Redundant operators for Ising spins

Ganpathy Murthy and R. Shankar
J. W. Gibbs Laboratory of Physics, Yale University, New Haven, Connecticut 06520
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We remind the reader of the important role that could be played in Monte Carlo renormalization-group studies by redundant operators, defined to be changes in the Hamiltonian due to infinitesimal changes of variables. We then overcome the problem of defining an *infinitesimal* change of variables for discrete Ising spins and construct redundant operators in $d=2$ and 3. We analyze how they may be seen in numerical experiments.

I. INTRODUCTION

The Monte Carlo renormalization-group (MCRG) technique is a powerful numerical scheme for studying the critical behavior of statistical mechanical systems or Euclidean field theories on a lattice.[1] By numerical methods one can study the flow of Hamiltonians under renormalization, locate fixed points, measure the eigenvalues of the linearized transformation (i.e., the critical exponents), and so on.

In the numerical studies one expects that if one starts with any critical Hamiltonian H_c, say critical nearest neighbor (NN), then the flow under renormalization will be towards the fixed point H^*, while if one begins with a slightly noncritical system, the renormalized H would initially stay close to the critical surface and move towards H^* but eventually veer away. This picture stems from the classification of perturbations of H^* into relevant and irrelevant ones.[2] (We ignore the rare case of truly marginal operators.)

There is, however, another class of operators, namely, *redundant* operators, which can alter this picture. These are defined by Wegner[3] as changes in H^* when an infinitesimal charge of variables is made in the functional integral. To be concrete, consider the example of

$$Z=\int\prod_x d\phi(x)e^{H(\phi(x))}\ . \tag{1.1}$$

Under a change

$$\phi(x)\to\phi'(x)=\phi(x)+\delta\ , \tag{1.2}$$

which leaves the measure invariant, we find

$$Z\to Z'=\int\prod_x d\phi'(x)e^{H(\phi')+\delta H(\phi')}=Z\ , \tag{1.3}$$

where

$$\delta H(\phi')=-\sum_x \left.\frac{\partial H}{\partial\phi}\right|_{\phi'}\delta\ . \tag{1.4}$$

The physics is unaffected by the change $H\to H+\delta H$. First, the free energy is unaffected since we can convert $H+\delta H$ to H by going backwards from Eq. (1.4) to Eq. (1.1). Secondly, any correlation function $\langle\phi'_1\cdots\phi'_n\rangle$ with respect to $H+\delta H$ has an image in the ensemble generated by H:

$$\begin{aligned}\langle\phi'_1\cdots\phi'_n\rangle_{H+\delta H}&=\int\prod_x d\phi'(\phi'_1\cdots\phi'_n)e^{[H(\phi')+\delta H(\phi')]}\\&=\int\prod d\phi[(\phi_1+\delta)\cdots(\phi_n+\delta)]e^{H(\phi)}\\&=\langle(\phi+\delta\phi)_1\cdots(\phi+\delta\phi)_n\rangle_H\ ,\end{aligned} \tag{1.5}$$

where $\delta\phi=\delta$. These ideas generalize readily to case where $\delta\phi=\delta\phi(x)$. Each change of variables $\phi\to\phi+\delta\phi$ generates its own redundant operator δH.

Let H^* be the fixed point of some renormalization-group transformation R. In the vicinity of H^*, the flow in a redundant direction δH^* is free of physical significance. This is reflected in the fact that the eigenvalue associated with the direction δH^* varies with R.[3] In the study of Pawley *et al.*[4] an odd operator with a spurious eigenvalue occured and was identified as a redundant operator. Fortunately, in this case the operator was odd and the flow to a fixed point (restricted to the even sector in MCRG) was not spoiled. On the other hand, one can envisage the possibility of an even redundant eigenvalue which is repulsive. In this case no fixed point will be found if one starts with a critical (NN say) system and generates a flow. The redundant operator is no longer just a curiosity or nuisance.

With the above remarks to motivate a study of redundant operators we ask how to generate them for Ising systems. Whereas for the case of continuous fields ϕ the notion of an infinitesimal change in ϕ (and hence in H^*) exists, no obvious candidate exists for Ising spins $S=\pm1$. In $d=2$, thinking of Ising systems in terms of continuous fields is not very useful. Besides, MCRG studies are done on $H(S)$, and it will be useful to have examples of $\delta H^*(S)$ that can be interpreted as redundant. These will be constructed in Sec. II. In Sec. III we will consider concrete examples of δH^* in $d=2$ and 3.

II. CONSTRUCTION OF REDUNDANT OPERATORS

Consider an Ising system described by

$$Z=\sum_S e^{H(S)}\equiv\sum_S e^{\sum_i S_i V(S_i)}\ , \tag{2.1}$$

where i is the site label, i is any site not equal to i, and $V(S)$ is any function. For example, in the case of H_{NN}, the nearest-neighbor interaction, $V=K\sum_j S_j$, where j is a NN of i.

We shall now derive the redundant operators δH for this H. Of course we are interested in redundant operators δH^* associated with a fixed point H^*. Since the derivation works for any H, let us first find δH in general.

As mentioned earlier, the problem with mimicking the derivation of δH associated with $H(\phi)$ is that in the case of $H(S)$, $S=\pm 1$, there is no obvious infinitesimal change of variables $S\rightarrow S+\delta S$ that respects $|S|=1$. We circumvent this problem as follows.

Let us introduce a matrix $Q(S',S)$ (where S and S' are each collective labels for all the spins) such that

$$\sum_{S'} Q(S',S)=1 \ . \tag{2.2}$$

Similar functions, $P(S',S)$, called projection operators, were introduced to generate block-spin transformations.[5,6] In those cases, however, S' was defined on a smaller lattice than S, i.e., one was trying to thin out degrees of freedom. Here we want S' and S to be defined on the same lattice. Clearly,

$$\begin{aligned} Z=\sum_S e^{H(S)}&=\sum_{S'S} Q(S',S)e^{H(S)} \\ &=\sum_{S'} e^{H'(S')} \ . \end{aligned} \tag{2.3}$$

By construction $S'=\pm 1$, and $H'(S')$ is defined by

$$e^{H'(S')}=\sum_S Q(S',S)e^{H(S)} \ . \tag{2.4}$$

In going from $H(S)$ to $H'(S')$ we have preserved the partition function, the lattice size, and the condition $|S|=1$, all of which allow us to view the change $H\rightarrow H'$ as the result of a change of variables. To generate an infinitesimal transformation we start with the identity transformation

$$I=\prod_i \left[\frac{1+S_i'S_i}{2}\right]\equiv\prod_i \delta(S_i',S_i) \tag{2.5}$$

and add an infinitesimal generator:

$$Q_\epsilon=\prod_i [\delta(S_i',S_i)+\epsilon S_i' f(S)] \ . \tag{2.6}$$

Here $f(S)$ is any function of the "old" spins S, and the S_i' in front of f ensures Eq. (2.2) is valid. We then have

$$\begin{aligned} e^{H'(S')}&=\sum_S\prod_i [\delta(S_i',S_i)+\epsilon S_i' f(S)]e^{H(S)} \\ &=e^{H(S')}+\epsilon\sum_i\sum_S \left[\prod_{\mathrm{i}}\delta(S_{\mathrm{i}}'S_{\mathrm{i}})\right]S_i' f(S) \\ &\quad\times e^{H(S_i,S_{\mathrm{i}})}+O(\epsilon^2) \ . \end{aligned} \tag{2.7}$$

Let us expand $f(S)$ as

$$f(S)\equiv f(S_i,S_{\mathrm{i}})=f_1(S_{\mathrm{i}})+S_i f_2(S_{\mathrm{i}}) \tag{2.8}$$

and define

$$\Delta H=H(S_i,S_{\mathrm{i}}')-H(S_i',S_{\mathrm{i}}') \tag{2.9}$$

$$=(S_i-S_i')V(S_{\mathrm{i}}') \ , \tag{2.10}$$

where V stands for the terms coupled to the spin at i [see Eq. (2.1)]. Then

$$\begin{aligned} e^{H'(S')}&=e^{H(S')}\left[1+\epsilon\sum_i\sum_{S_i} S_i'[f_1(S_{\mathrm{i}}')+S_i f_2(S_{\mathrm{i}}')]e^{(S_i-S_i')V(S_{\mathrm{i}}')}\right] \\ &=e^{H(S')}\left[1+\epsilon\sum_i S_i' e^{-S_i'V(S_{\mathrm{i}}')}\{2f_1(S_{\mathrm{i}}')\cosh[V(S_{\mathrm{i}}')]+2f_2(S_{\mathrm{i}}')\sinh[V(S_{\mathrm{i}}')]\}\right] \\ &\equiv e^{H(S')+\delta H(S')} \ , \end{aligned} \tag{2.11}$$

which gives us

$$\delta H(S')=2\epsilon\sum_i S_i' e^{-S_i'V(S_{\mathrm{i}}')}[f_1(S_{\mathrm{i}}')\cosh V(S_{\mathrm{i}}') + f_2(S_{\mathrm{i}}')\sinh V(S_{\mathrm{i}}')] \ . \tag{2.12}$$

Before considering explicit examples of δH, let us present the evidence that these are indeed redundant operators associated with H. First, they are generated by a change of coordinates $S\rightarrow S'$ and they do not affect Z (to first order in ϵ, just as is required in the case of continuous fields also) or the lattice size. The sum of any two redundant operators is redundant, as can be seen by compounding the infinitesimal transformation associated with each. For the case $H=H_{\mathrm{NN}}^*$, the critical nearest-neighbor system with $K=\tilde{K}=-\frac{1}{2}\ln\tanh K$, we have verified for the various cases we studied that $H_{\mathrm{NN}}^*+\delta H_{\mathrm{NN}}^*$ lies on the critical surface (CS) by using our earlier work[7] where the equation for the CS near H_{NN}^* was derived. (This test is useful because it eliminates operators ΔH which keep Z the same to first order because $\langle\Delta H\rangle=0$, but which take us out of the CS. These cannot be called redundant since they change by physics, i.e., the correlation length. The redundant δH are characterized by the fact that they are linear combinations of interactions with coefficients *analytic* in the parameters of H.) Next we would like to establish the analog of Eq. (1.5). Consider, for example,

$$\langle S_1'S_2'\rangle_{H+\delta H}=Z^{-1}\sum_{S'}\sum_{S}S_1'S_2'\left[\prod_i[\delta(S_i,S_i')+\epsilon S_i'f(S)]\right]e^{H(S)}$$
$$=\langle S_1,S_2\rangle_H+Z^{-1}\epsilon\sum_{SS_i'}\sum_i\left[\prod_{\mathrm{i}}\delta(S_{\mathrm{i}},S_{\mathrm{i}}')\right]S_i'f(S)e^{H(S)}S_1'S_2'+O(\epsilon^2)\ . \tag{2.13}$$

The first term comes from taking the δ from each site and doing the sum over S' first, while the second comes from taking δ at all sites but one where $\epsilon S'f$ is taken instead. Consider a given value of i. If we sum over S_i' we will get 0 (because of the factor S_i') unless S_1' or S_2' gets rid of it, i.e., unless $i=1$ or 2. It is evident that

$$\langle S_1'S_2'\rangle_{H+\delta H}=\langle S_1,S_2\rangle_H+\langle\epsilon f(S)_1,S_2\rangle+\langle S_1,\epsilon f(S)_2\rangle_H$$
$$=\langle[S+\epsilon f(S)]_1,[S+\epsilon f(S)]_2\rangle_H\ . \tag{2.14}$$

Thus the correlation functions transform as if the change $S\rightarrow S+\epsilon f(S)$ has been made. We say "as if ", because the literal change $S\rightarrow S+\epsilon f(S)$ cannot be made without violating $|S|=1$. For those readers who would like to see redundant operators arising from a literal infinitesimal change of variables (as in the case of ϕ) we provide the following alternate version, which for practical reasons is limited to the case H_{NN}. Let us work with the transfer matrix $T(\sigma)$, written in terms of Pauli matrices such that

$$A=\mathrm{Tr}T^R\ ,$$

where R is the number of rows (in $d=2$) or layers (in$d=3$). The trace is originally taken in the eigenbasis of σ_3, the eigenvalues $\pm S$ being the spins summed over in the nonoperator version. However, as the trace can be taken in any basis related by a unitary transformation U, let us take it in the basis of $\tilde{\sigma}=U^\dagger\sigma U$. Let us now express every σ in T in terms of $\tilde{\sigma}$. This amounts to making an infinitesimal (operator) change of variables $\sigma\rightarrow\sigma+\delta\sigma$, where $\delta\sigma=i\epsilon[\Omega,\sigma]$ for the case $U=I+i\epsilon\Omega$. (We restrict ourselves to unitary rather than similarity transformations since we want $\tilde{\sigma}$ to be isomorphic to σ and, in particular, Hermitian.) Under this change

$$T\rightarrow UTU^\dagger=T+\Delta T=T(1+\epsilon\rho)\ .$$

If we now go back to S language, $\epsilon\rho$ will correspond to the additional interaction δH_{NN} which is clearly redundant. In Appendix A we give a concrete example of such a δH for $d=2$ and it agrees with the redundant δH found via Eq. (2.12) for a certain choice of f. The transfer-matrix approach is much more cumbersome and for practical reasons limited to δH_{NN}. It is presented here only to corraborate the fact that what we are doing indeed corresponds to a change of variables.

Finally, we would like to show that if δH^* is redundant, then under any RG transformation it goes into $\delta H^{*'}$, which is also redundant in the sense that $\delta H^{*'}$ is generated from H^* by a change of variables.

Consider a vector $V(H)$ which has 2^N components labeled by the spins S on a lattice with N sites, and having values

$$V_S(H)=e^{H(S)}\ . \tag{2.15}$$

We may view Q_ϵ Eq. (2.6) as a square $(2^N\times 2^N)$ matrix in this space:

$$V_{S'}(H')=\sum_S Q_{S'S}V_S(H)\ ,$$

or in an compact notation

$$V(H')=QV(H)\ . \tag{2.16}$$

Although Q acts linearly on V the action on H is nonlinear. To find $Q{:}H\rightarrow H'$, we must start with $V_S(H)$, transform by Q, extract the couplings from $V_{S'}(H')$, and reconstruct H. (This can be done since there are 2^N known Boltzmann weights and 2^N interactions.) The usual block-spin transformation generated by $P(S',S)$ is represented in this notation as

$$V'(H')=PV(H)\ , \tag{2.17}$$

where P is a *rectangular* matrix with $2^{N/L^2}$ rows and 2^N columns where L is the "block size," i.e., L^2 equals the number of spins after divided by the number of spins before.

As the matrix P becomes infinitely large, the fact that $V(H)$ can carry more interactions than $V'(H')$ becomes unimportant. If H^* is a fixed point of P, we cannot of course say $V'(H^*)=V(H^*)$ since they have different dimensions, but we can say that if H' is extracted from V', it agrees with the H^* put into V for all couplings of finite range.

If we start with $V(H^*)$ and let $Q=I+\epsilon F$ act on it, it produces a redundant change

$$\Delta V(H^*)=\epsilon FV(H^*)\ . \tag{2.18}$$

We wish to show that if one implements a blocking P [which takes $V(H^*)$ to $V'(H^*)$] on $V(H^*)+\Delta V(H^*)$, the result is $V'(H^*)+\Delta V'(H^*)$, where $\Delta V'(H^*)=\epsilon F'V'(H^*)$ where F' is some generator, like F. Therefore, let us consider

$$P(V+\Delta V)=PV(H^*)+P\Delta V(H^*)$$
$$=V'(H^*)+\epsilon PFV(H^*)$$
$$\equiv V'(H^*)+\Delta V'(H^*)\ . \tag{2.19}$$

Therefore,

$$\Delta V'(H^*)=\epsilon PFV(H^*)\ . \tag{2.20}$$

Suppose we can write

$$PF=F'P\quad\text{for some } F'\ , \tag{2.21}$$

then

$$\Delta V'(H^*)=\epsilon F'PV(H^*)$$
$$=\epsilon F'V'(H^*)\ , \tag{2.22}$$

i.e., $\Delta V'(H^*)$ is the redundant change produced by the generator $\epsilon F'$. If we set $2^N = M$, $2^{N/L^2} = m$, then Q is $M \times M$; P is $m \times M$; F is $M \times M$, F' is $m \times m$. Equation (2.21) can be solved for F':

$$F' = PFP^{-1} , \tag{2.23}$$

where P^{-1} is the right inverse of P. There is no problem constructing such an inverse; in fact, there is an infinite number of them since P corresponds to m equations among M unknowns. (We are making the mild assumption that the m rows of P are linearly independent.) Different choices of F' will correspond to different changes of variables, but $\Delta V'$ generated by them will be the same in all cases since they differ by the annihilators of V'. One can also check that $Q' = I + \epsilon F'$ obeys Eq. (2.2), i.e.,

$$\sum_{S'} Q'(S',S) = 1$$

or

$$\sum_{S'} F'(S',S) = 0 .$$

III. CONCRETE EXAMPLES

Let us return to Eq. (2.12) and generate different examples of δH by choosing f_1 and f_2 at will. First, let us note that in Eq. (2.12) the quantity in brackets is just some function $A(S_i)$. The breakdown into f_1 and f_2 will no longer interest us since we do not want to follow the change of correlation functions as per Eq. (2.14). Therefore, we write

$$\delta H = \epsilon \sum_i S_i e^{-S_i V(S_i)} A(S_i) ; \tag{3.1}$$

written this way, δH coincides with Eq. (3) of Dekeyser and Rogiers,[8] who derived linear relations between correlation functions by finding operators X for which $\langle X \rangle = 0$ due to the symmetry of the measure. Such relations which also occured elsewhere[9,10] were, however, not seen as resulting from infinitesimal transformations, and X was not viewed as a redundant operator.

Since we are interested in a δH^* associated with a fixed point H^*, we must first pick our H^* and then each choice of A will generate a redundant operator. Clearly the simpler H^* and A are, the simples δH^* will be. We will choose $H^* = H_{\rm NN}$, the critical nearest-neighbor interaction first in $d=2$ and then in $d=3$. Although $H^*_{\rm NN}$ is not the fixed point for the usual 2×2 or 3×3 blockings,[11,12] Swendsen has shown[13] that "optimized" RG transformations for which this is the fixed point do exist.

In $d=2$, if we want redundant operators involving just the coupling of any spin and its four nearest neighbors, we can choose

$$A = A_{\alpha\beta\gamma\delta} = S_1^\alpha S_2^\beta S_3^\gamma S_4^\delta ,$$

where $S_1, \dots, S_4$ are the four neighbors and α through $\gamma = 0$ or 1. These 16 relations yield (upon using various symmetries) six redundant operators which are listed in Ref. 8. We mention just the even ones here:

$$\delta H_1 = \epsilon \sum_{\rm sites} [S^3 H_{1357} + 2C(2C^2-1)H_{01} - 2SC^2 H_{04} - SC^2 H_{02} - S(3C^2-1)] , \tag{3.2a}$$

where $C = \cosh(2K)$ and $S = \sinh(2K)$, so that at $K = 0.440\,68\dots$,

$$\delta H_1^* = \epsilon \sum_{\rm sites} (H_{1357} + 6\sqrt{2} H_{01} - 4H_{04} - 2H_{02} - 5) , \tag{3.2b}$$

$$\delta H_2 = \epsilon \sum [(C^3 + 2S^2 C + C)H_{01} - 2C^2 S H_{04} + \tfrac{1}{2} S^2 C H_{0124} - SC^2 H_{02} - 2C^2 S] , \tag{3.3a}$$

which implies

$$\delta H_2^* \equiv \epsilon \sum (5\sqrt{2} H_{01} - 4H_{04} + 2^{-1/2} H_{0124} - 2H_{02} - 4) . \tag{3.3b}$$

The convention for the naming and normalization of these operators $H_{ijk\dots}$ is given in Appendix A following Eq. (A19). Note incidentally that the δH in Eq. (A19), derived in the transfer-matrix approach, agrees with δH_2 above.

In $d=3$ we shall focus on odd redundant operators since one may have been seen in numerical studies.[4] Here are six with the shortest possible ranges of interactions:

$$\delta H_1 = (1-6a)A + a^2 B + a^2 C - a^3 D - a^3 E + a^4 F + a^4 G - a^5 H + a^6 M , \tag{3.4a}$$

$$\delta H_2 = (a^2 - 2a - 4a^3)A + (1 + 6a^2 + 5a^4)B/12 + (2a^2 + a^4)C/3 - (2a + 8a^3 + 2a^5)D/12 - (2a + 4a^3 + 2a^5)E/8 + (a^2 + 2a^4)F/3 + (5a^2 + 6a^4 + a^6)G/12 - (4a^3 + 2a^5)H/6 + a^4 M , \tag{3.4b}$$

$$\delta H_3 = (-2a + a^2 - 4a^3)A + (8a^2 + 4a^4)B/12 + (1 + 2a^4)C/3 - (4a + 4a^3 + 4a^5)D/12 - 8a^3 E/8 + (2a^2 + a^6)F/3 + (4a^2 + 8a^4)G/12 - (4a^3 + 2a^5)H/6 + a^4 M ,$$

where $a = \tanh K$ and should be set equal to tanh (0.221 654) to get δH^*. There are three more such operators δH_4, δH_5, and δH_7) obtained by changing $a \to 1/a$ and multiplying by a^6.

The operators A–M are defined as follows. Imagine spins numbered in a plane as in Appendix A [following Eq. A19)] with an identical layer above, numbered 0^+ through 9^+. Then $A = H_0$, $B = H_{013}$, $C = H_{012}$, $D = H_{135}$, $E = H_{130^+}$, $F = H_{13457}$, $G = H_{01241^+}$, $H = H_{13574^+}$, and M is the product of a spin with its six nearest neighbors. The multiplicity of each term is as in $d=2$. For example, there are 12 terms per site implied in D or G, eight per site in E, etc.

By forming linear combinations of these operators we can get new ones involving just three or four:

$$\delta H_7 = (1-6a+3a^2-12a^3+3a^4-6a^5+a^6)a^{-2}A + (1+a^2)B - aE \ , \tag{3.5}$$

$$\delta H_8 = (1-6a+3a^2-12a^3+3a^4-6a^5+a^6)a^{-2}A + 2(1+a^2)B/3 + 4(1+a^2)C/3 - 2aD/3 \ . \tag{3.6}$$

Now that we have a few short-range redundant operators, can we ask how they tie in with the numerical work? This turns out to be a difficult question. First of all, we do not have redundant eigenvectors, simply redundant operators. Next, we have no idea what the redundant eigenvalues will be since they depend on the optimized transformations that make H^*_{NN} the fixed point. Lastly, in the numerical work one finds the eigenvectors and eigenvalues of $T_{\alpha\beta}$, which is the linearized transformation matrix T^∞ truncated to a finite-dimensional subspace of interactions. The connection between the solution to this problem and that of the exact one in infinite dimensions is unclear. Also, given that in practice one must work with a finite number of couplings, say 15, it is not clear which 15 to choose to get the leading eigenvalues. For the physical (i.e, relevant and irrelevant couplings) there is reason to believe that one must start within the shortest-range couplings and work upwards in range, i.e., the belief is that the leading eigenvectors will be saturated by a few short-range interactions and hence recovered without much mutilation in a small subspace calculation. We do not know if this applies to the unphysical redundant ones, although it seems plausible in momentum space renormalization.

In view of all this we can say only the following. Consider the even redundant operators δH^*_1 and δH^*_2 in $d=2$. We know from the exact solution that all irrelevant eigenvalues are integers. If the eigenvalue problem of T^∞ is solved any noninteger eigenvalue must be associated with a redundant operator. If the eigenvector is saturated by short-range couplings one can reasonably hope to see it as an eigenvector of the truncated $T_{\alpha\beta}$. The only way in practice that we will know this is happening is that as $T_{\alpha\beta}$ is made larger, the eigenvector and eigenvalue remain stable. If the vector lies in the subspace H_{1357}, H_{01}, H_{04}, and H_{02}, we can expect it to be close to δH^*_1 [Eq. (3.2b)]; if it lies in the space H_{01}, H_{04}, H_{0124}, and H_{02} it will be close to δH^*_2; and if it involves all of the above interactions it must be a linear combination of δH^*_1 and δH^*_2. Conversely, any eigenvector (stable under enlargement of $T_{\alpha\beta}$) that lies in this subspace must be linearly independent of δH^*_1 and δH^*_2 if the corresponding eigenvalue is an integer. It is also clear that we must work in spaces involving the above couplings before a redundant operator could possibly enter.

What if no redundant eigenvectors (i.e., noninteger eigenvalues) are found if $T_{\alpha\beta}$ is studied in the space of the H's mentioned above? Then we would have to conclude that δH^*_1 and δH^*_2 are nowhere near being eigenvectors of T^∞, and that they are linear combinations of eigenvectors which are predominantly outside the subspace in question. (We are ignoring the freak case of a redundant eigenvalue being accidentally integer.)

In $d=3$, if a stable odd repulsive eigenvector besides the leading one is found, and it is saturated by short-range couplings, we can expect it to be a linear combination of the six δH's given earlier. It must also be true that the leading relevant one is linearly independent of these.

IV. CONCLUSION

We have shown that redundant operators can be derived for discrete spins despite the fact that there seems to be no obvious way to perform an infinitesimal change of variables. Although a literal infinitesimal change is still out of the question, we found a way to induce infinitesimal transformations in which H is charged by an infinitesimal amount δH. We showed that the δH's so obtained met all the tests for redundancy: They formed a subspace, they left the physics invariant, they were closed under renormalization.

Note added in proof. For clear evidence of redundant operators (that were derived by these methods) occurring in the odd sector in $d=2$, see Ref. 14.

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APPENDIX A

Here we will derive a redundant operator associated with H_{NN} in $d=2$ in the transfer-matrix formalism. As explained earlier the chief attraction of this approach is that a literal, infinitesimal change of (operator) variables generates δH.

Let us first write (for general K) the partition function on a lattice with M rows and N columns (numbered by integers m and n, respectively) in terms of the transfer matrix T:

$$Z = [2\sinh(2K)]^{M/2}\text{Tr}T^M \ , \tag{A1}$$

$$T = \exp\left[K\sum_n \sigma_3(n)\sigma_3(n+1)\right]\exp\left[\widetilde{K}\sum_n \sigma_1(n)\right] \ . \tag{A2}$$

Here $\widetilde{K} = -\frac{1}{2}\ln\tanh K$ and $\sigma(n)$ are the Pauli matrices at the sites of a one-dimensional lattice labeled by the interger n. The sum over S in Z is now viewed as the trace over the combined eigenvalues of the $\sigma_3(n)$'s. We now replace the sum over S by the sum over S' (again $=\pm 1$) which are the eigenvalues of the transformed operators

$$\widetilde{\sigma}_3(n) = U^\dagger \sigma_3(n) U \ . \tag{A3}$$

This leaves Z invariant since the trace is invariant under a unitary change of basis. If we now replace all the σ's by $\widetilde{\sigma}$'s in T and reconstruct the new Boltzmann weights we will find these correspond to $H_{\rm NN}(S')+\delta H(S')$. Since U can be chosen close to the identity

$$U=I+i\epsilon\Omega \quad (\Omega=\Omega^{\dagger}) , \tag{A4}$$

$\delta H(S')$ will really be infinitesimal and redundant by construction. At the end we will of course set $K=\widetilde{K}$, the critical value, to get the redundant operator for $H^{*}_{\rm NN}$.

To illustrate our procedure we choose

$$\Omega=\sum_n \sigma_2(n)\sigma_3(n+1) . \tag{A5}$$

Under the action of Ω, T changes as follows:

$$\begin{aligned} T &\to UTU^{\dagger}\\ &=T+i\epsilon[\Omega,T]\\ &=(1+i\epsilon T^{-1}[\Omega,T])\\ &=T[I+\epsilon(T^{-1}\Omega T-\Omega)]\\ &\equiv T[1+\epsilon\rho] . \end{aligned} \tag{A6}$$

In order to calculate

$$\rho=i[T^{-1}\Omega T-\Omega]\equiv i\hat{\Omega}-i\Omega , \tag{A7}$$

we shall exploit the linear relation between ρ and Ω. We will first calculate ρ due to just one term in Eq. (A5), with $n=0$; and then do the sum over n in the answer so obtained. In terms of $\sigma\equiv\sigma(0)$, $\bar{\sigma}=\sigma(-1)$, $\sigma'=\sigma(1)$,

$$\rho=i\hat{\sigma}_2\hat{\sigma}_3'-i\sigma_2\sigma_3' , \tag{A8}$$

where

$$\hat{\sigma}=T^{-1}\sigma T . \tag{A9}$$

Consider first

$$-i\sigma_2\sigma_3'=\sigma_1\sigma_3\sigma_3' . \tag{A10}$$

Now one can check,[15] by computing $\hat{\sigma}_3=T^{-1}\sigma_3 T$, that

$$\sigma_1=(C-S\hat{\sigma}_3\sigma_3) , \tag{A11}$$

where

$$C=\cosh 2K,\quad S=\sinh 2K . \tag{A12}$$

Setting this into Eq. (A10), we get

$$-i\sigma_2\sigma_3'=C\sigma_3\sigma_3'-S\hat{\sigma}_3\sigma_3' . \tag{A13}$$

Our motivation for writing everything in terms of σ_3 and $\hat{\sigma}_3$ is that one can read off the corresponding change δH in the Ising interaction by inspection. For illustration let us consider the $d=1$ case. Let $E_0(S',S)$ be the energy of two adjacent "rows" with spins S' and S. Then the transfer matrix T obeys

$$\exp E_0(S',S)=\langle S' \,|\, T \,|\, S\rangle . \tag{A14}$$

Let us now add a term $\epsilon E(S',S)$ to E_0. The new T given by $T(1+\epsilon\rho)$ obeys

$$\exp[E_0(S',S)][1+\epsilon E(S',S)]=\langle S' \,|\, T(1+\epsilon\rho) \,|\, S\rangle , \tag{A15}$$

so that

$$\epsilon e^{E_0(S',S)}E(S',S)=\langle S' \,|\, \epsilon T\rho \,|\, S\rangle . \tag{A16}$$

Let $E(S',S)=a(S+S')+bSS'$. We can get all these terms to occur on the right-hand side by replacing every S by a σ_3 acting to the right on $|S\rangle$ and every S' by a σ_3 acting to the left on $\langle S'|$. For example,

$$\begin{aligned} e^{E_0(S',S)}S'S=S'S\langle S' \,|\, T \,|\, S\rangle &=\langle S' \,|\, \sigma_3 T\sigma_3 \,|\, S\rangle\\ &=\langle S' \,|\, TT^{-1}\sigma_3 T\sigma_3 \,|\, S\rangle\\ &=\langle S' \,|\, T\hat{\sigma}_3\sigma_3 \,|\, S\rangle . \end{aligned} \tag{A17}$$

Conversely a term $\epsilon\hat{\sigma}_3\sigma_3$ in $\epsilon\rho$ translates into an interaction $\epsilon S'S$. The extension to $d=2$ (or 3) is direct, one lets S' and S be a shorthand for the spins in two adjacent rows (or layers). Given all this we can see that Eq. (A13) translates into the following interaction among the Ising spins (if we remember to reinstate the sum over n):

$$\delta H=\sum_{\rm sites}(CS_{n,m}S_{n,m+1}-SS_{n+1,m}S_{n,m+1}) . \tag{A18}$$

We must next treat the first term in Eq. (A8), $i\hat{\sigma}_2\hat{\sigma}_3$, in a similar fashion, expressing all operators in terms of $\hat{\sigma}_3$'s and σ_3's, with all quantities with carets on the left. In this "normal ordered" form, the classical interaction can be read off by inspection. The final outcome of the calculation (relegated to Appendix B) is that

$$\begin{aligned} \delta H=\epsilon\sum_{\rm sites}[&(C^3+2S^2C+C)H_{01}-2C^2SH_{04}\\ &+\tfrac{1}{2}S^2CH_{0124}-SC^2H_{02}-2C^2S] . \end{aligned} \tag{A19}$$

The interaction H_{0124}, for example, stands for a coupling $S_0S_1S_2S_4$ in the following notation:

$$\begin{bmatrix} 0 & 1 & 2\\ 3 & 4 & 5\\ 6 & 7 & 8 \end{bmatrix} ,$$

as well as three other terms (per site) obtained by rotating and reflecting the triangle 0124. Likewise H_{02} stands for the coupling S_0S_2 and its partner related by a $+90°$ rotation. (The multiplicity of each term is such that each interacting pair or quartet of spins is represented once and once only when the sum over sites is carried out.)

The careful reader may ask how we could get a coupling between two spins that are separated by two sites in the y direction when our transfer-matrix formalism can accomodate only coupling between adjacent rows. The answer is that the Ω we began with only generates the coupling S_0S_2 but not its partner, S_6S_0. Likewise it only generates the triangle 0124 but not its three companions. Thus if we imagine $H_{\rm NN}$ embedded in a space of generally anisotropic couplings, Ω generates a redundant direction δH_Ω that points in the space of anisotropic (but translationally invariant) interactions. What this means is that if one perturbes $H_{\rm NN}$ with anistropic terms and carries out the isotropic RG of Swendsen, one of the directions, δH_Ω, is redundant. Suppose we repeat the analysis using an Ω' which is obtained by reflecting Ω about a column. This will generate $\delta H_\Omega{}'$ which is related to δH_Ω

by the same reflection since T is reflection symmetric. Since the sum of two redundant operators is also redundant, $\delta H_\Omega + \delta H_\Omega'$ is also a redundant operator. Likewise if we use the column-to-column transfer matrix we will generate a δH that will be related to δH_Ω by a 90° rotation. More generally, we could argue that given a δH_Ω which is redundant, there must be other redundant operators related to it by the symmetries of H^* and the RG transformation. The sum of all such operators will then be redundant and lie in the space of symmetric interactions. This is what is given by Eq. (A19). Finally, we must put $C=\sqrt{2}$ and $S=1$ to get δH^*, the redundant operator assocated with H^*_{NN}.

APPENDIX B

Here we convert the term $i\hat{\sigma}_2\hat{\sigma}_3'$ in ρ [Eq. (A8)] to a classical interaction by expressing everything in terms of $\hat{\sigma}_3$'s and σ_3's, with all quantities with carets situated to the left. Therefore, let us begin with

$$i\hat{\sigma}_2\sigma_3' = \hat{\sigma}_3\hat{\sigma}_1\hat{\sigma}_3' \tag{B1}$$

and try to eliminate $\hat{\sigma}_1$. Since $\sigma_1 \equiv \sigma_1(0)$,

$$\hat{\sigma}_1 = T^{-1}\sigma_1 T = e^{-\tilde{K}\sum\sigma_1} e^{-K(\sigma_3\sigma_3'+\sigma_3\bar{\sigma}_3)} \times \sigma_1 e^{K(\sigma_3\bar{\sigma}_3+\sigma_3\sigma_3')} e^{\tilde{K}\sum\sigma_1} . \tag{B2}$$

If we now use relations like

$$e^{K\sigma_3\sigma_3'} = \cosh K + \sigma_3\sigma_3'\sinh K \tag{B3}$$

[because $(\sigma_3\sigma_3')^2=1$] we get

$$\hat{\sigma}_1 = e^{-\tilde{K}\sum\epsilon\sigma_1}(C^2\sigma_1 + SC\sigma_1\sigma_3\bar{\sigma}_3 + SC\sigma_1\sigma_3\sigma_3' + S^2\sigma_1\bar{\sigma}_3\sigma_3')e^{\tilde{K}\sum\sigma_1} . \tag{B4}$$

When we take $\exp(-\tilde{K}\sum\sigma_1)$ through the parentheses and pair it with $\exp(\tilde{K}\sum\sigma_1)$, (i) it leaves all σ_1's alone, and (ii) replaces every σ_3, $\bar{\sigma}_3$, and σ_3' by $\hat{\sigma}_3$, $\hat{\sigma}_3$, and $\hat{\sigma}_3'$ since as far as these are concerned, it is like taking T^{-1} through the parentheses and pairing it with T.

We next feed this expression with carets in the right places back in Eq. (A8), eliminate σ_1 using

$$\sigma_1 = C - S\hat{\sigma}_3\sigma_3 , \tag{B5}$$

and take all quantities with carets on the left using

$$\begin{gathered}[\hat{\sigma}_3,\ \sigma_3',\ \text{or}\ \hat{\sigma}_3]=0,\quad [\sigma_3(n),\hat{\sigma}_3(m)]=0,\quad m\neq n \\ \sigma_3\hat{\sigma}_3 = -\hat{\sigma}_3\sigma_3 + 2C/S ,\end{gathered} \tag{B6}$$

and read off the classical interaction. Finally, we add the two terms coming from the second piece, $-i\sigma_2\sigma_3'$ in Eq. (A8) for ρ, symmetrize the expression using reflections, rotations, etc., to get Eq. (A19).

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