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Scaling theory of the low-field Hall effect near the percolation threshold

David J. Bergman
School of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel

David Stroud
Department of Physics, The Ohio State University, Columbus, Ohio 43210
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A scaling theory of the low-field Hall effect in a two-component metal-nonmetal mixture near the percolation threshold of the metallic component is formulated and some of its physical consequences are examined. We predict that under certain conditions a peak in the Hall resistivity R_e versus metal volume fraction p_M can be observed near the threshold.

Investigations of the theory of the Hall effect in composite media were started nearly 20 years ago by Juretschke, Landauer, and Swanson.[1] They treated various types of microgeometries in three dimensions (3D) approximately, and they also described the exact general solution for the Hall effect at low magnetic field H in isotropic two-dimensional (2D) composites. The behavior predicted for the 2D case has recently been observed experimentally for the first time.[2] Significant further progress was then made by the introduction of an effective-medium approximation,[3] by the nodes-links approximation in two different forms,[4,5] by the exact solution of the Hall problem on a Cayley tree network,[6] and, more recently, by the exact solution of the Hall and transverse magnetoresistance problems for arbitrary field strength in 2D.[7] Also, recently, a comprehensive theory of the low-field Hall effect in isotropic two-component composites has been developed.[8] An important result of that work was the conclusion that the low-field Hall conductivities of the two components λ_M, λ_I and that of the composite λ_e satisfy the following exact relation:[9]

$$\frac{\lambda_e-\lambda_I}{\lambda_M-\lambda_I}=X\left(\frac{\sigma_I}{\sigma_M}\right), \tag{1}$$

where X is independent of the Hall conductivities; it is a function only of the ratio of the Ohmic conductivities of the two components, and its precise form depends on the microgeometry of the composite. An attempt to construct a scaling theory of the Hall effect was made previously by Shklovskii.[10] However, that involved an improper scaling ansatz and resulted in an inconsistent description of the critical behavior.

In this Rapid Communication we present a consistent scaling theory of the low-field Hall effect that is based upon the theory of Ref. 8, and this leads to some rather interesting predictions of critical behavior in an isotropic good conductor (σ_M, λ_M)–bad conductor (σ_I, λ_I) mixture near the percolation threshold $p_M=p_c$ of the former.

Equation (1) suggests that the particular combination of Hall conductances that appears on the left-hand side depends only on the Ohmic properties of the system. This is borne out by the fact that in order to evaluate the function X, one only needs to know the microscopic electric fields $\mathbf{E}^{(x)}(\mathbf{r})$, $\mathbf{E}^{(y)}(\mathbf{r})$ present in the system when an external potential difference is applied in the x,y directions, respectively, in the absence of a magnetic field:[8]

$$X=\frac{1}{V}\int dV\Theta_M(\mathbf{r})(\mathbf{E}^{(x)}\times\mathbf{E}^{(y)})_z \ . \tag{2}$$

Here V is the total volume, and $\Theta_M(\mathbf{r})$ is a characteristic step function equal to 1 when $\mathbf{r}$ is inside the σ_M component and equal to 0 otherwise, so that the integration is effectively restricted to the σ_M volume. As a consequence of these remarks, one is naturally led to assume that, near the percolation threshold of σ_M, the appropriate scaling variable would be the same as that which appears in the Ohmic conductivity, namely,[11] $(\sigma_I/\sigma_M)/|p_M-p_c|^{t+s}$.

We therefore make the following scaling ansatz for the bulk effective Hall conductivity λ_e:

$$\frac{\lambda_e-\lambda_I}{\lambda_M-\lambda_I}=|p_M-p_c|^{\tau}F\left(\frac{\sigma_I/\sigma_M}{|p_M-p_c|^{t+s}}\right), \tag{3}$$

$$\text{for } \sigma_I/\sigma_M \ll 1 \text{ and } |p_M-p_c|\ll 1 \ .$$

The exponent τ characterizes the critical behavior of λ_e for $p_M>p_c$ when σ_I (and therefore also λ_I) vanishes. In that case we have $\lambda_e/\lambda_M \propto (p_M-p_c)^{\tau}$. The value of τ is $\tau=2t\simeq 2.60$ in 2D; $\tau\simeq 3.7$ in 3D.[12]

As usual, there are three interesting limits for the scaling function $F(Z)$, namely,

$$F(Z)\propto\begin{cases}\text{const for } Z\ll 1,\ p_M>p_c\ \ \text{(Regime I)}\\ Z^2 \text{ for } Z\ll 1,\ p_M<p_c\ \ \text{(Regime II)}\\ Z^{\tau/(t+s)} \text{ for } Z\gg 1,\ p_M \lessgtr p_c\ \ \text{(Regime III)}\end{cases} \ . \tag{4}$$

The first of these limits has been discussed before,[12] while the last limit is obviously a consequence of the need to cancel the dependence of Eq. (3) on p_M-p_c. However, the second limit merits some discussion, since one might have expected $F(Z)\propto Z$ below the threshold. In fact, as $\sigma_I/\sigma_M\to 0$, the fields $\mathbf{E}^{(x)}$ and $\mathbf{E}^{(y)}$ will also tend to 0 linearly with σ_I/σ_M inside the σ_M component, whenever that component does not percolate. From Eq. (2) it then follows that $X\propto(\sigma_I/\sigma_M)^2$ when $p_M<p_c$.

The analogous scaling ansatz for the Ohmic conductivity of the mixture σ_e would be

$$\frac{\sigma_e-\sigma_I}{\sigma_M-\sigma_I}=|p_M-p_c|^{t}G\left(\frac{\sigma_I/\sigma_M}{|p_M-p_c|^{t+s}}\right), \tag{5}$$

$$\text{for } \sigma_I/\sigma_M\ll 1 \text{ and } |p_M-p_c|\ll 1 \ ,$$

where

$$G(Z)\propto\begin{cases}\text{const in Regime I}\\ Z \text{ in Regime II}\\ Z^{t/(t+s)} \text{ in Regime III}\end{cases} . \tag{6}$$

We note the difference in behavior between $G(Z)$ and $F(Z)$ in Regime II: the behavior of $G(Z)\propto Z$ is dictated by the fact that $\sigma_e\propto\sigma_I$ when $p_M<p_c$. While Eq. (6) is essentially equivalent to the scaling ansatz of Straley,[11] in which the left-hand side of Eq. (5) is replaced by σ_e/σ_M, Eqs. (3) and (4) differ in important respects from the scaling ansatz of Shklovskii,[10] which did not take into account the results included in Eqs. (1) and (2). In particular, we shall see below the crucial importance of making the scaling ansatz (3) for $(\lambda_e-\lambda_I)/(\lambda_M-\lambda_I)$ rather than for λ_e/λ_M.

The consequences of our scaling ansatz are best discussed in terms of the Hall resistivities R_M, R_I, R_e, which are related to the conductivities as follows:

$$R_i=\lambda_i/\sigma_i^2, \quad \text{for } i=M,I,e \ , \tag{7}$$

if the magnetic field is weak enough so that $\lambda_i \ll \sigma_i$ (or alternatively, so that the cyclotron frequency ω_c and the Ohmic relaxation time τ_0 satisfy $\omega_c\tau_0 \ll 1$). We will assume not only that $\sigma_M \gg \sigma_I$, but that $\lambda_M \gg \lambda_I$ and $R_M \ll R_I$ as well. However, no *a priori* assumption is made regarding σ_e or λ_e. In this way we find

$$R_e\propto\begin{cases}A_1R_M|p_M-p_c|^{-g}+B_1R_I\left(\dfrac{\sigma_I}{\sigma_M}\right)^2|p_M-p_c|^{-2t} \text{ in Regime I}\\ A_2R_M|p_M-p_c|^{-g}+B_2R_I|p_M-p_c|^{2s} \text{ in Regime II}\\ A_3R_M\left(\dfrac{\sigma_I}{\sigma_M}\right)^{-g/(t+s)}+B_3R_I\left(\dfrac{\sigma_I}{\sigma_M}\right)^{2s/(t+s)} \text{ in Regime III}\end{cases} , \tag{8}$$

where

$$g=2t-\tau \ , \tag{9}$$

and where A_i, and B_i are constants of order one. The critical exponent g has the values 0, 0.29 ± 0.05, and 1 in 2D, 3D, and 6D, respectively,[2,5,12] while t and s are the usual Ohmic-conductivity critical exponents.

In Regime I, the ratio of the second to the first term in R_e is of order $(\lambda_I/\lambda_M)|p_M-p_c|^{-\tau}$, and thus either of them may dominate, depending on the parameters of the system. However, both of them increase as p_M decreases towards p_c, and this will continue until $(p_M-p_c)^{t+s}\simeq\sigma_I/\sigma_M$, at which point Regime III is entered and R_e rounds off at a value independent of p_M. As p_M decreases below p_c, Regime II is eventually entered and there a nonmonotonic behavior is possible, since R_e is the sum of an increasing and a decreasing term: a minimum of R_e will occur at $p=p_{\min}$ where

$$|p_{\min}-p_c|\simeq\left(\frac{R_M}{R_I}\right)^{1/(2s+g)} , \tag{10}$$

provided that point lies in Regime II, i.e., if

$$|p_{\min}-p_c|^{t+s}\simeq\left(\frac{R_M}{R_I}\right)^{(t+s)/(2s+g)} > \frac{\sigma_I}{\sigma_M} . \tag{11}$$

(In these as well as in the subsequent approximate equalities, we ignore coefficients of order one.) Otherwise, R_e will continue to increase monotonically towards R_I as p_M decreases throughout Regime II. The (local) minimum value of R_e, which occurs for $p_M=p_{\min}$, is given by

$$R_{e,\min}\simeq R_M\left(\frac{R_I}{R_M}\right)^{g/(2s+g)} , \tag{12}$$

while the (local) maximum value, which must occur when $|p_M-p_c|^{t+s}\lesssim\sigma_I/\sigma_M$, is given by

$$R_{e,\max}\simeq R_M\left(\frac{\sigma_I}{\sigma_M}\right)^{-g/(t+s)} . \tag{13}$$

This peak can only be observed in 3D composites, since in the 2D case (i.e., thin films), $g=0$. A qualitative plot of R_e vs p_M is shown in Fig. 1.

An experimental test of these predictions would have to use a pair of components whose Ohmic conductivities are very different, e.g., a metal σ_M and a semiconductor σ_I, where clearly $\sigma_M \gg \sigma_I$. In order to observe the peak described above, R_M/R_I should then not be too small. This is necessary to ensure that Eq. (11) is satisfied, but also to

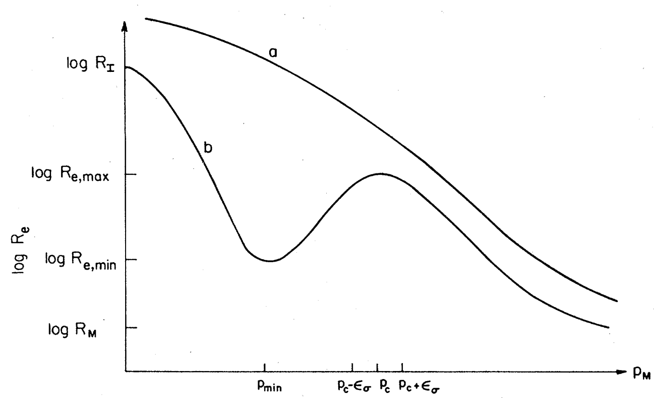

FIG. 1. Qualitative plot of log R_e (Hall resistivity) vs p_M (metallic volume fraction) a, for the case $(R_M/R_I)^{(t+s)/(2s+g)}<\sigma_I/\sigma_M$, b, for the opposite case. The width of the region where the peak in R_e gets rounded off is $\epsilon_\sigma\simeq(\sigma_I/\sigma_M)^{1/(t+s)}$. The other important quantities in this plot can be calculated from Eqs. (10), (12), and (13).

separate the positions of $R_{e,\min}$ and $R_{e,\max}$ sufficiently so that they will actually occur at experimentally distinguishable values of p_M. As an example, if we take

$$\frac{\sigma_I}{\sigma_M}=10^{-6},\quad \frac{R_M}{R_I}=10^{-3},\quad t=1.95,\quad s=0.7,\quad g=0.3$$

(see Refs. 11–13 for the values of t, s, and g), then we find that Eq. (11) is well satisfied and that

$$|p_{\min}-p_c|\simeq 0.017\ ,$$

$$\frac{R_{e,\min}}{R_M}\simeq 3.4\ ,$$

$$\frac{R_{e,\max}}{R_M}\simeq 4.8\ .$$

A somewhat better situation would occur if we took

$$\frac{\sigma_I}{\sigma_M}=10^{-9},\quad \frac{R_M}{R_I}=10^{-2}\ ,$$

and t, s, and g as before. In that case, Eq. (11) is again satisfied, and we find that

$$|p_{\min}-p_c|\simeq 0.067\ ,$$

$$\frac{R_{e,\min}}{R_M}\simeq 2.3\ ,$$

$$\frac{R_{e,\max}}{R_M}\simeq 10.4\ .$$

The reason why such extreme values of the conductivity ratio are needed in order to observe a sizable peak in R_e is that the critical exponent g, which controls the divergence of R_e, is so small.

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[9]A heuristic proof of this result can be given by noting first that when $\lambda_I=\lambda_M=\lambda$, then also $\lambda_e=\lambda$. Since we assume all λ's to be much smaller than all σ's (this is the low-field assumption), we can expand λ_e in powers of λ_I,λ_M, to linear order. It then follows that $\lambda_e-\lambda_I=(\lambda_M-\lambda_I)X$, where X must be a homogeneous function of order zero of σ_I,σ_M only.

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