by pair distribution functions, the results seem more general than those of Sears and reduce to his by a simple approximation consistent with the order of b to which the results are significant.

We find that the scattering method allows a clearer insight into the refractive index, especially on how correlations enter into local-field results, than do the constitutive equations. Apart from deriving a general propagator in an infinite medium, we derived the refractive index in a semi-infinite medium by explicitly regenerating plane waves in a Huygens treatment of all the spherical waves in the problem. In essence the task is reduced to a one-dimensional problem, an advantage which persists even when more complicated scattering events dominate (in Sec. IV where the mean scattering length is zero). Another advantage of using a semi-infinite medium is that one can explicitly demonstrate the possibility of reflection which we treated in Appendix B.

We discussed incoherent scattering and demonstrated that it is possible to generate a real part to the refractive index, even when the mean scattering length is zero. Although this is an interesting possibility we argue that it is seldom likely to matter in practice. An exception may be a slab of incoherently scattering resonant nuclei, a situation of some practical interest for monochromating epithermal neutron beams.

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APPENDIX A: MULTIPLE SCATTERING TREATMENT OF REFRACTIVE INDEX CONSIDERING CORRELATIONS

Returning to (3.1) for ψ^l, we replace the sums over intermediate points by integrals, but now put in the pairwise probabilities

$$g(|\mathbf{r}_l-\mathbf{r}_{l-1}|)g(|\mathbf{r}_{l-1}-\mathbf{r}_{l-2}|)\times\cdots\times g(|\mathbf{r}_2-\mathbf{r}_1|) \quad \text{(A1)}$$

that the trajectory $\mathbf{r}_l,\mathbf{r}_{l-1},\ldots,\mathbf{r}_1$ occurs. An element of the multiple sum then becomes

$$\rho\int dz_i\int d\sigma_i \frac{\exp\{ik_0[\sigma_{i,i-1}^2+(z_i-z_{i-1})^2]^{1/2}\}}{[\sigma_{i,i-1}^2+(z_i-z_{i-1})^2]^{1/2}} \times g([\sigma_{i,i-1}^2+(z_i-z_{i-1})^2]^{1/2}) . \quad \text{(A2)}$$

Because there is translational symmetry in the transverse direction, described by $\boldsymbol{\sigma}$, we can use $\mathbf{r}_{i-1}$ as an origin for $\boldsymbol{\sigma}$ and denote $\boldsymbol{\sigma}_{i,i-1}$ by $\boldsymbol{\sigma}$. After the variable change

$$\sigma^2+(z_i-z_{i-1})^2=y^2\rightarrow d\boldsymbol{\sigma}=d\mathbf{y}\equiv 2\pi y\,dy \quad \text{(A3)}$$

(A2) becomes

$$2\pi\rho\int dz_i\int_{|z_i-z_{i-1}|}^{\infty} dy\, e^{ik_0y}g(y) . \quad \text{(A4)}$$

To make contact with the analysis of Sec. III, we take out a factor of i/k_0 to produce

$$\frac{2\pi i\rho}{k_0}\int dz_i\, G(|z_i-z_{i-1}|) , \quad \text{(A5)}$$

where G is defined by

$$G(z)=-ik_0\int_z^{\infty} dy\, e^{ik_0y}g(y) . \quad \text{(A6)}$$

Putting (A5) back into the multiple sum and changing variables as before, we then obtain

$$\psi^l=\left[-\frac{2\pi i\rho b}{k_0}\right]^l\int_{-\infty}^{+\infty}\frac{d\lambda}{2\pi}\int_{-\infty}^{z_0}da_1\int_0^{\infty}da_{l+1}\int_{-\infty}^{\infty}\left[\prod_{i=2}^{l}da_i\right]\left[\prod_{j=2}^{l}G(|a_j|)\right]\exp\left[-i\lambda\left[\sum_{i=1}^{l+1}a_i-z_0\right]+ik_0z_0\right] . \quad \text{(A7)}$$

The integrals over a_i ($i=2,\ldots,l$) are normal Fourier transforms

$$\hat{G}(\lambda)=\int_{-\infty}^{\infty}da\, G(|a|)e^{i\lambda a} . \quad \text{(A8)}$$

We proceed by expanding the problem about the uniform state $g=1$ and write

$$g(y)=1+[g(y)-1]\equiv 1+h(y) , \quad \text{(A9)}$$

$$\hat{G}(\lambda)=f(\lambda)-\frac{2ik_0}{\lambda}\int_0^{\infty}e^{ik_0a}\sin(\lambda a)h(a)da . \quad \text{(A10)}$$

The first term $f(\lambda)$ is as in Sec. III and is the result of the uniform part $g=1$; the second comes from the correlations, that is, from $h(a)$. The lth wave is then

$$\psi^l=\left[-\frac{2\pi i\rho b}{k_0}\right]^l\int\frac{d\lambda}{2\pi}[\hat{G}(\lambda)]^{l-1}\frac{i^2e^{-i\lambda z_0}}{k_0+\lambda}\left[\frac{1}{k_0-\lambda}+\frac{1}{k_0+\lambda}(1-e^{iz_0(\lambda-k_0)})\right] , \quad \text{(A11)}$$

where the other factors come from propagations to $\mathbf{r}_{l+1}$ and from $\mathbf{r}_1$ to $\mathbf{r}_0$. The only difference with before is that for $g\neq 1$, $h\neq 0$, and there is the term in (A10) additional to $f(\lambda)$. The shift in the poles of the λ integrand, as a result of summing an infinite geometrical series with terms ψ^l, yields the new expression for the refractive index discussed in the text.

APPENDIX B: REFLECTION FROM A SEMI-INFINITE MEDIUM

When scattering events produce fields only at a point further in the direction of the propagation, one recovers the thin slab or "pasted" thin-slab approximations, but is unable to describe reflection. Reflection results from Huygens reconstructions of (multiple) scattering events over all possible distances into the medium. The calculation of Sec. III needs amendment when $z_0\leq 0$. In that case we obtain for (3.9)

$$\int_{-\infty}^{z_0}da_1\, e^{ik_0|a_1|+i\lambda a_1}=\frac{ie^{i(\lambda-k_0)z_0}}{k_0-\lambda_0} . \quad \text{(B1)}$$

The expression for ψ^l, the l-scattered coherent wave is then (3.11) with (B1) substituted for the { } contribution in (3.9). The sum of all the ψ^l yields the reflected wave ψ_R:

$$\psi_R(z_0)=-\frac{2\pi i\rho b}{k_0}\int\frac{d\lambda}{2\pi}\frac{i^2e^{ik_0z_0}}{k_0+\lambda}\frac{1}{k_0-\lambda} \times\frac{1}{1+(2\pi i\rho/k_0)f(\lambda)} . \quad \text{(B2)}$$

The differences from (3.13) are that $\psi_0(z_0)$ does not appear and the wave is a plane wave traveling in the negative z direction. The poles, however, are again located by the $f(\lambda)$ term from the infinite summation of waves, and the propagation constant is uneffected (since the $e^{-i\lambda z_0}$ term is absent) as it should be in free space. The λ integration simply determines the amplitude of the reflected wave. The result is

$$\psi_R(z_0)=\frac{1}{4}\frac{(n^2-1)}{n}e^{-ik_0 z_0} . \quad \text{(B3)}$$

We note that the sum of the transmission and reflection coefficients implied by this wave and the transmitted wave (3.18) is

$$T+R=\frac{(1+n)^4}{(4n)^2} ,$$

which is unity to within factors of $(n-1)^3$, that is b^3.

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